Equidimensionality of the Brauer loop scheme

Brian Rothbach
University of California, Berkeley
rothbach@math.berkeley.edu

Submitted: Aug 5, 2009; Accepted: May 11, 2010; Published: May 20, 2010
Mathematics Subject Classification: 14M99

Abstract

We give another description of certain subvarieties of the Brauer loop scheme of Knutson and Zinn-Justin. As a consequence, we show that the Brauer loop scheme is equidimensional.

Contents

1 Introduction

Let N be a positive integer. An integer sequence $(i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$ is said to be **cyclically ordered** if either $i_1 = i_2 = \cdots = i_k$, or $i_1 \neq i_k$ and for some $1 \leqslant l \leqslant k$, the cyclically rotated sequence $(i_l, i_{l+1}, \ldots, i_k, i_1, \ldots, i_{l-1})$ is weakly increasing. We will write $\circlearrowleft (i_1, \ldots, i_k)$ as shorthand for the statement "the sequence $(i_1, \ldots, i_k)$ is cyclically ordered".

Knutson and Zinn-Justin [1] defined a nonstandard multiplication $\bullet$ on $M_N(\mathbb{C})$, the set of $N \times N$ complex matrices, by setting $(P \bullet Q)_{ik} = \sum_{j:\circlearrowleft(i,j,k)} P_{ij}Q_{jk}$. We refer to their paper as a reference for several nice geometric models of this multiplication. We recall the following facts from their paper.

1. A matrix M is invertible under $\bullet$ if and only if the diagonal entries are nonzero. The set of invertible matrices under $\bullet$ is a solvable Lie group, with the invertible diagonal matrices T serving as a maximal torus, and with unipotent radical U the set of all matrices with ones along the diagonal.

2. Let $E = \{M \bullet M = 0 : M \in M_N(\mathbb{C})\}$, which can be described set theoretically by the (possibly nonreduced) equations $(M \bullet M)_{ij} = 0$ for $1 \leqslant i, j \leqslant N$ and $M_{ii} = 0$ for $1 \leqslant i \leqslant N$. Then

$$E = \coprod_{\pi \in \mathfrak{I}} F_\pi$$

where $\mathfrak{I} \subset S_N$ is the set of involutions in S_n, and for each $\pi \in \mathfrak{I}$, F_π is the set of all matrices $M \in E$ such that the upper triangular part of M is Borel conjugate to the strictly upper triangular part of π.

3. Each F_π is a union of $(U, \bullet)$ orbits; in other words, $U \bullet F_\pi = F_\pi$.

4. Suppose π has k fixed points. Then F_π is nonempty and irreducible of dimension $\frac{1}{2}(N^2 - k)$.

As a consequence, Knutson and Zinn-Justin were able to classify all the top dimensional irreducible components of E and to give a partial set of equations for the top dimensional components of E. Moreover, they compute the multidegree of these top dimensional components and connect that polynomial to the entries of the Frobenius-Perron eigenvector of a certain Markov process associated to the Brauer loop model.

The main theorem of this paper is a proof of the following conjecture of Knutson and Zinn-Justin.

Conjecture 1 *The Brauer loop scheme is equidimensional; that is the irreducible components of E are exactly $E_\pi = \overline{F_\pi}$ where $\pi \in S_N$ is an involution with maximal number of 2-cycles. In particular, E is equidimensional of dimension $\lfloor N^2/2 \rfloor$.*

Our method for proving this conjecture is to generalize a construction of Knutson and Zinn-Justin that gives a dense subvariety G_π of F_π for any involution π. As a consequence, we can generalize the equations for the top dimensional components and also prove the following characterization of the closure poset of the F_π's.

Theorem. Let π, π' be two involutions in S_N, and suppose that π has k 2-cycles $(i_1, j_1), \ldots (i_k, j_k)$ and $N - 2k$ fixed points $1 \leqslant a_1 < a_2 < \cdots < a_{N-2k} \leqslant N$. Then $F_\pi \subset \overline{F_{\pi'}}$ if and only if

a. Every two cycle (i_l, j_l) $(1 \leqslant l \leqslant k)$ of π occurs in the disjoint cycle decomposition of π'.

b. Every two cycle occurring in the disjoint cycle decomposition of π' is either of the form (i_l, j_l) or of the form $(a_i, a_{N-2k+1-i})$ for some $1 \leqslant i \leqslant \lfloor \frac{N-2k}{2} \rfloor$.

The paper proceeds as follows. Section 2 reviews the decomposition of the Brauer loop scheme into the finitely many irreducible locally closed schemes F_π. In section 3, we generalize a theorem of Knutson and Zinn-Justin to obtain for each involution π a parameterization of a dense subvariety G_π of $\overline{F_\pi}$. In section 4, we take a quick digression to analyze the effects of a natural cyclic action. In section 5, we show how to construct a partial set of equations for each F_π, and use this to characterize the closure poset of the F_π's. The conjecture of Knutson and Zinn-Justin is an immediately corollary of the classification of the poset.

2 A decomposition and the dimension of the F_π's

Given a matrix M, we define $M_\leqslant$ to be the upper triangular matrix associated to M; namely $(M_\leqslant)_{ij} = M_{ij}$ if $i \leqslant j$, and $(M_\leqslant)_{ij} = 0$ otherwise. Similarly, we will write $M_<$ and $M_>$ to refer to the strictly upper triangular matrix associated to M and the strictly lower triangular matrix associated to M respectively. Notice that $M = M_\leqslant + M_>$ for any matrix M.

Recall that $E = \{M : M \bullet M = 0\}$. From the definition of $\bullet$, $M \in E$ if and only if $M_\leqslant^2 = 0$ and $M_\leqslant M_> + M_> M_\leqslant$ is upper triangular. (The alternative characterizations of $\bullet$ given in [1] make this more apparent.) In particular, one can characterize the matrices $M_\leqslant$ arising from $M \in E$ by the following theorem of Melnikov [2].

Theorem 1 *Let $B \subset GL_N$ be the Borel subgroup of invertible upper triangular matrices. Then B acts by conjugation (under ordinary matrix multiplication) on the set $V = \{L : L$ is upper triangular and $L^2 = 0\}$. Under this action, V decomposes into a finite union of B orbits, indexed bijectively by involutions $\pi \in S_N$. The B orbit associated to π is $B \cdot \pi_<$.*

For each involution $\pi \in S_N$, define the locally closed subset F_π of E to be $\{M : M \bullet M = 0$ and $M_\leqslant \in B \cdot \pi_<\}$. We have the following results from Knutson and Zinn-Justin.

Theorem 2 *Let $\mathfrak{I} \subset S_N$ be the set of all involutions. Then,*

1. $E = \coprod_{\pi \in \mathfrak{I}} F_\pi$

2. Each F_π is a union of $(U, \bullet)$ orbits.

3. Suppose π has k fixed points. Then F_π is nonempty and irreducible of dimension $\frac{1}{2}(N^2 - k)$.

Since $E = \bigcup_{\pi \in \mathfrak{I}} \overline{F_\pi}$ decomposes into a union of finitely many irreducible closed subvarieties, we can immediately make the following observation about E.

Corollary 1 *The only possible irreducible components of E are the varieties $\overline{F_\pi}$ and the top dimensional components correspond bijectively with involutions having a maximal number of two cycles.*

Conceivably the lower dimensional $\overline{F_\pi}$'s could also be irreducible components of E. The point of the rest of the paper is to show that each of these subvarieties is contained in some top dimensional component.

3 A geometric description

Our next goal is to give a geometric description of the varieties $\overline{F_\pi}$; we will see that each such variety is the closure of the $\bullet$ conjugation orbit of a torus invariant subspace. This construction generalizes the parameterization of top dimensional components developed by Knutson and Zinn-Justin.

Let $\pi \in S_N$ be an involution with k 2-cycles $(i_1, j_1), \dots, (i_k, j_k)$, where $i_l < j_l$ for all $1 \leqslant l \leqslant k$, and $N - 2k$ fixed points $a_1 < a_2 < \cdots < a_{N-2k}$. We define a matrix $\underline{\pi}$ as follows.

1. If i is not a fixed point of π, $\underline{\pi}_{i,m} = \delta_{\pi(i),m}$.
2. For $\lfloor \frac{N-2k+1}{2} \rfloor + 1 \leqslant l \leqslant N - 2k$, $\underline{\pi}_{a_l,m} = \delta_{a_{N-2k+1-l},m}$.
3. For $1 \leqslant l \leqslant \lfloor \frac{N-2k+1}{2} \rfloor$, $\underline{\pi}_{a_l,m} = 0$.

Examples.

1. If N is even and π has a maximal number of two cycles, then $\underline{\pi}$ is the permutation matrix of π. If N is odd and π has a maximal number of two cycles, then $\underline{\pi}$ is the permutation matrix of π with the unique nonzero diagonal entry replaced by zero.

2. If $\pi = id_N$, then $\underline{\pi}$ is just $(w_0)_>$, where w_0 is the matrix with 1's on the antidiagonal and zeroes elsewhere. For example, $\underline{id_4}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

3. In general, one can obtain $\underline{\pi}$ from π by replacing the $k \times k$ square submatrix whose rows and columns are the fixed points of π with $(w_0)_>$, where w_0 again only has 1's on the antidiagonal. For example, if $\pi = (12) \in S_4$, then $\underline{(12)}$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Recall that the group of $\bullet$ invertible matrices contains a torus T given by the set of all invertible diagonal matrices and a unipotent factor U given by the set of all matrices with ones on the diagonals. For any element $t \in T$, we will write t_i as shorthand for t_{ii}, $1 \leqslant i \leqslant N$.

We are now ready to construct for each involution $\pi \in S_N$ an dense subvariety G_π of the $\overline{F_\pi}$.

Theorem 3 *Let $G_\pi = U \bullet \{\underline{\pi} t : t \in T\}$ (so the U action is by $\bullet$-conjugation, while $\underline{\pi} t$ is defined in terms of ordinary matrix multiplication.) Then $\overline{F_\pi} = \overline{G_\pi}$.*

Proof. Direct calculation shows $\underline{\pi}t \in E$ for all $t \in T$. The upper triangular part of $\underline{\pi}t$ is of the form $\pi_{<}t$ by construction, and thus $\underline{\pi}t \subset F_\pi$. Since F_π is invariant under $\bullet$ conjugation by U, $G_\pi \subset F_\pi$, and thus $\overline{G_\pi} \subset \overline{F_\pi}$. By the irreducibility of F_π, in order to prove $\overline{G_\pi} = \overline{F_\pi}$ we merely need to prove that both F_π and G_π both have the same dimension $\frac{1}{2}(N^2 - k)$.

Recall that the dimension of F_π was shown to be $\frac{1}{2}(N^2 - k)$ for any involution π by Knutson and Zinn-Justin [1].

To compute the dimension of G_π, we generalize an argument of Knutson and Zinn-Justin. First, we compute the dimension of the $(U, \bullet)$-orbit of a generic point in $\{\underline{\pi}t : t \in T\}$. Then we show the elements of $\{\underline{\pi}t : t \in T\}$ correspond to distinct U orbits, so that the dimension of G_π is the dimension of the generic orbit plus the dimension of $\{\underline{\pi}t : t \in T\}$.

We compute the dimension of the generic orbit by finding the size of the U-stabilizer. Let $\mathfrak{U} = \{M \in M_N(\mathbb{C}) : M_{ii} = 0\}$ denote the Lie algebra of U. In order to compute the dimension of the U stabilizer of $\underline{\pi}t$, it suffices to find the stabilizer of $\underline{\pi}t$ in $\mathfrak{U}$. Equivalently, we must find the dimension of the solution space of $\underline{\pi}t \bullet P = P \bullet \underline{\pi}t$ where $P \in \mathfrak{U}$. Note that then the dimension of the generic orbit will be equal to the generic number of linearly independent equations arising from the condition $\underline{\pi}t \bullet P = P \bullet \underline{\pi}t$.

Associate to $\underline{\pi}$ a partially directed link diagram $L_{\underline{\pi}}$ as follows: If i is not a fixed point of π, then connect the points i and $\pi(i)$ with an undirected edge. Recall that we have labeled the fixed points of π as $a_1 < a_2 < \cdots < a_{N-2k}$; to complete the partially directed link diagram, for each $\lfloor \frac{N-2k+1}{2} \rfloor + 1 \leqslant l \leqslant N - 2k$ draw a directed edge from a_l to $a_{N-2k+1-l}$ for all $\lfloor \frac{N-2k}{2} \rfloor \leqslant l \leqslant N - 2k$ (the arrow should point from the larger value to the smaller value). Note that if N is odd, there will be a unique fixed point in the diagram.

We make a few observations about the resulting diagrams. If $i_1 < i_2 < i_3 < i_4$ contain a pair of directed arrows, then those arrows do not cross and consist of an arrow pointing from i_4 to i_1 and an arrow pointing from i_3 to i_2. Similarly, if $i_1 < i_2 < i_3$ consist of a directed arrow and the unique fixed point, then the directed arrow points from i_3 to i_1, and i_2 is the fixed point.

In order to simplify notation in the upcoming discussion, we will introduce the involution π' associated to the link diagram obtained by replacing all directed edges in $L_{\underline{\pi}}$ with undirected edges; the main convenience is that an edge connects two distinct points i, j in the link diagram $L_{\underline{\pi}}$ if and only if $\pi'(i) = j$. In addition, $\underline{\pi}t$ can be obtained from $\pi't$ by setting $t_i = 0$ for each i that is the tail of a directed edge and setting $t_i = 0$ for the unique fixed point of π' if N is odd. Similarly the equations for the stabilizer of $\underline{\pi}t$ can be obtained from those of $\pi't$ by setting the same t_i's equal to 0.

The equation arising from $\pi't \bullet P = P \bullet \pi't$ in coordinate (i, j) is of the form

$$t_{\pi'(i)} P_{\pi'(i)j} [\circlearrowleft (i \leqslant \pi'(i) \leqslant j)] = P_{i\pi'(j)} t_j [\circlearrowleft (i \leqslant \pi'(j) \leqslant j)]$$

where $1 \leqslant i, j \leqslant N$ and $[S]$ is defined by $[S] = 1$ if S is a true statement and $[S] = 0$ otherwise. As previously observed, the equations for the stabilizer of $\underline{\pi}t$ are obtained from the equations for the stabilizer of $\pi' t$ by setting $t_i = 0$ for all i at the tail of a directed edge and for the unique fixed point of the link diagram when N is odd.

After setting the appropriate t's to 0, the equations corresponding to $i = j$ or $i = \pi'(j)$ hold trivially, since either the logical condition is 0 or we have set $t_i = t_j = 0$. So we can assume that i and j lie on distinct orbits of π' and we group the equations by the corresponding pair of orbits $(i, \pi'(i))$, $(j, \pi'(j))$ (note that if N is odd, the one of these orbits may be a fixed point, but not both.) We will show that each pair of edges of the link diagram contributes four linearly independent equations to the stabilizer of a generic $\underline{\pi} t$, and if N is odd, each pair of an edge and the unique unmatched point generically contributes two linearly independent equations.

Let us start with two crossing edges, so we may assume $\circlearrowleft (i < j < \pi'(i) < \pi'(j))$. Looking first at the stabilizer of $\pi' t$, we get the four equations:

$$t_i P_{ij} = P_{\pi'(i)\pi'(j)} t_j$$

$$t_j P_{j\pi'(i)} = P_{\pi'(j)i} t_{\pi'(i)}$$

$$t_{\pi'(i)} P_{\pi'(i)\pi'(j)} = P_{ij} t_{\pi'(j)}$$

$$t_{\pi'(j)} P_{\pi'(j)i} = P_{j\pi'(i)} t_i$$

These equations are linearly independent unless $t_i t_{\pi(i)} = t_j t_{\pi(j)}$, and so for a generic choice of t_i's we get four linearly independent equations.

Now we consider what happens to these equations when we set $t_i = 0$ as described above to get the equations of the stabilizer of $\underline{\pi} t$. By the previous observations, at most one of the crossing edges is directed. No matter which I is at the head of a directed edge, at most one of the t's will be set equal to 0. If no t_i's are set equal to 0, then we will still generically have four linearly independent equations. If exactly one t_i is set equal to 0, then we still have that $t_i t_{\pi(i)} \neq t_j t_{\pi(j)}$ generically (since one side will be zero, and the other generically nonzero), and so there will still generically be four linearly independent equations as desired.

If we have a pair of edges that do not cross, we can assume $\circlearrowleft (i < j < \pi'(j) < \pi'(i))$. Then for the stabilizer of $\pi' t$ we get the following six equations:

(a) $t_i P_{i\pi'(j)} = P_{\pi'(i)j} t_{\pi'(j)}$

(b) $t_{\pi'(j)} P_{\pi'(j)i} = P_{j\pi'(i)} t_i$

(c) $0 = P_{i,j} t_{\pi'(j)}$

(d) $t_i P_{ij} = 0$

(e) $0 = P_{\pi'(j)\pi'(i)}t_i$

(f) $t_{\pi' j}P_{\pi'(j)\pi'(i)} = 0$

Again, we obtain equations of $\underline{\pi}t$ by setting some of the t_i's equal to 0. Clearly, the pair of equations (c) and (d) contribute at most one linearly independent equation, as does the pair (e) and (f). However, as long as at most one of the edges is directed, so at most one t is equal to zero, the equations $(a)-(f)$ generically contribute four linearly independent equations. Suppose that both edges are directed, so two t's have been set equal to zero. By changing the roles of $i, j, \pi'(i)$, and $\pi'(j)$ and using our previous observations about link diagrams, we may assume that $i < j < \pi'(j) < \pi'(i)$, that $t_{\pi'(i)} = t_{\pi'(j)} = 0$ and that t_i and t_j are nonzero. Then the equations simplify to:

$t_iP_{i\pi'(j)} = 0$

$0 = P_{j\pi'(i)}t_i$

$t_iP_{ij} = 0$

$0 = P_{\pi'(j)\pi'(i)}t_i$

which again is generically four linearly independent equations.

Finally, if we have an edge and a fixed point, we may assume that j is the fixed point, so $i \neq \pi'(i)$. We may assume $i < j < \pi'(i)$ and that t_i is nonzero by construction of $\underline{\pi}$. Then we get the two equations for the stabilizer of $\underline{\pi}t$:

$t_iP_{ij} = 0$

$0 = P_{j\pi'(i)}t_i$

which are by construction generically linearly independent.

Note that for each pair of edges and for each pair of and edge an a fixed point, we have found a collection of linear independent equations in the corresponding variables. Since this partitions the variables into distinct nonoverlapping sets, the corresponding sets of equations are all mutually independent.

Let $N = 2n + r$ (n an integer, $r = 0$ or 1). Counting the set of independent equations shows that the dimension of the generic orbit is $4\frac{n(n-1)}{2} + 2nr = 2n^2 - 2n + 2nr$. Now, $\frac{1}{2}(N^2 - k) = \frac{1}{2}(4n^2 - 4nr + r^2 - k) = (2n^2 - 2n + 2nr) + (2n + r - k) =$ the dimension of the generic U-orbit of $\underline{\pi}t$ plus the dimension of $\underline{\pi}t$. Thus if we can show that each orbit contains at most one element of $\underline{\pi}t$, we are done.

So suppose $P \bullet \underline{\pi}t = \underline{\pi}t' \bullet P$ for some $P \in U$. We must show that $t_i = t'_i$ for all i lying on either an undirected edge or the head of an edge of the corresponding link diagram.

In either case the equation in entry $(\pi'(i), i)$ reads $P_{\pi'(i)i}t_i = t_iP_{ii}$ and since $P \in U$, one gets $t_i = t'_i$ for the required indices. In particular, $\underline{\pi}t = \underline{\pi}t'$ as desired.□

4 A cyclic action on the $\overline{F_\pi}$

Given an integer k, we define $[[k]]$ to be the unique number in $\{1, \dots, N\}$ such that $k = [[k]] \mod N$.

Knutson and Zinn-Justin [1] observe that there is a natural continuous cyclic action acting on $M_N(\mathbb{C})$ that preserves the nonstandard multiplication $\bullet$, given by sending the matrix M to $c(M)$, where $c(M)_{ij} = M_{[[i-1]],[[j-1]]}$. Such a cyclic rotation preserves the relation $\circlearrowleft$, and thus preserves the multiplication $\bullet$. Alternatively one can visualize this action as a translation in their infinite strip model, which again makes it clear that c is a ring homomorphism. The action c fixes the zero matrix, hence also the variety $E = \{M | M \bullet M = 0\}$.

While the $\overline{F_\pi}$ are not invariant under the action of c, Knutson and Zinn-Justin were able to show that c maps top dimensional components of E to other top dimensional components. Moreover, for these top dimensional $\overline{F_\pi}$, c corresponds to rotating the link diagram associated to a π.

Our goal is to prove the following weaker version of the above statement for general $\overline{F_\pi}$.

Theorem 4 *Suppose we fix an involution π and an integer d. Then $c^d(\overline{F_\pi})$ is of the form $\overline{U \bullet \{c^d(\underline{\pi})t | t \in T\}}$. Let π^* be the unique involution such that $c^d(\underline{\pi})_< = \pi^*_<$. Then $c^d(\underline{\pi})$ can be obtained from $\underline{\pi^*}$ by setting certain nonzero entries of $\underline{\pi^*}$ to zero. In particular, $c^d(\overline{F_\pi}) \subset \overline{F_{\pi^*}}$, and $c^d(\underline{\pi}) = \underline{\pi^*}$ if rank$(c^d(\underline{\pi})_<)$ =rank$(\pi^*_<)$.*

Proof. Note $c(U) = U$, since U is the set of $\bullet$-invertible matrices, and c fixes the identity. Then $U \bullet \{c^d(\underline{\pi})t | t \in T\} = c^d(G_\pi)$ is contained in $c^d(\overline{F_\pi})$, and since c is continuous, taking closures gives us the first statement.

Fix π, let $L_{\underline{\pi}}$ be the link diagram associated to $\underline{\pi}$, and let $a_1 < a_2 < \cdots < a_{N-2k}$ be the fixed points of π. We can naturally associate to $c^d(\underline{\pi})$ the link diagram $L_{c^d(\underline{\pi})}$ obtained by rotating the link diagram $L_{\underline{\pi}}$ d times; we observe that $c^d(\underline{\pi})$ is a partial permutation matrix such that $c^d(\underline{\pi})_{ij} = 1$ if either i and j are connected by an undirected edge of $L_{c^d(\underline{\pi})}$ or if there is a directed edge of $L_{c^d(\underline{\pi})}$ pointing from i to j.

Now $c^d(\underline{\pi})_<$ is a partial permutation matrix, with nonzero entries in coordinates (i, j) with $i < j$, and either i and j connected by an unmatched edge of $L_{c^d(\underline{\pi})}$ or a directed edge of $L_{c^d(\underline{\pi})}$ pointing from i to j (in this case the edge points from the smaller number to the larger number.) In particular, when we construct $\underline{\pi^*}$, $L_{\underline{\pi^*}}$ has undirected edges corresponding to the undirected edges of $L_{c^d(\underline{\pi})}$ and the directed edges of $L_{c^d(\underline{\pi})}$ that point from a smaller number to a larger numbers.

From the above discussion, the only reason why $c^d(\underline{\pi})$ might not be obtained from $\underline{\pi^*}$ by setting certain nonzero entries of $\underline{\pi^*}$ to zero is that fixed points of $\underline{\pi^*}$ aren't matched together by directed edges in the proper way. Let $a_{t_1} < \cdots < a_{t_j}$ be the fixed points of π that when rotated by c give rises to to the fixed points of π^* (so $[[a_{t_1}+d]], \ldots, [[a_{t_j}+d]]$ are the fixed points of π^*.) In $L_{\underline{\pi}}$ there is a directed arrow from a_{t_m} to $a_{t_{j-m+1}}$ for $\lfloor \frac{j}{2} \rfloor \leqslant m \leqslant j$; this implies that the there is a directed arrow from $[[a_{t_m}+d]]$ to $[[a_{t_{j-m+1}}+d]]$ for $\lfloor \frac{j}{2} \rfloor \leqslant m \leqslant j$. We must show $[[a_{t_1}+d]] < \cdots < [[a_{t_j}+d]]$ if it does, this means that the directed edges in $L_{\underline{\pi^*}}$ agree with the directed edges of $L_{c^d(\underline{\pi})}$ on this set, and we are done. No matter what, we have $\circlearrowleft ([[a_{t_1}+d]], \ldots, [[a_{t_j}+d]])$, since we took a cyclic ordered set and rotated it. To finish, we observe $[[a_{t_1}+d]] < [[a_{t_j}+d]]$, otherwise we would have replaced that directed edge with an undirected edge in creating $L_{\underline{\pi^*}}$, and thus we can conclude that $a_{t_1}+d < \cdots < a_{t_j}+d \mod N$ as desired.

The final statement follows immediately from the second statement.□

5 Equations for the $\overline{F_\pi}$'s

The geometric description of the $\overline{F_\pi}$'s allows one to construct equations satisfied by these varieties. We have the following generalization of a theorem of Knutson and Zinn-Justin [1]. (Because the proofs of the theorems are identical, we refer the reader to their paper for both the proof and a description of the strip model mentioned below.)

Proposition 1 *Fix an involution $\pi \in S_N$. The variety $\overline{F_\pi}$ satisfies the following equations:*

(1) $M \bullet M = 0$.

(2) (diagonal conditions) $(M^2)_{ii} = 0$ if i is a fixed point of π. (Notice that this equation is defined in terms of ordinary matrix multiplication, not in terms of $\bullet$.)

(3) (more diagonal conditions) $(M^2)_{ii} = (M^2)_{\pi(i)\pi(i)}$ if i is not a fixed point of π

(4) for any (i,j), $r_{ij}(M) \leqslant r_{ij}(\underline{\pi})$. (equivalently, require the vanishing of all $r_{ij}(\underline{\pi})+1$ minors of the submatrix southwest of (i,j) in the strip model of Knutson and Zinn-Justin.

We conjecture that these equations define the F_π as a reduced scheme. As supporting evidence, we have the following:

Theorem 5 *Let π, π' be two involutions in S_N, and suppose that π has k 2-cycles $(i_1,j_1), \ldots (i_k,j_k)$ and $N-2k$ fixed points $1 \leqslant a_1 < a_2 < \cdots < a_{N-2k} \leqslant N$. Then $F_\pi \subset \overline{F_{\pi'}}$ if and only if*

a. Every two cycle (i_l,j_l) $(1 \leqslant l \leqslant k)$ of π occurs in the disjoint cycle decomposition of π'.

b. Every two cycle occurring in the disjoint cycle decomposition of π' is either of the form (i_l,j_l) or of the form $(a_i, a_{N-2k+1-i})$ for some $1 \leqslant i \leqslant \lfloor \frac{N-2k}{2} \rfloor$.

For the if direction, notice conditions (a) and (b) imply that $\underline{\pi}T \subset \underline{\pi'}T$ and then the statement follows immediately from Theorem 3.

Conversely, suppose $F_\pi \subset \overline{F_{\pi'}}$. Note that for all $t \in T$, $(\underline{\pi}t)^2_{ii} = t_i t_{\pi(i)}$ if $i \neq \pi(i)$ and $= 0$ if i is a fixed point of π. Now the equations from Theorem 4 hold on $\overline{F_{\pi'}}$. Suppose that i is a fixed point of π'. Then $(M^2)_{ii} = 0$ on $F_{\pi'}$ and since $\underline{\pi}t \in \overline{F_{\pi'}}$, we have $(\underline{\pi}t)^2_{ii} = 0$ as well for all $t \in T$. By the previous computation i must be a fixed point of π also (otherwise $t_i t_{\pi(i)}$ is generically nonzero.) Suppose (i, j) is a 2-cycle of π'. Then $(M^2)_{ii} = (M^2)_{jj}$ on $\overline{F_{\pi'}}$, and thus for all $\underline{\pi}t$. By the previous calculation, the equality only happen if either (i, j) is a 2-cycle of F_π or i and j are both fixed points of π (otherwise we are trying to set $t_i t_{\pi(i)} = 0$ or $t_i t_{\pi(i)} = t_j t_{\pi(j)}$ where $(i, \pi(i)), (j, \pi(j))$ are distinct orbits of π.) In particular, this implies condition a.

So we may assume π' satisfies condition a. Now we show inductively that for each $1 \leqslant l \leqslant \lfloor \frac{N-2k}{2} \rfloor$ that either $(a_l, a_{N-2k+1-l})$ is a 2 cycle of π' or both a_l and $a_{N-2k+1-l}$ are fixed points of π'. By induction we may assume the statement is true for $l = 1, 2, \ldots, j-1$. Now look at the equation of type 4 as defined in Theorem 4 corresponding to the coordinates $(a_{N-2k+1-j}, a_j)$ in $\underline{\pi'}$. This gives a maximum for the rank of the corresponding matrix for any point in $\overline{F_{\pi'}}$. By induction, the submatrices of $\underline{\pi'}$ and $\underline{\pi}$ lying southwest of $(a_{N-2k+1-j}, a_j)$ are identical except possibly at coordinate $(a_{N-2k+1-j}, a_j)$, and that the rank of this pair of matrices is equal if and only if $\underline{\pi'}$ is nonzero in position $(a_{N-2k+1-j}, a_j)$; if the rank is not equal that $\underline{\pi}$ has a bigger rank and cannot be contained in $\overline{F_{\pi'}}$, which is a contradiction. But the requirement that $\underline{\pi'}$ is nonzero in position $(a_{N-2k+1-j}, a_j)$ is exactly the requirement that either $(a_j, a_{N-2k+1-j})$ is a 2-cycle of π' or both of $a_j, a_{N-2k+1-j}$ are fixed points of π', as desired. $\square$

Note that one can summarize the above theorem by $\overline{F_\pi} \subset \overline{F_{\pi'}}$ if and only if the support of $\underline{\pi}$ is properly contained in the support of $\underline{\pi'}$; here the support of a matrix is the set of coordinates that have a nonzero entry.

One may worry that the shift action c might induce more equations on $\overline{F_\pi}$. But one can modify the above proof to show that for any d, if $c^d(\overline{F_\pi}) \subset \overline{F_{\pi'}}$, then $\overline{F_{\pi^*}} \subset \overline{F_{\pi'}}$, where π^* was defined in section 4. (One uses the diagonal conditions to force the undirected edges to imply that π' inherits 2-cycles from $c^d(\underline{\pi})$ as above, and the rank conditions as above on the directed edges; that is, one shows that the support of $\underline{\pi'}$ contains the support of $c^d(\underline{\pi})$ by using a cyclic shift of the proof of the above theorem, and then notes that any $\underline{\pi'}$ which contains the support of $c^d(\underline{\pi})$ also contains the support of $\underline{\pi^*}$.) Thus we need only check that $\overline{F_{\pi^*}}$ doesn't induce any new equations on $c^d(\overline{F_\pi})$ that we haven't already described, which follows from the fact that the support $c^d(\underline{\pi})$ is contained in the support $\underline{\pi^*}$, as shown in section 4.

Finally, since the irreducible components of E correspond to the maximal F_π under closure, the conjecture of Knutson and Zinn-Justin follows immediately as a corollary.

Corollary 2 *The Brauer loop scheme E is equidimensional, with irreducible components indexed bijectively by involutions with maximal number of 2-cycles. For any F_π, there is a unique involution π' with maximal number of 2-cycles such that $F_\pi \subset \overline{F_{\pi'}}$.*

Proof. For any F_π that does not have a maximal number of 2-cycles, Theorem 4 describes how to construct a π' with maximal number of 2-cycles such that $F_\pi \subset \overline{F_{\pi'}}$. Moreover, this construction is unique (if π has fixed points $a_1 < a_2 < ... < a_{N-2k}$, then $\pi' = \pi(a_1 a_{N-2k}) \ldots (a_{\lfloor \frac{N-2k}{2} \rfloor} a_{N-2k+1-\lfloor \frac{N-2k}{2} \rfloor})$.$\square$

References

[1] A. Knutson and P. Zinn-Justin. A scheme related to the brauer loop model. *Advances in Mathematics*, 214:40–77, 2007.

[2] A. Melnikov. B-orbits in solutions to the equation $x^2 = 0$ in triangular matrices. *J. of Algebra*, 223:101–108, 2000.