

# The Spectra of Certain Classes of Room Frames: The Last Cases

Jeffrey H. Dinitz and Gregory S. Warrington

Department of Mathematics and Statistics  
University of Vermont  
Burlington, Vermont  
U.S.A. 05405

Submitted: Nov 13, 2009; Accepted: May 5, 2010; Published: May 20, 2010  
Mathematics Subject Classification: 05B15

## Abstract

In this paper we study the spectra of certain classes of Room frames. The three spectra which we study are incomplete Room squares, uniform Room frames and Room frames of type  $2^u t^1$ . These problems have been studied in numerous papers over the years; in this paper, we complete the three spectra. In addition we find a Howell cube of type  $H_3(6, 10)$ . This corrects a previous claim of nonexistence of this design.

## 1 Introduction

Room squares and generalizations have been extensively studied for over 40 years. In 1974, Mullin and Wallis [12] showed that the spectrum of Room squares consists of all odd positive integers other than 3 or 5; however, many other related questions have remained unsolved. (For an extensive survey from 1992 of Room squares and related designs, we refer the reader to [5].) In the 1994 paper by Dinitz, Stinson and Zhu [6], the authors studied three well-known spectra of designs closely related to Room squares and in each instance left exactly one unsolved case. In this paper we will prove the existence of each of these designs.

Also, in the 1986 paper by A. Rosa and D. Stinson [13] it was claimed that there is no Howell cube on any 6-regular graph on 10 points. In Section 2 we disprove this claim by exhibiting such a cube.

We begin with the definitions. Let  $S$  be a set, and let  $\{S_1, \dots, S_n\}$  be a partition of  $S$ . An  $\{S_1, \dots, S_n\}$ -*Room frame* is an  $|S| \times |S|$  array,  $F$ , indexed by  $S$ , which satisfies the following properties:

1. every cell of  $F$  either is empty or contains an unordered pair of symbols of  $S$ ,
2. the subarrays  $S_i \times S_i$  are empty, for  $1 \leq i \leq n$  (these subarrays are referred to as *holes*),
3. each symbol  $x \notin S_i$  occurs once in row (or column)  $s$ , for any  $s \in S_i$ ,
4. the pairs occurring in  $F$  are those  $\{s, t\}$ , where  $(s, t) \in (S \times S) \setminus \bigcup_{i=1}^n (S_i \times S_i)$ .

The *type* of a Room frame  $F$  is defined to be the multiset  $\{|S_i| : 1 \leq i \leq n\}$ . Typically an “exponential” notation is used to describe types: type  $t_1^{u_1}t_2^{u_2}\cdots t_k^{u_k}$  denotes  $u_i$  occurrences of  $t_i$ ,  $1 \leq i \leq k$ . We note that a Room square of side  $n$  is equivalent to a Room frame of type  $1^n$ . Examples of Room frames of types  $1^83^1$  and  $2^5$  are given below.

		48			37	6X			59
		69			5X	38			47
			39	4X			57	68	
67	8X				04	15	29		
58	79			03				2X	16
9X		78	06			24		13	
			05	7X	89		14		23
		46	3X		25	19			08
		35	49	1X		26			07
34		56			17	28	0X		
			27	18			09	36	45

A Room frame of type  $1^83^1$

			79	68		35	24		
		69	78			34			25
59	48							17	06
16	07	58	49						
26						19	08		37
	27	18	09					36	
39	04	15	28						
38				29				05	14
		57	46		13	02			
47	56		03				12		

A (uniform) Room frame of type  $2^5$

Note that the pairs of elements contained in a Room square naturally define a graph (the *underlying graph*). Each row (column) of the Room frame is a 1-factor of the underlying graph and the set of all rows (columns) is a 1-factorization of the underlying graph. The row 1-factorization and column 1-factorization are *orthogonal 1-factorizations* in the sense that any two edges that are in the same row 1-factor are in different column 1-factors. It is straightforward to see that the existence of a Room frame of type  $t_1^{u_1}t_2^{u_2}\cdots t_k^{u_k}$  is equivalent to a pair of orthogonal 1-factorizations of the complete graph on  $\sum t_i \times u_i$  points which is missing a spanning set of  $u_i$  complete graphs on  $t_i$  points for  $1 \leq i \leq k$ .

Three important spectra of Room frames were considered in [6] and will be discussed in this paper. Specifically, they are Room frames of types  $1^{n-s}s^1$  (incomplete Room squares),  $t^u$  (uniform Room frames) and  $2^ut^1$ . We describe each one and give the current state of knowledge for each.

The existence of a Room frame of type  $1^{n-s}s^1$  is equivalent to the existence of an object called an  $(n, s)$ -*incomplete Room square* which is essentially a Room square of side  $n$  containing a Room square of side  $s$  as a subarray. (By considering “incomplete” Room squares, one can allow  $s = 3$  or  $5$ , as well.) The existence of these objects is a fundamental question in this area. See [5, 6, 7, 14] for prior results. The following theorem summarizes the known results regarding these objects.

**Theorem 1.1** [6] Suppose  $n$  and  $s$  are odd positive integers,  $n \geq 3s + 2$ , and  $(n, s) \neq (5, 1)$ . Then there exists an  $1^{n-s} s^1$  Room frame (equivalently an  $(n, s)$ -incomplete Room square) except possibly when  $n = 67$  and  $s = 21$ .

A Room frame of type  $t^u$  (i.e. a Room frame having  $u$  holes, each of size  $t$ ) is termed a *uniform* Room frame. A systematic study of the spectrum for uniform Room frames was begun in 1981 in [3]. Other results can be found in [2, 5, 6, 9]. The following theorem summarizes the known results regarding uniform Room frames.

**Theorem 1.2** [6] Suppose  $t$  and  $u$  are positive integers,  $u \geq 4$ , and  $(t, u) \neq (1, 5), (2, 4)$ . Then there exists a Room frame of type  $t^u$  if and only if  $t(u - 1)$  is even, except possibly when  $u = 4$  and  $t = 14$  (i.e. of type  $14^4$ ).

Room frames of type  $2^u t^1$  are Room frames with one hole of size  $t$  and  $u$  holes of size 2. This problem can be thought of as an even-side analogue of the incomplete Room square problem. The known results on this problem can be found in [6, 8, 10]. The following theorem summarizes the known results regarding this type of frame.

**Theorem 1.3** [6] Suppose  $t$  and  $u$  are positive integers. If  $t \geq 4$ , then there exists a frame of type  $2^u t^1$  if and only if  $t$  is even and  $u \geq t + 1$ , except possibly when  $u = 19$  and  $t = 18$ .

In this paper we find all three of the exceptional cases, allowing us to complete each of the spectra mentioned above. Our method entails the use of the hill-climbing algorithm for finding Room frames. This algorithm is described in [4] and was used previously to find many of the frames of the types mentioned in this paper. An additional description of the algorithm can be found at [5] and a general description of hill-climbing algorithms used in design theory can be found at [11]. We should mention that the current success of the algorithm is mainly attributable to the speedup of computers (and the existence of large clusters) over the past 15 years. There were no new heuristics or algorithmic speedups used in these searches over what was used in [6].

We ran the algorithm many times and below we give a statistical analysis of expected number times the algorithm must be restarted before finding each frame. It also estimates the expected time for a single process to finish on a single 3GHz cpu. In each case a restart begins when the search reaches the threshold of  $8000 * \text{side}$  operations without a decrease in the deficit. A rough estimate of the time this algorithm could have taken to complete these searches in 1994 would be to multiply the value given below by 40.

frame	number of restarts	expected time to finish (hours)
$1^{46} 21^1$	480,000	16
$2^{19} 18^1$	550,000	18
$14^4$	5,260,000	180

In the Appendix we give a  $(67, 21)$ -incomplete Room square, a Room frame of type  $14^4$  and a Room frame of type  $2^{19}18^1$ . We present each as a pair of orthogonal one-factorizations  $F_r$  and  $F_c$  of the appropriate underlying graph (for example  $K_{68} - K_{22}$  for the  $(67, 21)$ -incomplete Room square). In order to construct the square just note that if pair  $\{x, y\}$  is in the  $i$ th factor of  $F_r$  and the  $j$ th factor of  $F_c$ , then the pair  $\{x, y\}$  is in the  $i$ th row and  $j$ th column of the Room frame. The addition of these frames to the three theorems above completes each of their spectra. We record this in the following three theorems.

**Theorem 1.4** *There exists an  $1^{n-s}s^1$  Room frame (equivalently an  $(n, s)$ -incomplete Room square) if and only if  $n$  and  $s$  are odd positive integers,  $n \geq 3s + 2$ , and  $(n, s) \neq (5, 1)$ .*

**Theorem 1.5** *There exists a Room frame of type  $t^u$  if and only if  $u \geq 4$ ,  $t(u-1)$  is even, and  $(t, u) \neq (1, 5), (2, 4)$ .*

**Theorem 1.6** *There exists a frame of type  $2^u t^1$  if and only if  $t$  and  $u$  are positive integers,  $t$  is even and  $u \geq t+1$ .*

## 2 A Howell cube

An object that is very closely related to a Room square and slightly more general is a Howell design. Let  $S$  be a set of  $2n$  symbols. A *Howell design*  $H(s, 2n)$  (on symbol set  $S$ ) is an  $s \times s$  array,  $H$ , which satisfies the properties:

1. every cell of  $H$  either is empty or contains an unordered pair of symbols from  $S$ ,
2. each symbol of  $S$  occurs once in each row and column of  $H$ , and
3. every unordered pair of symbols occurs in at most one cell of  $H$ .

We note that a Room square of side  $2n - 1$  is an  $H(2n - 1, 2n)$ . As was the case for Room frames, the pairs of symbols in the cells of an  $H(s, 2n)$  can be thought of as the edges of a  $s$  regular graph on  $2n$  symbols, the *underlying graph* of the Howell design. An  $H^*(s, 2n)$  is defined to be an  $H(s, 2n)$  whose underlying graph contains a maximal independent set, i.e. one of size  $2n - s$ . The rows and columns of an  $H(s, 2n)$  form orthogonal 1-factorizations of the underlying graph. As with Room frames, the existence of a pair of orthogonal 1-factorizations of an  $s$ -regular graph on  $2n$  vertices is equivalent to the existence of an  $H(s, 2n)$ . Below we give examples of two small Howell designs, an  $H(4, 6)$  and an  $H^*(4, 8)$  with independent set  $\{1, 2, 3, 4\}$ .

04		13	25
23	14	05	
	35	24	01
15	02		34

An  $H(4, 6)$

15	26	37	48
47	38	25	16
28	17	46	35
36	45	18	27

An  $H^*(4, 8)$

A  $d$ -dimensional Howell design  $H_d(s, 2n)$  is a  $d$ -dimensional array in which every cell either is empty or contains an unordered pair of symbols from an  $s$ -set and such that each two-dimensional projection is an  $H(s, 2n)$ . An  $H_3(s, 2n)$  is a *Howell cube*. An  $H_d(s, 2n)$  is equivalent to  $d$  mutually orthogonal 1-factorizations of the underlying graph (an  $s$  regular graph on  $2n$  vertices).  $\nu(s, 2n)$  denotes the maximum value of  $d$  such that an  $H_d(s, 2n)$  exists. Information on  $\nu(s, 2n)$  can be found in [1].

In 1986, A. Rosa and D. Stinson [13] studied orthogonal 1-factorizations of  $s$ -regular graphs on 10 or fewer vertices. In that paper it was claimed that there is no set of three orthogonal 1-factorizations of *any* 6-regular graph on 10 vertices. In other words, there is no Howell cube  $H_3(6, 10)$ . This is not correct. Below we give three orthogonal 1-factorizations,  $F_1, F_2$  and  $F_3$ , of the graph which is the complement of  $K_4 \cup K_{3,3}$ . Since this graph has an independent set of size four (namely  $\{3, 4, 6, 8\}$ ), the existence of these three orthogonal 1-factorizations implies there exists an  $H_3^*(6, 10)$ .

**Theorem 2.1** *There exists an  $H_3^*(6, 10)$ .*

**Proof:** We display three orthogonal 1-factorizations of the 6-regular graph on 10 vertices which is the complement of  $K_4 \cup K_{3,3}$ .

$F_1$

$$\begin{aligned} & \{1, 10\}, \{2, 3\}, \{4, 9\}, \{5, 6\}, \{7, 8\} \\ & \{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 10\}, \{8, 9\} \\ & \{1, 4\}, \{2, 6\}, \{3, 7\}, \{5, 8\}, \{9, 10\} \\ & \{1, 6\}, \{2, 7\}, \{3, 9\}, \{4, 5\}, \{8, 10\} \\ & \{1, 9\}, \{2, 8\}, \{3, 5\}, \{4, 10\}, \{6, 7\} \\ & \{1, 8\}, \{2, 5\}, \{3, 10\}, \{4, 7\}, \{6, 9\} \end{aligned}$$

$F_2$

$$\begin{aligned} & \{1, 3\}, \{2, 8\}, \{4, 7\}, \{5, 6\}, \{9, 10\} \\ & \{1, 10\}, \{2, 4\}, \{3, 9\}, \{5, 8\}, \{6, 7\} \\ & \{1, 9\}, \{2, 6\}, \{3, 10\}, \{4, 5\}, \{7, 8\} \\ & \{1, 8\}, \{2, 7\}, \{3, 5\}, \{4, 9\}, \{6, 10\} \\ & \{1, 4\}, \{2, 3\}, \{5, 7\}, \{6, 9\}, \{8, 10\} \\ & \{1, 6\}, \{2, 5\}, \{3, 7\}, \{4, 10\}, \{8, 9\} \end{aligned}$$

$F_3$

$$\begin{aligned} & \{1, 10\}, \{8, 9\}, \{2, 6\}, \{3, 5\}, \{4, 7\} \\ & \{2, 3\}, \{9, 10\}, \{4, 5\}, \{6, 7\}, \{1, 8\} \\ & \{4, 9\}, \{5, 7\}, \{1, 6\}, \{2, 8\}, \{3, 10\} \\ & \{5, 6\}, \{2, 4\}, \{3, 7\}, \{8, 10\}, \{1, 9\} \\ & \{7, 8\}, \{6, 10\}, \{1, 4\}, \{3, 9\}, \{2, 5\} \\ & \{1, 3\}, \{5, 8\}, \{2, 7\}, \{4, 10\}, \{6, 9\} \end{aligned}$$

## References

- [1] J. H. DINITZ. Howell Designs in *The Handbook of Combinatorial Designs (2nd Edition)*, (C. J. Colbourn and J. H. Dinitz, eds.) Chapman & Hall/CRC, Boca Raton, FL, 2007, 499–504.
- [2] J. H. DINITZ AND E. R. LAMKEN. Uniform Room frames with five holes. *J. Combin. Designs* **1** (1993), 323–328.
- [3] J. H. DINITZ AND D. R. STINSON. Further results on frames. *Ars Combinatoria* **11** (1981), 275–288.
- [4] J. H. DINITZ AND D. R. STINSON. A hill-climbing algorithm for the construction of one-factorizations and Room squares. *SIAM J. on Algebraic and Discrete Methods* **8** (1987), 430–438.
- [5] J. H. DINITZ AND D. R. STINSON. Room squares and related designs. In “Contemporary Design Theory: A Collection of Surveys”, John Wiley & Sons, 1992, pp. 137–204.
- [6] J. H. DINITZ, D. R. STINSON AND L. ZHU. On the spectra of certain classes of Room frames, *Electron. J. Combin.* **1** (1994), Research Paper 7, 21 pp
- [7] B. DU AND L. ZHU. The existence of incomplete Room squares. *J. Combin. Math. Combin. Comput.* **14** (1993), 183–192.
- [8] G. GE. On the existence of Room frames of type  $2^n u^1$ . *J. Statist. Plan. Infer.*, **94** (2001), 219–230.
- [9] G. GE AND L. ZHU. On the existence of Room frames of type  $t^u$  for  $u = 4$  and 5. *J. Combin. Designs* **1** (1993), 183–191.
- [10] G. GE AND L. ZHU. On the existence of Room frames of type  $2^n u^1$ . *J. Combin. Math. Combin. Comput.* **18** (1995), 65–82.
- [11] P. B. GIBBONS AND PATRIC R. J. ÖSTERGÅRD, Computational methods in design theory in *The Handbook of Combinatorial Designs (2nd Edition)*, (C. J. Colbourn and J. H. Dinitz, eds.) Chapman & Hall/CRC, Boca Raton, FL, 2007, 755–782.
- [12] R. C. MULLIN AND W. D. WALLIS. The existence of Room squares. *Aequationes Math.* **13** (1975), 1–7.
- [13] A. ROSA AND D. R. STINSON. One-factorizations of regular graphs and Howell designs of small orders *Utilitas Math.* **29** (1986), 99–124.
- [14] D. R. STINSON AND L. ZHU. Towards the spectrum of Room squares with sub-squares. *J. Combin. Theory A* **63** (1993), 129–142.









## A Room frame of type $2^{19}18^1$

$F_r$

3,20 4,40 5,17 6,10 7,24 8,50 9,18 11,46 12,42 13,52 14,48 15,49 16,31 19,55 21,41 22,26 23,47 25,38 27,54 28,44 29,43 30,45 32,51 33,39 34,35  
 36,53 37,56  
 3,45 4,38 5,26 6,52 7,43 8,39 9,21 10,17 11,40 12,49 13,23 14,51 15,41 16,42 18,50 19,24 20,22 25,53 27,33 28,47 29,37 30,46 31,44 32,54 34,56  
 35,48 36,55  
 1,47 2,16 5,44 6,14 7,28 8,30 9,29 10,49 11,41 12,55 13,15 17,48 18,53 19,42 20,45 21,39 22,43 23,56 24,38 25,40 26,32 27,50 31,34 33,52 35,51  
 36,54 37,46  
 1,41 2,30 5,24 6,21 7,39 8,31 9,50 10,43 11,25 12,18 13,53 14,32 15,46 16,36 17,47 19,56 20,55 22,52 23,44 26,49 27,51 28,45 29,42 33,37 34,54  
 35,40 38,48  
 1,38 2,34 3,19 4,55 7,53 8,40 9,13 10,54 11,45 12,17 14,35 15,25 16,28 18,48 20,39 21,50 22,51 23,31 24,42 26,43 27,52 29,49 30,41 32,46 33,56  
 36,44 37,47  
 1,18 2,9 3,49 4,11 7,51 8,48 10,56 12,33 13,29 14,47 15,27 16,40 17,52 19,45 20,21 22,34 23,46 24,32 25,54 26,55 28,41 30,39 31,50 35,53 36,42  
 37,43 38,44  
 1,39 2,43 3,42 4,6 5,37 9,41 10,44 11,15 12,52 13,56 14,36 16,23 17,45 18,47 19,25 20,48 21,24 22,46 26,53 27,49 28,50 29,34 30,40 31,51 32,55  
 33,35 38,54  
 1,17 2,55 3,51 4,27 5,31 6,35 9,45 10,29 11,30 12,50 13,41 14,52 15,39 16,20 18,44 19,49 21,40 22,48 23,53 24,47 25,36 26,46 28,42 32,43 33,54  
 34,37 38,56  
 1,50 2,39 3,56 4,44 5,54 6,28 7,26 8,25 11,23 12,45 13,51 14,19 15,40 16,29 17,53 18,31 20,52 21,43 22,47 24,46 27,42 30,38 32,35 33,55 34,49  
 36,48 37,41  
 1,44 2,21 3,18 4,26 5,36 6,24 7,29 8,28 11,51 12,48 13,27 14,56 15,45 16,43 17,41 19,40 20,30 22,54 23,42 25,39 31,46 32,50 33,53 34,52 35,47  
 37,49 38,55  
 1,46 2,40 3,35 4,43 5,7 6,44 8,42 9,19 10,45 13,33 14,53 15,56 16,41 17,55 18,32 20,28 21,47 22,29 23,48 24,27 25,49 26,50 30,34 31,54 36,52  
 37,51 38,39  
 1,20 2,46 3,54 4,34 5,53 6,55 7,56 8,16 9,51 10,50 13,28 14,27 15,26 17,30 18,45 19,47 21,36 22,41 23,38 24,49 25,42 29,52 31,48 32,40 33,44  
 35,43 37,39  
 1,34 2,36 3,46 4,24 5,43 6,27 7,32 8,55 9,42 10,20 11,44 12,39 15,47 16,38 17,56 18,51 19,22 21,53 23,52 25,45 26,40 28,48 29,54 30,37 31,49  
 33,50 35,41  
 1,11 2,53 3,43 4,39 5,15 6,49 7,46 8,37 9,48 10,36 12,32 16,34 17,42 18,41 19,30 20,24 21,25 22,45 23,54 26,52 27,44 28,56 29,51 31,55 33,47  
 35,50 38,40  
 1,43 2,35 3,44 4,29 5,10 6,46 7,9 8,27 11,39 12,53 13,42 14,33 17,19 18,54 20,50 21,56 22,55 23,40 24,36 25,47 26,31 28,52 30,51 32,45 34,41  
 37,48 38,49  
 1,45 2,44 3,25 4,51 5,42 6,41 7,55 8,32 9,46 10,33 11,36 12,19 13,31 14,30 17,39 18,21 20,43 22,23 24,48 26,56 27,40 28,49 29,53 34,47 35,54  
 37,52 38,50  
 1,40 2,51 3,14 4,47 5,52 6,56 7,37 8,44 9,49 10,39 11,50 12,35 13,43 15,53 16,26 19,33 20,54 21,42 22,32 23,34 24,41 25,31 27,46 28,55 29,48  
 30,36 38,45  
 1,7 2,49 3,11 4,53 5,51 6,50 8,20 9,43 10,47 12,40 13,32 14,31 15,55 16,39 19,46 21,38 22,35 23,27 24,52 25,37 26,48 28,54 29,45 30,42 33,41  
 34,44 36,56  
 1,26 2,41 3,48 4,49 5,55 6,30 7,52 8,56 9,53 10,37 11,38 12,22 13,54 14,39 15,28 16,46 17,34 18,43 21,44 23,51 24,35 25,50 27,47 29,36 31,40  
 32,42 33,45  
 1,54 2,12 3,39 4,31 5,35 6,42 7,45 8,22 9,23 10,32 11,49 13,25 14,21 15,33 16,48 17,46 18,56 24,44 26,51 27,55 28,43 29,41 30,52 34,50 36,47  
 37,40 38,53  
 1,52 2,15 3,50 4,18 5,40 6,20 7,35 8,33 9,54 10,51 11,55 12,41 13,45 14,49 16,37 17,24 19,53 23,39 25,46 26,47 27,31 28,36 29,44 30,48 32,56  
 34,42 38,43  
 1,42 2,14 3,36 4,56 5,39 6,29 7,41 8,49 9,24 10,23 11,52 12,46 13,38 15,50 16,55 17,27 18,28 19,51 20,40 25,44 26,33 30,43 31,47 32,48 34,53  
 35,45 37,54  
 1,28 2,42 3,53 4,19 5,16 6,11 7,44 8,47 9,36 10,30 12,54 13,48 14,40 15,32 17,49 18,52 20,41 21,29 22,39 25,43 26,45 27,38 31,56 33,51 34,55  
 35,46 37,50  
 1,13 2,4 3,40 5,21 6,12 7,42 8,34 9,52 10,55 11,53 14,44 15,48 16,47 17,20 18,39 19,54 22,30 25,56 26,41 27,32 28,33 29,50 31,43 35,49 36,51  
 37,45 38,46  
 1,16 2,52 3,38 4,17 5,56 6,53 7,14 8,45 9,27 10,48 11,42 12,29 13,46 15,44 18,23 19,50 20,47 21,54 22,49 24,43 28,39 30,35 31,41 32,33 34,51  
 36,40 37,55  
 1,4 2,11 3,29 5,49 6,45 7,38 8,41 9,33 10,16 12,44 13,50 14,17 15,23 18,40 19,43 20,51 21,52 22,56 24,54 27,48 28,32 30,47 31,42 34,46 35,55  
 36,39 37,53  
 1,56 2,47 3,13 4,52 5,50 6,19 7,49 8,51 9,16 10,12 11,48 14,26 15,18 17,32 20,23 21,45 22,33 24,55 25,41 29,40 30,54 31,53 34,43 35,39 36,46  
 37,44 38,42  
 1,3 2,50 4,54 5,19 6,39 7,17 8,35 9,55 10,21 11,31 12,26 13,47 14,42 15,43 16,45 18,38 20,53 22,37 23,49 24,40 25,51 29,56 30,44 32,52 33,46  
 34,48 36,41  
 1,55 2,25 3,47 4,46 5,27 6,51 7,12 8,26 9,44 10,28 11,54 13,18 14,50 15,22 16,56 17,40 19,37 20,31 21,48 23,41 24,53 32,39 33,43 34,45 35,42  
 36,49 38,52  
 1,9 2,48 3,8 4,41 5,34 6,17 7,47 10,52 11,43 12,25 13,36 14,46 15,35 16,51 18,42 19,44 20,56 21,55 22,50 23,37 24,45 26,54 27,53 28,40 31,39  
 32,49 33,38  
 1,5 2,45 3,28 4,50 6,48 7,40 8,53 9,56 10,41 11,13 12,37 14,54 15,34 16,44 17,51 18,55 19,52 20,46 21,26 22,38 23,35 24,25 27,39 29,47 30,49  
 33,42 36,43  
 1,30 2,3 4,28 5,22 6,34 7,48 8,38 9,39 10,53 11,47 12,56 13,17 14,45 15,54 16,52 18,27 19,41 20,49 21,51 23,43 24,26 25,55 29,46 33,40 35,44  
 36,50 37,42  
 1,48 2,32 3,55 4,42 5,45 6,54 7,10 8,43 9,37 11,56 12,47 13,40 14,25 15,36 16,49 17,50 18,30 19,38 20,44 21,23 22,53 24,51 26,39 27,41 28,46  
 29,31 35,52  
 1,49 2,37 3,26 4,48 5,41 6,8 7,50 9,25 10,40 11,19 12,20 13,39 14,43 15,42 16,53 17,54 18,22 21,46 23,45 24,56 27,36 28,51 29,35 31,52  
 32,44 38,47  
 1,21 2,54 3,31 4,45 5,47 6,40 7,15 8,52 9,20 10,46 11,34 12,51 13,24 14,55 16,33 17,44 18,49 19,39 22,42 23,50 25,48 26,37 27,43 28,29 30,56  
 32,53 38,41  
 1,37 2,56 3,15 4,21 5,46 6,43 7,54 8,17 9,40 10,31 11,20 12,30 13,44 14,41 16,50 18,29 19,48 22,24 23,55 25,52 26,42 27,45 28,53 32,47 33,49  
 34,39 38,51  
 1,51 2,27 3,41 4,20 5,48 6,23 7,31 8,54 9,47 10,42 11,33 12,43 13,55 14,16 15,52 17,28 18,46 19,34 21,49 22,40 24,50 25,32 26,44 29,39 30,53  
 35,56 36,45  
 1,53 2,19 3,52 4,35 5,9 6,47 7,21 8,46 10,25 11,28 12,23 13,49 14,18 15,51 16,54 17,43 20,42 22,44 24,39 26,36 27,56 29,55 30,50 31,45 32,41  
 33,48 34,40  
 1,8 2,17 3,5 4,13 6,16 7,33 9,30 10,35 11,18 12,38 14,15 19,36 20,26 21,32 22,27 23,29 24,34 25,28 31,37  
 1,10 2,5 3,12 4,32 6,25 7,11 8,36 9,22 13,30 14,38 15,17 16,27 18,35 19,23 20,37 21,34 24,33 26,29 28,31  
 1,25 2,24 3,6 4,15 5,30 7,36 8,23 9,28 10,34 11,26 12,27 13,22 14,37 16,32 17,21 18,20 19,29 31,33 35,38  
 1,6 2,28 3,10 4,22 5,18 7,23 8,14 9,11 12,31 13,21 15,38 16,19 17,29 20,32 24,37 25,34 26,30 27,35 33,36  
 1,24 2,22 3,32 4,37 5,14 6,31 7,25 8,19 9,35 10,15 11,21 12,28 13,16 17,38 18,26 20,33 23,30 27,29 34,36  
 1,36 2,7 3,27 4,12 5,20 6,33 8,15 9,31 10,38 11,29 13,35 14,23 16,22 17,37 18,25 19,21 24,30 26,28 32,34  
 1,32 2,23 3,9 4,33 5,29 6,26 7,19 8,18 10,13 11,24 12,16 14,28 15,37 17,31 20,35 21,30 22,25 27,34 36,38  
 1,33 2,38 3,30 4,14 5,11 6,7 8,9 10,22 12,36 13,37 15,21 16,18 17,25 19,35 20,27 23,26 24,31 28,34 29,32

1,27 2,33 3,17 4,16 5,32 6,15 7,30 8,21 9,26 10,11 12,14 13,20 18,19 22,28 23,25 24,29 31,36 34,38 35,37  
 1,29 2,10 3,7 4,8 5,33 6,38 9,14 11,16 12,24 13,34 15,30 17,26 18,36 19,32 20,25 21,35 22,31 23,28 27,37  
 1,12 2,8 3,21 4,23 5,13 6,22 7,18 9,34 10,27 11,37 14,29 15,24 16,25 17,33 19,28 20,36 26,35 30,31 32,38  
 1,19 2,20 3,34 4,10 5,28 6,13 7,16 8,12 9,15 11,32 14,24 17,35 18,37 21,31 22,36 23,33 25,30 26,27 29,38  
 1,22 2,6 3,23 4,7 5,12 8,11 9,38 10,26 13,19 14,20 15,31 16,24 17,36 18,34 21,37 25,27 28,35 29,33 30,32  
 1,35 2,13 3,37 4,36 5,23 6,32 7,27 8,29 9,17 10,14 11,22 12,15 16,30 18,24 19,31 20,34 21,28 25,33 26,38  
 1,14 2,26 3,22 4,5, 6,9 7,34 8,24 10,18 11,27 12,13 15,19 16,21 17,23 20,29 25,35 28,37 30,33 31,38 32,36  
 1,15 2,18 3,16 4,30 5,38 6,37 7,20 8,10 9,32 11,17 12,34 13,26 14,22 19,27 21,33 23,36 24,28 25,29 31,35  
 1,23 2,31 3,24 4,25 5,8 6,36 7,13 9,12 10,19 11,14 15,29 16,35 17,22 18,33 20,38 21,27 26,34 28,30 32,37  
 1,31 2,29 3,33 4,9 5,25 6,18 7,22 8,13 10,24 11,35 12,21 14,34 15,20 16,17 19,26 23,32 27,30 28,38 36,37

$F_c$

3,39 4,13 5,20 6,9 7,44 8,29 10,42 11,26 12,31 14,43 15,55 16,24 17,41 18,47 19,35 21,48 22,53 23,36 25,50 27,52 28,45 30,49 32,56 33,40 34,51  
 37,54 38,46  
 3,49 4,53 5,33 6,47 7,54 8,10 9,42 11,21 12,38 13,41 14,34 15,29 16,32 17,50 18,56 19,28 20,52 22,45 23,43 24,55 25,48 26,44 27,46 30,40 31,39  
 35,37 36,51  
 1,39 2,45 5,30 6,37 7,55 8,13 9,38 10,47 11,27 12,54 14,41 15,43 16,51 17,26 18,40 19,50 20,35 21,46 22,42 23,29 24,44 25,56 28,52 31,48 32,34  
 33,53 36,49  
 1,25 2,46 4,51 6,33 7,53 8,12 9,48 10,22 11,43 13,54 14,45 15,19 16,17 18,44 20,40 21,28 23,30 24,39 26,51 27,49 29,56 31,36 32,47 34,52 35,55  
 37,50 38,42  
 1,32 2,44 3,47 4,8 7,41 9,52 10,51 11,50 12,27 13,20 14,29 15,42 16,21 17,43 18,25 19,40 22,49 23,54 24,45 26,30 28,35 31,56 33,55 34,46 36,53  
 37,48 38,39  
 1,42 2,13 3,22 4,46 7,11 8,14 9,49 10,55 12,16 15,56 17,37 18,54 19,48 20,36 21,32 23,50 24,28 25,45 26,53 27,47 29,41 30,44 31,52 33,39 34,43  
 35,51 38,40  
 1,33 2,22 3,16 4,44 5,28 6,45 9,53 10,35 11,42 12,55 13,30 14,50 15,52 17,51 18,41 19,21 20,32 23,48 24,49 25,54 26,38 27,40 29,46 31,43 34,39  
 36,47 37,56  
 1,31 2,10 3,50 4,43 5,46 6,25 9,51 11,22 12,44 13,40 14,39 15,37 16,56 17,48 18,19 20,41 21,53 23,33 24,30 26,54 27,55 28,47 29,42 32,52 34,49  
 35,38 36,45  
 1,46 2,51 3,32 4,52 5,49 6,53 7,13 8,50 11,44 12,43 14,22 15,39 16,27 17,31 18,20 19,26 21,55 23,56 24,48 25,47 28,54 29,38 30,41 33,36 34,42  
 35,40 37,45  
 1,24 2,18 3,52 4,47 5,25 6,13 7,45 8,55 11,54 12,14 15,30 16,40 17,23 19,43 20,53 21,34 22,27 26,48 28,39 29,50 31,44 32,49 33,42 35,56 36,46  
 37,51 38,41  
 1,54 2,47 3,33 4,37 5,42 6,51 7,40 8,15 9,50 10,39 13,21 14,49 16,46 17,35 18,53 19,36 20,38 22,31 23,55 24,52 25,43 26,29 27,41 28,48 30,45  
 32,44 34,56  
 1,19 2,52 3,7 4,41 5,47 6,22 8,18 9,46 10,54 13,53 14,48 15,40 16,43 17,36 20,27 21,42 23,32 24,37 25,28 26,39 29,51 30,50 31,49 33,45 34,55  
 35,44 38,56  
 1,50 2,7 3,42 4,15 5,51 6,52 8,45 9,44 10,41 11,24 12,47 16,18 17,29 19,54 20,56 21,27 22,39 23,53 25,46 26,43 28,55 30,32 31,35 33,49 34,40  
 36,37 38,48  
 1,53 2,24 3,12 4,7 5,8 6,32 9,15 10,50 11,29 16,44 17,52 18,39 19,51 20,33 21,41 22,55 23,45 25,40 26,46 27,34 28,49 30,56 31,54 35,47 36,48  
 37,42 38,43  
 1,51 2,6 3,37 4,22 5,40 7,18 8,19 9,56 10,43 11,48 12,39 13,49 14,55 17,46 20,25 21,31 23,26 24,41 27,45 28,50 29,52 30,42 32,54 33,47 34,38  
 35,53 36,44  
 1,52 2,54 3,27 4,9 5,48 6,39 7,20 8,11 10,49 12,36 13,34 14,38 17,22 18,42 19,56 21,50 23,41 24,51 25,33 26,40 28,53 29,45 30,47 31,55 32,43  
 35,46 37,44  
 1,44 2,40 3,43 4,12 5,23 6,7 8,53 9,39 10,46 11,56 13,42 14,52 15,24 16,48 19,45 20,51 21,35 22,28 25,27 26,41 29,49 30,55 31,38 32,37 33,50  
 34,47 36,54  
 1,6 2,55 3,24 4,33 5,38 7,25 8,9 10,53 11,49 12,21 13,46 14,37 15,41 16,50 19,52 20,42 22,56 23,40 26,45 27,44 28,51 29,43 30,39 31,47 32,36  
 34,48 35,54  
 1,15 2,26 3,17 4,42 5,54 6,43 7,16 8,36 9,31 10,45 11,40 12,41 13,50 14,47 18,52 21,30 22,46 23,44 24,53 25,51 27,29 28,34 32,48 33,56 35,39  
 37,55 38,49  
 1,14 2,23 3,40 4,36 5,32 6,16 7,49 8,47 9,12 10,44 11,39 13,26 15,53 17,21 18,50 22,41 24,42 25,55 27,43 28,56 29,54 30,31 33,48 34,45 35,52  
 37,46 38,51  
 1,47 2,33 3,53 4,30 5,55 6,56 7,48 8,43 9,34 10,18 11,16 12,28 13,52 14,51 15,38 17,44 19,46 20,39 23,42 24,54 25,49 26,27 29,32 31,45 35,50  
 36,41 37,40  
 1,56 2,20 3,23 4,49 5,43 6,40 7,30 8,41 9,35 10,24 11,47 12,45 13,44 14,42 15,51 16,39 17,54 18,55 19,27 25,53 26,34 28,37 29,48 31,50 32,46  
 33,52 36,38  
 1,10 2,49 3,30 4,54 5,12 6,26 7,56 8,21 9,32 11,52 13,55 14,15 16,41 17,42 18,51 19,31 20,46 22,48 25,35 27,53 28,44 29,40 33,43 34,50 36,39  
 37,47 38,45  
 1,48 2,41 3,46 4,56 5,29 6,49 7,33 8,52 9,14 10,34 11,51 12,42 13,16 15,20 17,25 18,37 19,44 21,40 22,47 26,50 27,39 28,31 30,53 32,45 35,43  
 36,55 38,54  
 1,55 2,31 3,54 4,23 5,18 6,15 7,19 8,51 9,30 10,38 11,17 12,48 13,22 14,40 16,53 20,49 21,56 24,47 27,50 28,43 29,39 32,41 33,46 34,44 35,45  
 36,42 37,52  
 1,40 2,43 3,56 4,14 5,13 6,55 7,50 8,49 9,54 10,19 11,41 12,51 15,45 16,47 17,39 18,24 20,29 21,33 22,44 23,52 27,30 28,46 31,37 32,42 34,36  
 35,48 38,53  
 1,29 2,8 3,5 4,40 6,41 7,47 9,22 10,56 11,32 12,46 13,39 14,44 15,54 16,35 17,53 18,45 19,42 20,50 21,37 23,51 24,43 25,34 26,49 30,48 31,33  
 36,52 38,55  
 1,22 2,56 3,10 4,25 5,52 6,48 7,46 8,39 9,45 11,55 12,15 13,51 14,23 16,49 17,40 18,36 19,29 20,26 21,44 24,50 30,33 31,42 32,53 34,54 35,41  
 37,43 38,47  
 1,45 2,17 3,55 4,39 5,50 6,38 7,36 8,24 9,26 10,27 11,35 12,56 13,48 14,53 15,47 16,19 18,34 20,54 21,43 22,40 23,46 25,52 28,42 31,41 32,51  
 33,44 37,49  
 1,36 2,39 3,34 4,51 5,44 6,46 7,23 8,40 9,47 10,48 11,18 12,52 13,43 14,56 15,50 16,42 17,49 19,32 20,55 21,45 22,25 24,31 26,35 27,54 28,38  
 33,41 37,53  
 1,41 2,38 3,45 4,5 6,50 7,27 8,48 9,28 10,11 12,34 13,56 14,24 15,17 16,25 18,46 19,55 20,44 21,47 22,43 23,49 26,52 29,53 30,51 33,54 35,42  
 36,40 37,39  
 1,12 2,50 3,48 4,55 5,45 6,42 7,51 8,54 9,43 10,40 11,53 13,35 14,46 15,49 16,30 17,38 18,33 19,39 20,37 21,52 22,36 23,28 24,29 25,44 26,47  
 27,56 34,41  
 1,35 2,5 3,41 4,10 6,36 7,52 8,42 9,11 12,50 13,37 14,20 15,44 16,45 17,47 18,49 19,53 21,39 22,54 23,25 24,46 26,56 27,48 28,40 29,55 30,43  
 31,51 32,38  
 1,43 2,29 3,6 4,16 5,53 7,42 8,44 9,17 10,15 11,45 12,49 13,19 14,54 18,35 20,48 21,51 22,52 23,47 24,40 25,41 26,55 27,37 28,30 31,46 32,39  
 36,56 38,50  
 1,49 2,28 3,9 4,50 5,14 6,18 7,43 8,46 10,26 11,37 12,40 13,47 15,48 16,22 17,56 19,41 20,45 21,54 23,39 24,34 25,29 27,42 30,52 31,53 32,55  
 33,51 38,44  
 1,27 2,42 3,51 4,32 5,39 6,44 7,34 8,23 9,55 10,13 11,14 12,53 15,46 16,54 17,45 18,48 19,47 20,43 21,49 22,50 24,56 25,30 26,28 29,33 31,40  
 37,41 38,52  
 1,8 2,53 3,44 4,48 5,56 6,31 7,22 9,41 10,52 11,46 12,24 13,45 14,28 15,21 16,55 17,33 18,43 19,23 20,34 25,39 26,42 27,51 29,47 30,54 32,40  
 35,49 36,50  
 1,23 2,48 3,21 4,45 5,11 6,54 7,39 8,56 9,40 10,14 12,13 15,31 16,52 17,55 18,26 19,49 20,47 22,51 24,33 25,42 27,35 28,41 29,44 30,46 32,50  
 34,53 36,43

1,20 2,21 3,31 4,11 5,35 6,14 7,9 8,37 10,16 12,29 13,36 15,32 17,28 18,22 19,33 23,27 24,26 25,38 30,34  
 1,26 2,30 3,8 4,34 5,19 6,12 7,38 9,37 10,33 11,28 13,23 14,36 15,18 16,29 17,27 20,31 21,25 22,35 24,32  
 1,3 2,36 4,29 5,26 6,23 7,21 8,33 9,16 10,37 11,30 12,35 13,17 14,31 15,25 18,28 19,38 20,24 22,34 27,32  
 1,9 2,4 3,14 5,36 6,27 7,10 8,38 11,31 12,20 13,25 15,23 16,34 17,24 18,21 19,37 22,33 26,32 28,29 30,35  
 1,38 2,15 3,29 4,6 5,21 7,28 8,26 9,36 10,31 11,19 12,18 13,32 14,25 16,33 17,30 20,22 23,37 24,27 34,35  
 1,21 2,16 3,26 4,35 5,37 6,24 7,17 8,32 9,27 10,30 11,33 12,22 13,18 14,19 15,34 20,28 23,38 25,36 29,31  
 1,37 2,14 3,13 4,27 5,15 6,17 7,35 8,28 9,33 10,20 11,38 12,19 16,26 18,31 21,23 22,30 24,36 25,32 29,34  
 1,7 2,3 4,26 5,17 6,11 8,25 9,23 10,28 12,32 13,33 14,18 15,35 16,38 19,34 20,21 22,24 27,31 29,37 30,36  
 1,34 2,37 3,20 4,18 5,31 6,19 7,26 8,30 9,21 10,36 11,25 12,17 13,29 14,33 15,22 16,23 24,35 27,38 28,32  
 1,16 2,9 3,15 4,28 5,10 6,8 7,32 11,34 12,23 13,27 14,35 17,20 18,30 19,24 21,26 22,37 25,31 29,36 33,38  
 1,4 2,25 3,11 5,9 6,10 7,31 8,34 12,30 13,24 14,26 15,28 16,20 17,19 18,23 21,29 22,38 27,36 32,35 33,37  
 1,17 2,27 3,28 4,19 5,7 6,34 8,22 9,13 10,12 11,23 14,32 15,36 16,31 18,38 20,30 21,24 25,37 26,33 29,35  
 1,28 2,34 3,38 4,31 5,24 6,20 7,37 8,17 9,25 10,29 11,13 12,26 14,16 15,27 18,32 19,30 21,36 22,23 33,35  
 1,13 2,32 3,18 4,20 5,22 6,29 7,24 8,27 9,19 10,25 11,36 12,33 14,17 15,26 16,28 21,38 23,35 30,37 31,34  
 1,11 2,12 3,35 4,21 5,16 6,28 7,15 8,31 9,29 10,17 13,38 14,30 18,27 19,22 20,23 24,25 26,36 32,33 34,37  
 1,30 2,19 3,25 4,17 5,34 6,35 7,29 8,20 9,18 10,23 11,15 12,37 13,28 14,21 16,36 22,32 24,38 26,31 27,33  
 1,18 2,35 3,19 4,38 5,27 6,30 7,14 8,16 9,24 10,21 11,20 12,25 13,31 15,33 17,32 22,29 23,34 26,37 28,36  
 1,5 2,11 3,36 4,24 6,21 7,12 8,35 9,20 10,32 13,15 14,27 16,37 17,34 18,29 19,25 22,26 23,31 28,33 30,38