The centers of gravity of the associahedron and of the permutahedron are the same

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Abstract

In this article, we show that Loday's realization of the associahedron has the the same center of gravity as the permutahedron. This proves an observation made by F. Chapoton.

We also prove that this result holds for the associahedron and the cyclohedron as realized by the first author and C. Lange.

1 Introduction.

This article is the continuation of previous work [7, 8, 1] devoted to the study of generalized associahedra via geometric and combinatorial tools arising from finite Coxeter groups.

In 1963, J. Stasheff discovered a cell complex [21, 22] of great importance in algebraic topology, geometric topology and combinatorics ([2, 6, 10]). This cell complex can be realized as a simple $n-1$-dimensional convex polytope in $\mathbb{R}^n$: the *associahedron*. Many realizations of the associahedron were given over the last thirty years ([13]). In

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2004, J. L. Loday ([9]) computed the classical realization of the associahedron given by S. Shnider and S. Sternberg in [19]. Loday gave a beautiful combinatorial algorithm to compute the integer coordinates of the vertices of the associahedron, and showed that his realization can be obtained naturally from the *classical permutahedron* of dimension $n-1$, i.e. the convex hull of all possible permutations of the point $(1, \ldots, n)$ in $\mathbb{R}^n$. The permutahedron encodes the combinatorics and the geometry behind the symmetric group S_n. The main motivation behind Loday's realization is to study the geometric and combinatorial properties of the associahedron via those of the well-known permutahedron.

Among the many areas of mathematics where the associahedron appears, cluster algebra and Cambrian fans are the ones motivating our work. In 2000, S. Fomin and A. Zelevinsky introduced a new family of fans indexed by Weyl groups called *cluster fans*, whose structure encodes the one of finitely generated cluster algebras [5]. These fans are polytopal and the corresponding polytopes, called *generalized associahedra*, have been first realized by F. Chapoton, S. Fomin and A. Zelevinsky in [4]. In this realization, not only is the face structure of generalized associahedra relevant, but so is their geometry. It turned out that generalized associahedra associated to symmetric groups are combinatorially isomorphic to Stasheff's associahedra, but different from Loday's realization.

In 2006, N. Reading came up with an elegant framework that suggested how to study generalized associahedra via the theory of Coxeter groups, a superclass of Weyl groups. The key objects introduced by N. Reading are called *Cambrian fans and lattices* [14, 15, 16, 17]. For each orientation of the Coxeter graph of a given finite Coxeter group, there is a Cambrian fan which provides a combinatorial and geometric interpretation of cluster fans as well as an explanation of their links with quiver theory as discussed by R. Marsh, M. Reineke and A. Zelevinsky in [12].

The discovery of polytopal realizations of Cambrian fans built from W-permutahedra [8] strengthened the relationship between Coxeter groups theory and Cambrian theory. It is now worth mentioning that the permutahedron can be realized for any finite Coxeter group W simply by taking the convex hull of the orbit of a generic point under the reflective action of W. These realizations enjoy two important properties. First, they generalize Loday's realization to any finite Coxeter group as they are obtained by simply removing some of the defining halfspaces of the W-permutahedron (Figure 1). Second, they provide combinatorial and geometric interpretations of cluster fans and their links to quiver theory since their normal fans are of Cambrian nature. We name these realizations *c-generalized associahedra*, where c refers to the orientation of the Coxeter graph of W.

We investigate the following question : what are the geometric properties that are preserved when we construct a *c*-generalized associahedron from a given W-permutahedron? We believe that the answer to this question would help us to highlight and refine the links between finite Coxeter groups theory and Cambrian theory. Except for the isometry classes of these realizations [1], little is known about them. What are their volumes? The number of integer points they contain? Their isometry groups?

It has been observed in numerous and large example (see [7, 8]) that the center of gravity remains unchanged whatever the orientation is chosen to be. J. L. Loday already reported in [9, Section 2.11] an observation made by F. Chapoton that the centers of
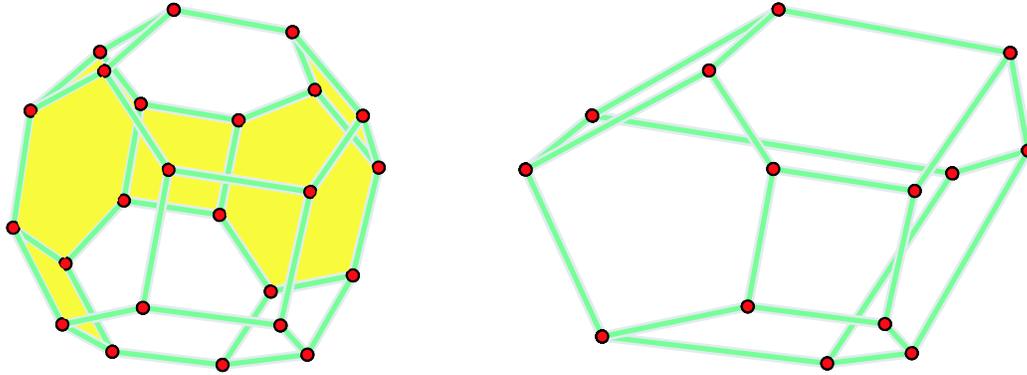
Figure 1: We obtain the associahedron (right) from the permutahedron (left) for the Coxeter group S_4 and the left-to-right orientation by removing all shaded halfspaces.

gravity of the vertices of the associahedron and of the permutahedron are the same. No proof is given, and after asking both F. Chapoton and J. L. Loday, it seems that this property of Loday's realization has never been proven until now. For symmetric groups (type A) and hyperoctahedral groups (type B), c-generalized associahedra were first realized by C. Hohlweg and C. Lange [7], a realization that recovers Loday's for a particular choice of orientation of the Coxeter graph of type A.

In this article we prove that for type A and B, the center of gravity remains unchanged when we remove halfspaces from the permutahedron to build a c-generalized associahedron.

The center of gravity (also known as isobarycenter or centroid) is a classical invariant of configuration of points whose significance in mechanical physics and classical euclidean geometry is not disputed. It also appears as a powerful tool in computational geometry and computer science (see [18, 11]).

Our result highlights an interesting partition of the set of vertices of c-generalized associahedra. A classical way to prove that a point is the center of gravity is to find enough isometries of the polytope whose axes intersect in this point. Unfortunately, already in the case A_2, there is only one nontrivial isometry of c-generalized associahedra, so we may only conclude that the center of gravity takes place on this axe. In order to overcome this problem, we find a partition of the set of vertices, which are parameterized in this case by the triangulations of a regular polygon, and by their isometry classes under the action of the corresponding dihedral group. Then, we show that the center of gravity for each of these classes is the same as the center of gravity of the permutahedron. For the reader familiar with Cambrian fans and c-clusters, we want to note that preliminary computations on general cases where the vertices are parameterized by c-clusters, allow us to observe the same phenomenon and identify orbits of a particular nonlinear action of a group on subsets of almost positive roots. The study of this group and its action will be the subject of future publications.

The article is organized as follows. In §2, we first recall the realization of the permutahedron and how to compute its center of gravity. Then we compute the center of gravity of Loday's realization of the associahedron. In order to do this, we partition its vertices into isometry classes of triangulations, which parameterize the vertices, and we show that the center of gravity for each of those classes is the center of gravity of the permutahedron.

In §3, we show that the computation of the center of gravity of any of the realizations given by the first author and C. Lange is reduced to the computation of the center of gravity of the Loday's classical realization of the associahedron. We do the same for the cyclohedron in §4.

We are grateful to Carsten Lange for allowing us to use some of the pictures he made in [7].

2 Center of gravity of the classical permutahedron and associahedron

2.1 The permutahedron

Let S_n be the symmetric group acting on the set $[n] = \{1, 2, \ldots, n\}$. The *permutahedron* $\mathsf{Perm}(S_n)$ is the classical $n-1$-dimensional simple convex polytope defined as the convex hull of the points

$$M(\sigma) = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{R}^n, \qquad \forall \sigma \in S_n.$$

The *center of gravity* (or *isobarycenter*) is the unique point G of $\mathbb{R}^n$ such that

$$\sum_{\sigma \in S_n} \overrightarrow{GM(\sigma)} = \overrightarrow{0}.$$

Since the permutation $w_0 : i \mapsto n+1-i$ preserves $\mathsf{Perm}(S_n)$, we see, by sending $M(\sigma)$ to

$$M(w_0\sigma) = (n+1-\sigma(1), n+1-\sigma(2), \ldots, n+1-\sigma(n)),$$

that the center of gravity is $G = (\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2})$.

2.2 Loday's realization

We present here the realization of the associahedron given by J. L. Loday [9]. However, instead of using planar binary trees, we use triangulations of a regular polygon to parameterize the vertices of the associahedron (see [7, Remark 1.2]).

2.2.1 Triangulations of a regular polygon

Let P be a regular $(n+2)$-gon in the Euclidean plane with vertices $A_0, A_1, \ldots, A_{n+1}$ in counterclockwise direction. A *triangulation of P* is a set of n noncrossing diagonals of P.

Let us be more explicit. A *triangle of P* is a triangle whose vertices are vertices of P. Therefore a side of a triangle of P is either an edge or a diagonal of P. A triangulation of P is then a collection of n distinct triangles of P with noncrossing sides. Any of the triangles in T can be described as $A_i A_j A_k$ with $0 \leqslant i < j < k \leqslant n+1$. Each $1 \leqslant j \leqslant n$ corresponds to a unique triangle $\Delta_j(T)$ in T because the sides of triangles in T are noncrossing.

Therefore we write $T = \{\Delta_1(T), \ldots, \Delta_n(T)\}$ for a triangulation T, where $\Delta_j(T)$ is the unique triangle in T with vertex A_j and the two other vertices A_i and A_k satisfying the inequation $0 \leqslant i < j < k \leqslant n+1$.

Denote by $\mathcal{T}_{n+2}$ the set of triangulations of P.

2.2.2 Loday's realization of the associahedron

Let T be a triangulation of P. The *weight $\delta_j(T)$* of the triangle $\Delta_j(T) = A_i A_j A_k$, where $i < j < k$, is the positive number

$$\delta_j(T) = (j - i)(k - j).$$

The weight $\delta_j(T)$ of $\Delta_j(T)$ represents the product of the number of boundary edges of P between A_i and A_j passing through vertices indexed by smaller numbers than j with the number of boundary edges of P between A_j and A_k passing through vertices indexed by larger numbers than j.

The *classical associahedron* $\mathsf{Asso}(S_n)$ is obtained as the convex hull of the points

$$M(T) = (\delta_1(T), \delta_2(T), \ldots, \delta_n(T)) \in \mathbb{R}^n, \quad \forall T \in \mathcal{T}_{n+2}.$$

We are now able to state our first result.

Theorem 2.1. *The center of gravity of $\mathsf{Asso}(S_n)$ is $G = (\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2})$.*

In order to prove this theorem, we need to study closely a certain partition of the vertices of P.

2.3 Isometry classes of triangulations

As P is a regular $(n+2)$-gon, its isometry group is the dihedral group $\mathcal{D}_{n+2}$ of order $2(n+2)$. So $\mathcal{D}_{n+2}$ acts on the set $\mathcal{T}_{n+2}$ of all triangulations of P: for $f \in \mathcal{D}_{n+2}$ and $T \in \mathcal{T}_{n+2}$, we have $f \cdot T \in \mathcal{T}_{n+2}$. We denote by $\mathcal{O}(T)$ the orbit of $T \in \mathcal{T}_{n+2}$ under the action of $\mathcal{D}_{n+2}$.

We know that G is the center of gravity of $\mathsf{Asso}(S_n)$ if and only if

$$\sum_{T \in \mathcal{T}_{n+2}} \overrightarrow{GM(T)} = \overrightarrow{0}.$$

As the orbits of the action of $\mathcal{D}_{n+2}$ on $\mathcal{T}_{n+2}$ form a partition of the set $\mathcal{T}_{n+2}$, it is sufficient to compute

$$\sum_{T \in \mathcal{O}} \overrightarrow{GM(T)}$$

for any orbit $\mathcal{O}$. The following key observation implies directly Theorem 2.1.

Theorem 2.2. *Let $\mathcal{O}$ be an orbit of the action of $\mathcal{D}_{n+2}$ on $\mathcal{T}_{n+2}$, then G is the center of gravity of $\{M(T) \,|\, T \in \mathcal{O}\}$. In particular, $\sum_{T \in \mathcal{O}} \overrightarrow{GM(T)} = \overrightarrow{0}$.*

Before proving this theorem, we need to prove the following result.

Proposition 2.3. *Let $T \in \mathcal{T}_{n+2}$ and $j \in [n]$, then $\displaystyle\sum_{f \in \mathcal{D}_{n+2}} \delta_j(f \cdot T) = (n+1)(n+2)$.*

Proof. We prove this proposition by induction on $j \in [n]$. For any triangulation T', we denote by $a_j(T') < j < b_j(T')$ the indices of the vertices of $\Delta_j(T')$. Let H be the group of rotations in $\mathcal{D}_{n+2}$. It is well-known that for any reflection $s \in \mathcal{D}_{n+2}$, the classes H and sH form a partition of $\mathcal{D}_{n+2}$ and that $|H| = n+2$. We consider also the unique reflection $s_k \in \mathcal{D}_{n+2}$ which maps A_x to $A_{n+3+k-x}$, where the values of the indices are taken in modulo $n+2$. In particular, $s_k(A_0) = A_{n+3+k} = A_{k+1}$, $s_k(A_1) = A_k$, $s_k(A_{k+1}) = A_{n+2} = A_0$, and so on.

Basic step $j = 1$: We know that $a_1(T') = 0$ for any triangulation T', hence the weight of $\Delta_1(T')$ is $\delta_1(T') = (1-0)(b_1(T') - 1) = b_1(T') - 1$.

The reflection $s_0 \in \mathcal{D}_{n+2}$ maps A_x to A_{n+3-x} (where $A_{n+2} = A_0$ and $A_{n+3} = A_1$). In other words, $s_0(A_0) = A_1$ and $s_0(\Delta_1(T'))$ is a triangle in $s_0 \cdot T'$. Since

$$s_0(\Delta_1(T')) = s_0(A_0 A_1 A_{b_1(T')}) = A_0 A_1 A_{n+3-b_1(T')}$$

and $0 < 1 < n + 3 - b_1(T')$, $s_0(\Delta_1(T'))$ has to be $\Delta_1(s_0 \cdot T')$. In consequence, we obtain that

$$\delta_1(T') + \delta_1(s_0 \cdot T') = (b_1(T') - 1) + (n + 3 - b_1(T') - 1) = n + 1,$$

for any triangulation T'. Therefore

$$\sum_{f \in \mathcal{D}_{n+2}} \delta_1(f \cdot T) = \sum_{g \in H} \big((\delta_1(g \cdot T) + \delta_1(s_0 \cdot (g \cdot T))) \big) = |H|(n+1) = (n+1)(n+2),$$

proving the initial case of the induction.

Inductive step: Assume that, for a given $1 \leqslant j < n$, we have

$$\sum_{f \in \mathcal{D}_{n+2}} \delta_j(f \cdot T) = (n+1)(n+2).$$

We will show that

$$\sum_{f \in \mathcal{D}_{n+2}} \delta_{j+1}(f \cdot T) = \sum_{f \in \mathcal{D}_{n+2}} \delta_j(f \cdot T).$$

Let $r \in H \subseteq \mathcal{D}_{n+2}$ be the unique rotation mapping A_{j+1} to A_j. In particular, $r(A_0) = A_{n+1}$. Let T' be a triangulation of P. We have two cases:

Case 1. If $a_{j+1}(T') > 0$ then $a_{j+1}(T') - 1 < j < b_{j+1}(T') - 1$ are the indices of the vertices of the triangle $r(\Delta_{j+1}(T'))$ in $r \cdot T'$. Therefore, by unicity, $r(\Delta_{j+1}(T'))$ must be $\Delta_j(r \cdot T')$. Thus

$$
\begin{aligned}
\delta_{j+1}(T') &= (b_{j+1}(T') - (j+1))(j+1 - a_{j+1}(T')) \\
&= ((b_{j+1}(T') - 1) - j)(j - (a_{i+1}(T') - 1)) \\
&= \delta_j(r \cdot T').
\end{aligned}
$$

In other words:

$$
\sum_{\substack{f \in \mathcal{D}_{n+2}, \\ a_{j+1}(f \cdot T) \neq 0}} \delta_{j+1}(f \cdot T) = \sum_{\substack{f \in \mathcal{D}_{n+2}, \\ a_{j+1}(f \cdot T) \neq 0}} \delta_j(r \cdot (f \cdot T)) \tag{1}
$$

$$
= \sum_{\substack{g \in \mathcal{D}_{n+2}, \\ b_j(g \cdot T) \neq n+1}} \delta_j(g \cdot T).
$$

Case 2. If $a_{j+1}(T') = 0$, then $j < b_{j+1}(T') - 1 < n+1$ are the indices of the vertices of $r(\Delta_{j+1}(T'))$, which is therefore not $\Delta_j(r \cdot T')$: it is $\Delta_{b_{j+1}(T') - 1}(r \cdot T')$. To handle this, we need to use the reflections s_j and s_{j-2}.

On one hand, observe that $j + 1 < n + 3 + j - b_{j+1}(T')$ because $b_{j+1}(T') < n+1$. Therefore

$$
s_j(\Delta_{j+1}(T')) = A_{j+1} A_0 A_{n+3+j-b_{j+1}(T')} = \Delta_{j+1}(s_j \cdot T').
$$

Hence

$$
\begin{aligned}
\delta_{j+1}(T') + \delta_{j+1}(s_j \cdot T') &= (j+1)(b_{j+1}(T') - (j+1)) \\
&\quad + (j+1)(n + 3 + j - b_{j+1}(T') - (j+1)) \\
&= (j+1)(n+1-j).
\end{aligned}
$$

On the other hand, consider the triangle $\Delta_j(r \cdot T')$ in $r \cdot T'$. Since

$$
r(\Delta_{j+1}(T')) = A_j A_{b_{j+1}(T')-1} A_{n+1} = \Delta_{b_{j+1}(T')-1}(r \cdot T')
$$

is in $r \cdot T'$, $[j, n+1]$ is a diagonal in $r \cdot T'$. Hence $b_j(r \cdot T') = n+1$. Thus $\Delta_j(r \cdot T') = A_{a_j(r \cdot T')} A_j A_{n+1}$ and $\delta_j(r \cdot T') = (j - a_j(r \cdot T'))(n+1-j)$. We have $s_{j-2}(A_j) = A_{n+1}$, $s_{j-2}(A_{n+2}) = A_j$ and $s_{j-2}(A_{a_j(r \cdot T')}) = A_{n+1+j-a_j(r \cdot T')} = A_{j-a_j(r \cdot T')-1}$ since $a_j(r \cdot T') < j$. Therefore $s_{j-2}(\Delta_j(r \cdot T')) = A_{j-a_j(r \cdot T')-1} A_j A_{n+1} = \Delta_j(s_{j-2} r \cdot T')$ and $\delta_j(s_{j-2} r \cdot T') = (a_j(r \cdot T') + 1)(n+1-j)$. Finally we obtain that

$$
\begin{aligned}
\delta_j(r \cdot T') + \delta_j(s_{j-2} r \cdot T') &= (j - a_j(r \cdot T'))(n+1-j) + (a_j(r \cdot T') + 1)(n+1-j) \\
&= (j+1)(n+1-j).
\end{aligned}
$$

Since $\{H, s_k H\}$ forms a partition of $\mathcal{D}_{n+2}$ for any k, we have

$$
\begin{aligned}
\sum_{\substack{f \in \mathcal{D}_{n+2}, \\ a_{j+1}(f \cdot T)=0}} \delta_{j+1}(f \cdot T) &= \sum_{\substack{f \in H, \\ a_{j+1}(f \cdot T)=0}} \big(\delta_{j+1}(f \cdot T) + \delta_{j+1}(s_j f \cdot T) \big) \qquad (2) \\
&= \sum_{\substack{f \in H, \\ a_{j+1}(f \cdot T)=0}} (j+1)(n+1-j) \\
&= \sum_{\substack{rf \in H, \\ b_j(rf \cdot T)=n+1}} \big(\delta_j(rf \cdot T) + \delta_j(s_{j-2} rf \cdot T) \big), \text{ since } r \in H \\
&= \sum_{\substack{g \in H, \\ b_j(g \cdot T)=n+1}} \delta_j(g \cdot T).
\end{aligned}
$$

We conclude the induction by adding Equations (1) and (2). $\qquad \square$

Proof of Theorem 2.2. We have to prove that

$$
\overrightarrow{u} = \sum_{T' \in \mathcal{O}(T)} \overrightarrow{GM(T')} = \overrightarrow{0}.
$$

Denote by $\mathrm{Stab}(T') = \{f \in \mathcal{D}_{n+2} \,|\, f \cdot T' = T'\}$ the stabilizer of T', then

$$
\sum_{f \in \mathcal{D}_{n+2}} M(f \cdot T) = \sum_{T' \in \mathcal{O}(T)} |\mathrm{Stab}(T')| M(T').
$$

Since $T' \in \mathcal{O}(T)$, $|\mathrm{Stab}(T')| = |\mathrm{Stab}(T)| = \frac{2(n+2)}{|\mathcal{O}(T)|}$, we have

$$
\sum_{f \in \mathcal{D}_{n+2}} M(f \cdot T) = \frac{2(n+2)}{|\mathcal{O}(T)|} \sum_{T' \in \mathcal{O}(T)} M(T').
$$

Therefore by Proposition 2.3 we have for any $i \in [n]$

$$
\sum_{T' \in \mathcal{O}(T)} \delta_i(T') = \frac{|\mathcal{O}(T)|}{2(n+2)}(n+1)(n+2) = \frac{|\mathcal{O}(T)|(n+1)}{2}. \qquad (3)
$$

Denote by O the point of origin of $\mathbb{R}^n$. Then $\overrightarrow{OM} = M$ for any point M of $\mathbb{R}^n$. By Chasles' relation we have finally
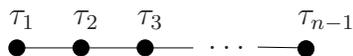
$$
\overrightarrow{u} = \sum_{T' \in \mathcal{O}(T)} \overrightarrow{GM(T')} = \sum_{T' \in \mathcal{O}(T)} (M(T') - G) = \sum_{T' \in \mathcal{O}(T)} M(T') - |\mathcal{O}(T)| G.
$$

So the ith coordinate of $\overrightarrow{u}$ is $\sum_{T' \in \mathcal{O}(T)} \delta_i(T') - \frac{|\mathcal{O}(T)|(n+1)}{2} = 0$, hence $\overrightarrow{u} = \overrightarrow{0}$ by (3). $\quad \square$

3 Center of gravity of generalized associahedra of type A and B

~~3.1 Realizations~~ of associahedra

As a Coxeter group (of type A), S_n is generated by the simple transpositions $\tau_i = (i,\ i+1)$, $i \in [n-1]$. The Coxeter graph Γ_{n-1} is then

$$\overset{\tau_1}{\bullet}\!-\!\overset{\tau_2}{\bullet}\!-\!\overset{\tau_3}{\bullet}\!\!-\cdots-\!\overset{\tau_{n-1}}{\bullet}$$

Let $\mathscr{A}$ be an orientation of Γ_{n-1}. We distinguish between *up* and *down* elements of $[n]$: an element $i \in [n]$ is *up* if the edge $\{\tau_{i-1}, \tau_i\}$ is directed from τ_i to τ_{i-1} and *down* otherwise (we set 1 and n to be down). Let $\mathsf{D}_{\mathscr{A}}$ be the set of down elements and let $\mathsf{U}_{\mathscr{A}}$ be the set of up elements (possibly empty).

The notion of up and down induces a labeling of the $(n+2)$-gon P as follows. Label A_0 by 0. Then the vertices of P are, in counterclockwise direction, labeled by the down elements in increasing order, then by $n+1$, and finally by the up elements in decreasing order. An example is given in Figure 2.
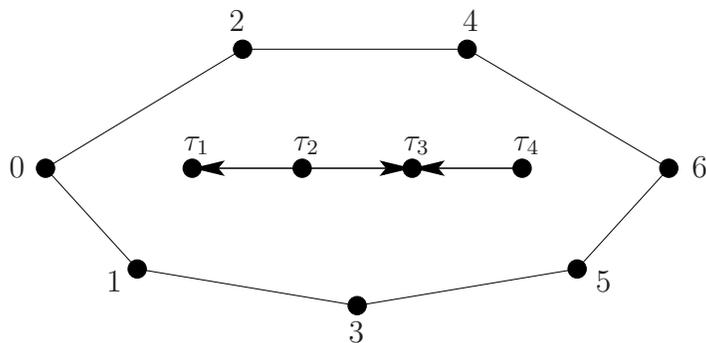

Figure 2: A labeling of a heptagon that corresponds to the orientation $\mathscr{A}$ of Γ_4 shown inside the heptagon. We have $\mathsf{D}_{\mathscr{A}} = \{1, 3, 5\}$ and $\mathsf{U}_{\mathscr{A}} = \{2, 4\}$.

We recall here a construction due to Hohlweg and Lange [7]. Consider P labeled according to a fixed orientation $\mathscr{A}$ of Γ_{n-1}. For each $l \in [n]$ and any triangulation T of P, there is a unique triangle $\Delta_l^{\mathscr{A}}(T)$ whose vertices are labeled by $k < l < m$. Now, count the number of edges of P between i and k, whose vertices are labeled by smaller numbers than l. Then multiply it by the number of edges of P between l and m, whose vertices

are labeled by greater numbers than l. The result $\omega_l^{\mathscr{A}}(T)$ is called the *weight* of $\Delta_l^{\mathscr{A}}(T)$. The injective map

$$M_{\mathscr{A}} : \mathcal{T}_{n+2} \longrightarrow \mathbb{R}^n$$
$$T \longmapsto (x_1^{\mathscr{A}}(T), x_2^{\mathscr{A}}(T), \dots, x_n^{\mathscr{A}}(T))$$

that assigns explicit coordinates to a triangulation is defined as follows:

$$x_j^{\mathscr{A}}(T) := \begin{cases} \omega_j^{\mathscr{A}}(T) & \text{if } j \in \mathsf{D}_{\mathscr{A}} \\ n + 1 - \omega_j^{\mathscr{A}}(T) & \text{if } j \in \mathsf{U}_{\mathscr{A}}. \end{cases}$$

Hohlweg and Lange showed that the convex hull $\mathsf{Asso}_{\mathscr{A}}(S_n)$ of $\{M_{\mathscr{A}}(T) \mid T \in \mathcal{T}_{n+2}\}$ is a realization of the associahedron with integer coordinates [7, Theorem 1.1]. Observe that if the orientation $\mathscr{A}$ is *canonic*, that is, if $\mathsf{U}_{\mathscr{A}} = \emptyset$, then $\mathsf{Asso}_{\mathscr{A}}(S_n) = \mathsf{Asso}(S_n)$.

The key is now to observe that the weight of $\Delta_j^{\mathscr{A}}(T)$ in T is precisely the weight of $\Delta_j(T')$ where T' is a triangulation in the orbit of T under the action of $\mathcal{D}_{n+2}$, as stated in the next proposition.

Proposition 3.1. *Let $\mathscr{A}$ be an orientation of Γ_{n-1}. Let $j \in [n]$ and let A_l be the vertex of P labeled by j. There is an isometry $r_j^{\mathscr{A}} \in \mathcal{D}_{n+2}$ such that:*

(i) $r_j^{\mathscr{A}}(A_l) = A_j$;

(ii) *the label of the vertex A_k is smaller than j if and only if the index i of the vertex $A_i = r_j^{\mathscr{A}}(A_k)$ is smaller than j.*

Moreover, for any triangulation T of P we have $\omega_j^{\mathscr{A}}(T) = \delta_j(r_j^{\mathscr{A}} \cdot T)$.

Proof. If $\mathscr{A}$ is the canonical orientation, then $r_j^{\mathscr{A}}$ is the identity, and the proposition is straightforward. In the following proof, we suppose therefore that $\mathsf{U}_{\mathscr{A}} \neq \emptyset$.

Case 1: Assume that $j \in \mathsf{D}_{\mathscr{A}}$. Let α be the greatest up element smaller than j and let $A_{\alpha+1}$ be the vertex of P labeled by α. Then by construction of the labeling, A_α is labeled by a larger number than j, and $[A_\alpha, A_{\alpha+1}]$ is the unique edge of P such that $A_{\alpha+1}$ is labeled by a smaller number than j. Denote by $\Lambda_{\mathscr{A}}$ the path from A_l to $A_{\alpha+1}$ passing through vertices of P labeled by smaller numbers than j. This is the path going from A_l to $A_{\alpha+1}$ in clockwise direction on the boundary of P.

By construction, $A_k \in \Lambda_{\mathscr{A}}$ if and only if the label of A_k is smaller than j. In other words, the path $\Lambda_{\mathscr{A}}$ consists of *all* vertices of P labeled by smaller numbers than j. Therefore the cardinality of $\Lambda_{\mathscr{A}}$ is $j + 1$.

Consider $r_j^{\mathscr{A}}$ to be the rotation mapping A_l to A_j. Recall that a rotation is an isometry preserving the orientation of the plane. Then the path $\Lambda_{\mathscr{A}}$, which is obtained by walking on the boundary of P from A_l to $A_{\alpha+1}$ in clockwise direction, is sent to the path Λ obtained by walking on the boundary of P in clockwise direction from A_j and going through $j + 1 = |\Lambda_{\mathscr{A}}|$ vertices of P. Therefore $\Lambda = \{A_0, A_1, \dots, A_j\}$, thus proving the first claim of our proposition in this case.

Case 2: assume that $j \in U_{\mathscr{A}}$. The proof is almost the same as in the case of a down element. Let α be the greatest down element smaller than j and let A_α be the vertex of P labeled by α. Then by construction of the labeling, $A_{\alpha+1}$ is labeled by a larger number than j, and $[A_\alpha, A_{\alpha+1}]$ is the unique edge of P such that A_α is labeled by a smaller number than j. Denote by $\Lambda_{\mathscr{A}}$ the path from A_l to A_α passing through vertices of P labeled by smaller numbers than j. This is the path going from A_α to A_l in clockwise direction on the boundary of P.

As above, $A_k \in \Lambda_{\mathscr{A}}$ if and only if the label of A_k is smaller than j. In other words, the path $\Lambda_{\mathscr{A}}$ consists of all the vertices of P labeled by smaller numbers than j. Therefore, again, the cardinality of $\Lambda_{\mathscr{A}}$ is $j+1$.

Let $r_j^{\mathscr{A}}$ be the reflection mapping A_α to A_0 and $A_{\alpha+1}$ to A_{n+1}. Recall that a reflection is an isometry reversing the orientation of the plane. Then the path $\Lambda_{\mathscr{A}}$, which is obtained by walking on the boundary of P from A_α to A_l in clockwise direction, is sent to the path Λ obtained by walking on the boundary of P in clockwise direction from A_α and going through $j+1 = |\Lambda_{\mathscr{A}}|$ vertices of P. Therefore $\Lambda = \{A_0, A_1, \ldots, A_j\}$. Hence $r_j^{\mathscr{A}}(A_l)$ is sent on the final vertex of the path Λ which is A_j, proving the first claim of our proposition.

Thus it remains to show that for a triangulation T of P we have $\omega_j^{\mathscr{A}}(T) = \delta_j(r_j^{\mathscr{A}} \cdot T)$. We know that $\Delta_j^{\mathscr{A}}(T) = A_k A_l A_m$ such that the label of A_k is smaller than j, which is smaller than the label of A_m. Write $A_a = r_j^{\mathscr{A}}(A_k)$ and $A_b = r_j^{\mathscr{A}}(A_m)$. Because of Proposition 3.1, $a < j < b$ and therefore

$$r_j^{\mathscr{A}}(\Delta_j^{\mathscr{A}}(T)) = A_a A_j A_b = \Delta_j(r_j^{\mathscr{A}} \cdot T).$$

So $(j - a)$ is the number of edges of P between A_l and A_k, whose vertices are labeled by smaller numbers than j. Similarly, $(b - j)$ is the number of edges between A_l and A_m, whose vertices are labeled by smaller numbers than j, and $(b - j)$ is the number of edges of P between A_l and A_m and whose vertices are labeled by larger numbers than j. So $\omega_l^{\mathscr{A}}(T) = (j - a)(b - j) = \delta_j(r_j^{\mathscr{A}} \cdot T)$. $\square$

Corollary 3.2. *For any orientation $\mathscr{A}$ of the Coxeter graph of S_n and for any $j \in [n]$, we have*

$$\sum_{f \in \mathcal{D}_{n+2}} x_j^{\mathscr{A}}(f \cdot T) = (n+1)(n+2).$$

Proof. Let $r_j^{\mathscr{A}} \in \mathcal{D}_{n+2}$ be as in Proposition 3.1.

Suppose first that $j \in U_{\mathscr{A}}$, then

$$
\begin{aligned}
\sum_{f \in \mathcal{D}_{n+2}} x_i^{\mathscr{A}}(f \cdot T) &= 2(n+2)(n+1) - \sum_{f \in \mathcal{D}_{n+2}} \omega_i^{\mathscr{A}}(f \cdot T) \\
&= 2(n+2)(n+1) - \sum_{f \in \mathcal{D}_{n+2}} \delta_j(fr_j^{\mathscr{A}} \cdot T), \text{ by Proposition 3.1} \\
&= 2(n+2)(n+1) - \sum_{g \in \mathcal{D}_{n+2}} \delta_j(g^{\mathscr{A}} \cdot T), \text{ since } r_j^{\mathscr{A}} \in \mathcal{D}_{n+2} \\
&= (n+1)(n+2), \text{ by Proposition 2.3}
\end{aligned}
$$

If $i \in D_{\mathscr{A}}$, the result follows from a similar calculation. $\square$

3.2 Center of gravity of associahedra

Theorem 3.3. *The center of gravity of* $\mathsf{Asso}_{\mathscr{A}}(S_n)$ *is* $G = (\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2})$ *for any orientation* $\mathscr{A}$.

By following precisely the same arguments as in §2.3, we just have to show the following generalization of Theorem 2.2.

Theorem 3.4. *Let* $\mathcal{O}$ *be an orbit of the action of* $\mathcal{D}_{n+2}$ *on* $\mathcal{T}_{n+2}$, *then* G *is the center of gravity of* $\{M_{\mathscr{A}}(T) \,|\, T \in \mathcal{O}\}$. *In particular,* $\sum_{T \in \mathcal{O}} \overrightarrow{GM_{\mathscr{A}}(T)} = \overrightarrow{0}$.

Proof. The proof is entirely similar to the proof of Theorem 2.2, using Corollary 3.2 instead of Proposition 2.3. $\qquad\square$

4 Center of gravity of the cyclohedron

In 1994, R. Bott and C. Taubes discovered the cyclohedron [3] in connection with knot theory. It was rediscovered independently by R. Simion [20]. In [7], the first author and C. Lange also gave a family of realizations for the cyclohedron, starting with the permutahedron of type B. We show in this section that the centers of gravity of the cyclohedron and of the permutahedron of type B are the same.

4.1 The type B-permutahedron

The hyperoctahedral group W_n is defined by $W_n = \{\sigma \in S_{2n} \,|\, \sigma(i) + \sigma(2n + 1 - i) = 2n + 1, \ \forall i \in [n]\}$. The *type B-permutahedron* $\mathsf{Perm}(W_n)$ is the simple n-dimensional convex polytope defined as the convex hull of the points

$$M(\sigma) = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{R}^{2n}, \qquad \forall \sigma \in W_n.$$

As $w_0 = (2n, 2n - 1, \ldots, 3, 2, 1) \in W_n$, we deduce from the same argument as in the case of $\mathsf{Perm}(S_n)$ that the center of gravity of $\mathsf{Perm}(W_n)$ is

$$G = (\frac{2n + 1}{2}, \frac{2n + 1}{2}, \ldots, \frac{2n + 1}{2}).$$

4.2 Realizations of the associahedron

An orientation $\mathscr{A}$ of Γ_{2n-1} is *symmetric* if the edges $\{\tau_i, \tau_{i+1}\}$ and $\{\tau_{2n-i-1}, \tau_{2n-i}\}$ are oriented in *opposite directions* for all $i \in [2n-2]$. There is a bijection between symmetric orientations of Γ_{2n-1} and orientations of the Coxeter graph of W_n (see [7, §1.2]). A triangulation $T \in \mathcal{T}_{2n+2}$ is *centrally symmetric* if T, viewed as a triangulation of P, is centrally symmetric. Let $\mathcal{T}_{2n+2}^B$ be the set of the centrally symmetric triangulations of P. In [7, Theorem 1.5] the authors show that for any symmetric orientation $\mathscr{A}$ of Γ_{2n-1}. The convex hull $\mathsf{Asso}_{\mathscr{A}}(W_n)$ of $\{M_{\mathscr{A}}(T) \,|\, T \in \mathcal{T}_{2n+2}^B\}$ is a realization of the cyclohedron with integer coordinates.

Since the full orbit of symmetric triangulations under the action of $\mathcal{D}_{2n+2}$ on triangulations provides vertices of $\mathsf{Asso}_{\mathscr{A}}(W_n)$, and vice-versa, Theorem 3.4 implies the following corollary.

Corollary 4.1. *Let $\mathscr{A}$ be a symmetric orientation of Γ_{2n-1}, then the center of gravity of $\mathsf{Asso}_{\mathscr{A}}(W_n)$ is $G = (\frac{2n+1}{2}, \frac{2n+1}{2}, \ldots, \frac{2n+1}{2})$.*

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