Integral Cayley graphs over abelian groups

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Abstract

Let Γ be a finite, additive group, $S \subseteq \Gamma,\ 0 \notin S,\ -S = \{-s:\ s \in S\} = S$. The undirected *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ has vertex set Γ and edge set $\{\{a, b\}:\ a, b \in \Gamma, a - b \in S\}$. A graph is called *integral*, if all of its eigenvalues are integers. For an abelian group Γ we show that $\mathrm{Cay}(\Gamma, S)$ is integral, if S belongs to the Boolean algebra $B(\Gamma)$ generated by the subgroups of Γ. The converse is proven for cyclic groups. A finite group Γ is called Cayley integral, if every undirected Cayley graph over Γ is integral. We determine all abelian Cayley integral groups.

1 Introduction

Eigenvalues of an undirected graph G are the eigenvalues of an arbitrary adjacency matrix of G. Harary and Schwenk [9] defined G to be *integral*, if all of its eigenvalues are integers. Since then many integral graphs have been discovered, for a survey see [4]. Nevertheless, as is shown in [2], the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for sufficiently large n. Known characterizations of integral graphs are restricted to certain graph classes. Here we proceed towards a characterization of integral Cayley graphs over abelian groups.

Let Γ be a finite, additive group, $S \subseteq \Gamma,\ 0 \notin S,\ -S = \{-s:\ s \in S\} = S$. The undirected *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ has vertex set Γ. Vertices $a,\ b \in \Gamma$ are adjacent if $a - b \in S$. For general properties of Cayley graphs we refer to Godsil and Royle [8] or Biggs [5]. Abdollahi and Vatandoost [1] show that there are exactly seven connected cubic integral Cayley graphs. So [15] presents a characterization of integral circulant graphs, which are Cayley graphs over cyclic groups. In this paper we prove for an abelian group Γ that $\mathrm{Cay}(\Gamma, S)$ is integral, if S belongs to the Boolean algebra $B(\Gamma)$ generated by the subgroups of Γ. By the result of So the converse turns out to be true for cyclic groups. We conjecture it to be true for abelian groups in general.

A finite group Γ is called *Cayley integral*, if every undirected Cayley graph over Γ is integral. We show that all nontrivial abelian Cayley integral groups are represented by

$$Z_2^n,\ Z_3^n,\ Z_4^n,\ Z_2^m \otimes Z_3^n,\ Z_2^m \otimes Z_4^n,\quad m \geqslant 1,\ n \geqslant 1.$$

Here $Z_k = \{0, 1, \ldots, k-1\}$ denotes the (additive) cyclic group of integers modulo k.

The *Hamming graph* $Ham(m_1, ..., m_r; D)$ has vertex set $Z_{m_1} \otimes \cdots \otimes Z_{m_r}$. Vertices $x \neq y$ are adjacent, if their *Hamming distance* is in a list D of possible distances. All Hamming graphs are proven to be integral Cayley graphs, which extends a partial result in [14]. Moreover, we show that certain graphs associated with the Sudoku puzzle and with pandiagonal Latin squares are integral Cayley graphs.

We remark that every set S in the Boolean algebra $B(\Gamma)$ satisfies $S = -S$. For the construction of a Cayley graph $Cay(\Gamma, S)$ we use only those $S \in B(\Gamma)$ which do not contain the additive identity 0 of Γ.

2 Integral subsets

Let Z be the set of all integers, M a finite, nonempty set, and f a complex valued function on M, $f : M \to \mathbb{C}$. A subset $A \subseteq M$ is called *f-integral*, if

$$f(A) \;=\; \sum_{a \in A} f(a) \;\in Z\ .$$

We agree upon $f(\emptyset) = 0$. So the empty set is always f-integral. The *complement* of $A \subseteq M$ is $\bar{A} = M \backslash A$. The following simple Lemma is due to $f(\bar{A}) = f(M) - f(A)$.

Lemma 1. *Let M be f-integral and $A \subseteq M$. Then A is f-integral, if and only if $\bar{A}$ is f-integral.*

A family $\Omega = \{A_1, \ldots, A_n\}$ of subsets of M is called *intersection stable*, if $A_i \cap A_j \in \Omega$ for every $i, j \in \{1, \ldots, n\}$.

Lemma 2. *Let $\Omega = \{A_1, \ldots, A_n\}$ be an intersection stable family of f-integral subsets of M. If M is f-integral, then:*

1. *$\bar{A}_i \cap A_j$ is f-integral for every $i, j \in \{1, \ldots, n\}$.*
2. *$A_{i_0} \cap \bar{A}_{i_1} \cap \ldots \cap \bar{A}_{i_k}$ is f-integral for every $k \geqslant 1$ and $\{i_0,\ i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. This remains true, if A_{i_0} is missing, respectively $A_{i_0} = M$.*

Proof. 1. For $i \neq j$ the set $A_i \cap A_j \in \Omega$ is f-integral and so

$$f(\bar{A}_i \cap A_j) \;=\; f(A_j) - f(A_i \cap A_j) \in Z.$$

2. Without loss of generality let $i_j = j$ for $j = 0, \ldots, k$, $A_0 \in \Omega$ or $A_0 = M$,

$$A \;=\; A_0 \cap \bar{A}_1 \cap \ldots \cap \bar{A}_k\ . \tag{1}$$

In view of Lemma 1 it is sufficient to show that $\bar{A}$ is f-integral. By de Morgan's rule we have

$$\bar{A} \;=\; \bar{A}_0 \cup A_1 \cup \ldots \cup A_k \;.$$

We apply the principle of inclusion and exclusion (see [11]) to determine $f(\bar{A})$.

$$f(\bar{A}) \;=\; \sum_{p=1}^{k+1} \,(-1)^{p-1}\, s_p \;,$$

$$s_p \;=\; \sum_{0\leqslant j_1<\ldots<j_p\leqslant k} f(B_{j_1}\cap \ldots \cap B_{j_p}) \;, \tag{2}$$

where $B_0 = \bar{A}_0$ and $B_j = A_j$ for every $j \in \{1,\ldots,k\}$. We show that all terms in the sum (2) represent integers. For $j_1 > 0$ we have

$$f(B_{j_1}\cap \ldots \cap B_{j_p}) \;=\; f(A_{j_1}\cap \ldots \cap A_{j_p}) \;\in Z \;,$$

because $A_{j_1}\cap \ldots \cap A_{j_p} \in \Omega$ is f-integral. If $j_1 = 0$ then $B_{j_1}\cap \ldots \cap B_{j_p}$ reduces to

$$\bar{A}_0 \cap A_{j_2}\cap \ldots \cap A_{j_p} \;=\; \bar{A}_0 \cap A_j \text{ for some } A_j \in \Omega \;,$$

because Ω is intersection stable. If $A_0 = M$ then $f(\bar{A}_0 \cap A_j) = f(\emptyset) = 0$. If $A_0 \in \Omega$ then $f(\bar{A}_0\cap A_j) \in Z$ follows from part 1 of this lemma. □

Let $A_1,\ldots,A_n$ be subsets of M. We denote the Boolean algebra generated by $A_1,\ldots,$ A_n in M by $B(A_1,\ldots,A_n;M)$. It is the smallest system of subsets of M that contains $A_1,\ldots,A_n$ and is invariant under the set operations union, intersection, and forming the complement. It is well known (see e.g Cohn [6]) that $B(A_1,\ldots,A_n;M)$ consists exactly of those sets $A \subseteq M$ which can be represented in disjunctive normal form by $A_1,\ldots,A_n$:

$$\begin{aligned} &A \;=\; \bigcup_{j=1}^{k} D_j,\; D_j \;=\; \bigcap_{l=1}^{n_j} W_{j,l} \;, \\ &W_{j,l} \in \{A_1,\ldots,A_n,\bar{A}_1,\ldots,\bar{A}_n\} \text{ for every } j,l. \end{aligned} \tag{3}$$

If the set system $\Omega = \{A_1,\ldots,A_n\}$ is intersection stable, then the sets D_j in (3) can be reduced to

$$D_j \;=\; A_{j,0}\cap \bar{A}_{j,1}\cap \ldots \cap \bar{A}_{j,m_j} \;, \tag{4}$$

where every $A_{j,l} \in \Omega$. We may have $m_j = 0$ and the term $A_{j,0}$ may be missing.

Lemma 3. *Let $\Omega = \{A_1,\ldots,A_n\}$ be an intersection stable family of subsets of M. If M and all sets $A_1,\ldots,A_n$ are f-integral, then every set $A \in B(A_1,\ldots,A_n;M)$ is also f-integral.*

Proof. Every set $A \in B(A_1, \ldots, A_n; M)$ can be written in disjunctive normal form according to (3). Once more we apply the inclusion-exclusion principle, this time to determine $f(A)$.

$$f(A) \;=\; \sum_{p=1}^{k} \;(-1)^{p-1}\; s_p$$

$$s_p \;=\; \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant k} f(D_{i_1} \cap \ldots \cap D_{i_p}) \tag{5}$$

We show that all terms in the sum (5) represent integers. By the intersection stability of Ω and the form (4) of the sets D_j we see that $T = D_{i_1} \cap \ldots \cap D_{i_p}$ takes a form corresponding to (4).

$$T \;=\; A_{i_0} \cap \bar{A}_{i_1} \cap \ldots \cap \bar{A}_{i_r} \tag{6}$$

Every set A_{i_j} that occurs in (6) belongs to Ω with the possible exception $A_{i_0} = M$. Now we conclude by Lemma 2 that $f(T)$ is integral. □

3 Group characters and Cayley graphs

Lovász [12] (see also Babai [3]) developed a method to express the eigenvalues of a graph in terms of the characters of a transitive subgroup of its automorphism group. This theory simplifies considerably for Cayley graphs over abelian groups. To improve the readability of our paper we include Lovász's arguments reduced to our purposes (Lemma 4, 6, and 7). For more algebraic background we refer to Cohn [6].

Let Γ be a finite additive group with n elements, $|\Gamma| = n$. For a positive integer k and $a \in \Gamma$ we denote as usual by ka the k-fold sum of a to itself, $(-k)a = k(-a),\ 0a = 0$. A *character* ψ of Γ is a homomorphism from Γ into the multiplicative group of complex numbers, $\psi : \Gamma \to \mathbb{C}\backslash\{0\}$,

$$\psi(\mu a + \nu b) \;=\; (\psi(a))^{\mu}\;(\psi(b))^{\nu} \text{ for every } a,\ b \in \Gamma \text{ and } \mu,\ \nu \in Z.$$

Fermat's little theorem yields

$$(\psi(a))^n \;=\; \psi(na) \;=\; \psi(0) \;=\; 1.$$

Therefore, $\psi(a)$ is an n-th root of unity for every $a \in \Gamma$.

Lemma 4. *Let H be a subgroup of Γ and ψ a character of Γ. If H contains an element g with $\psi(g) \neq 1$, then $\psi(H) = 0$ else $\psi(H) = |H|$.*

Proof. If $g \in H$ and $\psi(g) \neq 1$ then we have

$$\psi(H) \;=\; \sum_{h \in H} \psi(h+g) \;=\; \psi(g)\psi(H)\;,\; (1-\psi(g))\psi(H) \;=\; 0,$$

which implies $\psi(H) = 0$. If $\psi(g) = 1$ for every $g \in H$ then $\psi(H) = |H|$. □

We denote by $B(\Gamma)$ the Boolean algebra generated by the subgroups of Γ.

Lemma 5. *For an arbitrary character ψ of Γ every set $S \in B(\Gamma)$ is ψ-integral.*

Proof. According to Lemma 4 every subgroup H of Γ is ψ-integral. The subgroups of Γ constitute an intersection stable set system including Γ itself. Lemma 3 implies that every set $S \in B(\Gamma)$ is ψ-integral. □

Lemma 6. *Let ψ be a character of the additive group $\Gamma = \{v_1, \dots, v_n\}$, $S \subseteq \Gamma$, $0 \notin S$, $-S = S$. Assume that $A = (a_{i,j})$ is the adjacency matrix of $G = Cay(\Gamma, S)$ with respect to the given ordering of the vertex set $V(G) = \Gamma$. Then the column vector $(\psi(v_j))_{j=1,\dots,n}$ is an eigenvector of A with eigenvalue $\psi(S)$.*

Proof. We evaluate the product of the i-th row of A and $(\psi(v_j))_{j=1,\dots,n}$.

$$\begin{aligned}\sum_{j=1}^{n} a_{i,j}\psi(v_j) \;&=\; \sum_{1 \leqslant j \leqslant n,\ v_j - v_i \in S} \psi(v_j) \;=\; \sum_{s \in S} \psi(s + v_i) \\ &=\; \psi(v_i) \sum_{s \in S} \psi(s) \;=\; \psi(v_i)\psi(S)\end{aligned}$$

□

From now on we assume that the finite additive group Γ is abelian. Then Γ can be represented as the direct product of cyclic groups of prime power order (see Cohn [6]).

$$\Gamma \;=\; Z_{n_1} \otimes \cdots \otimes Z_{n_k},\ |\Gamma| = n = n_1 \cdots n_k \tag{7}$$

We consider the elements $x \in \Gamma$ as elements of the cartesian product $Z_{n_1} \times \cdots \times Z_{n_k}$,

$$x = (x_i),\ x_i \in Z_{n_i} = \{0, 1, \dots, n_i - 1\},\ 1 \leqslant i \leqslant k.$$

Addition is coordinatewise modulo n_i. Denote by e_i the unit vector with entry 1 in position i and entry 0 in all positions $j \neq i$. A character ψ of Γ is uniquely determined by its values $\psi(e_i)$, $1 \leqslant i \leqslant k$.

$$x = (x_i) \;=\; \sum_{i=1}^{k} x_i e_i, \quad \psi(x) \;=\; \prod_{i=1}^{k} (\psi(e_i))^{x_i} \tag{8}$$

As $e_i \in \Gamma$ has order n_i, the value $\psi(e_i)$ must be a complex n_i-th root of unity. So there are n_i possible choices for the value of $\psi(e_i)$. Let ζ_i be a primitive n_i-th root of unity for every i, $1 \leqslant i \leqslant k$. For every $\alpha = (\alpha_i) \in \Gamma$ a character ψ_α can be uniquely defined by

$$\psi_\alpha(e_i) \;=\; \zeta_i^{\alpha_i},\ 1 \leqslant i \leqslant k. \tag{9}$$

Thus all $|\Gamma| = n$ characters of the abelian group Γ can be obtained.

Lemma 7. *Let $\psi_1, \ldots, \psi_n$ be the distinct characters of the additive abelian group $\Gamma = \{v_1, \ldots, v_n\}$, $S \subseteq \Gamma$, $0 \notin S$, $-S = S$. Assume that $A = (a_{i,j})$ is the adjacency matrix of $G = Cay(\Gamma, S)$ with respect to the given ordering of the vertex set $V(G) = \Gamma$. Then the column vectors $(\psi_i(v_j))_{j=1,\ldots,n}$, $1 \leqslant i \leqslant n$, constitute an orthogonal basis of $\mathbb{C}^n$ consisting of eigenvectors of A. To the eigenvector $(\psi_i(v_j))_{j=1,\ldots,n}$ belongs the eigenvalue $\psi_i(S)$.*

Proof. By Lemma 6 and the considerations above it remains to prove that for $\alpha = (\alpha_i) \in \Gamma$, $\beta = (\beta_i) \in \Gamma$, $\alpha \neq \beta$, the eigenvectors $(\psi_\alpha(v_j))_{j=1,\ldots,n}$ and $(\psi_\beta(v_j))_{j=1,\ldots,n}$ are orthogonal (with respect to the standard inner product of $\mathbb{C}^n$). We represent Γ by (7) and define ψ_α and ψ_β according to (8) and (9). Observe that the complex conjugate $\bar{\zeta}$ of a root of unity ζ satisfies $\bar{\zeta} = \zeta^{-1}$.

$$\begin{aligned} \sigma &= \sum_{j=1}^{n} \psi_\alpha(v_j)\overline{\psi_\beta(v_j)} = \sum_{x=(x_i)\in\Gamma} \prod_{i=1}^{k} (\zeta_i)^{\alpha_i x_i} \prod_{i=1}^{k} (\bar{\zeta}_i)^{\beta_i x_i} \\ &= \sum_{0\leqslant x_1<n_1} \cdots \sum_{0\leqslant x_k<n_k} \prod_{i=1}^{k} \zeta_i^{(\alpha_i-\beta_i)x_i} = \prod_{i=1}^{k} \sum_{0\leqslant x_i<n_i} \zeta_i^{(\alpha_i-\beta_i)x_i} \end{aligned} \tag{10}$$

As $\alpha \neq \beta$ we may assume e.g. $\alpha_1 \neq \beta_1$. Then

$$\sum_{0\leqslant x_1<n_1} \zeta_1^{(\alpha_1-\beta_1)x_1} = \frac{\zeta_1^{(\alpha_1-\beta_1)n_1} - 1}{\zeta_1^{(\alpha_1-\beta_1)} - 1} = 0$$

implies $\sigma = 0$ by (10). ☐

Our main result is stated in the next theorem.

Theorem 8. *Let Γ be a finite abelian group and $B(\Gamma)$ the Boolean algebra generated by the subgroups of Γ. For every set $S \in B(\Gamma)$, $0 \notin S$, the Cayley graph $Cay(\Gamma, S)$ is integral.*

Proof. According to Lemma 7 all eigenvalues of $Cay(\Gamma, S)$ have the form $\psi(S)$ with a character ψ of Γ. By Lemma 5 we know that $\psi(S)$ is integral for every $S \in B(\Gamma)$. ☐

For an integer $n \geqslant 2$ and a proper divisor d of n we define

$$G_n(d) = \{k \in Z_n : \ gcd(k, n) = d\}.$$

The following result of So [15] leads to the converse of Theorem 8 for cyclic groups.

Lemma 9. *Let n be an integer, $n \geqslant 2$, $S \subseteq Z_n$, $0 \notin S$, $-S = S$. The Cayley graph $Cay(Z_n, S)$ is integral, if and only if there are proper divisors $d_1, \ldots, d_r$ of n such that*

$$S = \bigcup_{j=1}^{r} G_n(d_j) \ . \tag{11}$$

Theorem 10. *Let n be an integer, $n \geqslant 2,\ S \subseteq Z_n,\ 0 \notin S,\ -S = S$. The Cayley graph $Cay(Z_n, S)$ is integral, if and only if $S \in B(Z_n)$.*

Proof. If $S \in B(Z_n)$ then $Cay(Z_n, S)$ is integral by Theorem 8. To prove the converse let $Cay(Z_n, S)$ be integral. By Lemma 9 there are proper divisors $d_1, \ldots, d_r$ of n such that S satisfies (11). To prove $S \in B(Z_n)$ it is sufficient to show that $G_n(d) \in B(Z_n)$ for every proper divisor d of n. To every proper divisor d of n the cyclic group Z_n has exactly one subgroup of order $(n/d) > 1$, namely the cyclic group $[d]$ generated by d. If we define

$$M_n(d) \ = \ [d]\backslash\{0\} \ = \ \{qd : \ 1 \leqslant q < \frac{n}{d}\}$$

then $M_n(d) \in B(Z_n)$. Now we obtain

$$\begin{aligned} G_n(d) \ &= \ \{qd : \ 1 \leqslant q < \frac{n}{d},\ gcd(q, \frac{n}{d}) = 1\} \\ &= \ M_n(d) \setminus \bigcup \{M_n(\delta d) : \ 1 < \delta < \frac{n}{d},\ \delta \text{ divides } \frac{n}{d}\} \ , \end{aligned}$$

which implies $G_n(d) \in B(Z_n)$. □

In the introductory section we defined a finite additive group Γ to be *Cayley integral*, if for every $S \subseteq \Gamma,\ 0 \notin S,\ -S = S$, the Cayley graph $Cay(\Gamma, S)$ is integral. Observe that for this definition Γ may be nonabelian. By $ord(a)$ we denote the order of $a \in \Gamma$.

Lemma 11. *If the finite group Γ is Cayley integral then*

$$ord(a) \in \{2, 3, 4, 6\} \text{ for every } a \in \Gamma,\ a \neq 0.$$

Proof. The eigenvalues of a circuit C_n of length $n \geqslant 3$ are (see [5])

$$\lambda_j \ = \ 2\ cos(\frac{2\pi}{n} j),\ j = 0, 1, \ldots, n-1 \ .$$

This implies that C_n is integral only for $n = 3, 4$, or 6. Assume that Γ is Cayley integral and contains an element $a \neq 0,\ ord(a) \notin \{2, 3, 4, 6\}$. Let $U = [a]$ and $S = \{a, -a\}$. Then $|S| = 2$ and the subgroup generated by S is $[S] = U$. The Cayley graph $G = Cay(\Gamma, S)$ is regular of degree $|S| = 2$. Its connected components are generated by the right cosets of U. They are circuits of length $|U| = ord(a) \notin \{3, 4, 6\}$. Therefore, G is not integral, contradicting our assumption on Γ. □

Lemma 12. *Let Γ be a finite additive group, $S \subseteq \Gamma,\ -S = S$. If $ord(a) \in \{2, 3, 4, 6\}$ for every $a \in S$ then $S \in B(\Gamma)$.*

Proof. We show $\{a, -a\} \in B(\Gamma)$ for every $a \in S$. This leads to

$$S \ = \ \bigcup_{a \in S} \{a, -a\} \ \in \ B(\Gamma) \ .$$

According to the four possible orders of $a \in S$ we consider four cases.
1) $ord(a) = 2 : \ \{a, -a\} \ = \ \{a\} \ = \ [a]\backslash\{0\} \ \in \ B(\Gamma)$.
2) $ord(a) = 3 : \ \{a, -a\} \ = \ [a]\backslash\{0\} \ \in \ B(\Gamma)$.
3) $ord(a) = 4 : \ \{a, -a\} \ = \ [a]\backslash\{0, 2a\} \ = \ [a]\backslash[2a] \ \in \ B(\Gamma)$.
4) $ord(a) = 6 : \ \{a, -a\} \ = \ [a]\backslash\{0, 2a, 3a, 4a\} \ = \ [a]\backslash([2a] \cup [3a]) \ \in \ B(\Gamma)$. □

Denote by Z_k^n the n-fold direct product of Z_k with itself. It was already noticed by Lovász [12] that all Cayley graphs over Z_2^n ("cubelike graphs") are integral. The following theorem extends this result.

Theorem 13. *All nontrivial abelian Cayley integral groups are represented by*

$$Z_2^n,\ Z_3^n,\ Z_4^n,\ Z_2^m \otimes Z_3^n,\ Z_2^m \otimes Z_4^n,\quad m \geqslant 1,\ n \geqslant 1. \tag{12}$$

Proof. Lemma 11, Lemma 12, and Theorem 8 imply that the abelian group Γ is Cayley integral, if and only if

$$ord(a) \in \{2, 3, 4, 6\} \text{ for every } a \in \Gamma, a \neq 0. \tag{13}$$

The abelian group Γ is the direct product of cyclic groups of prime power order. In (12) we have listed all types of nontrivial abelian groups which satisfy (13). □

4 Examples

4.1 Hamming graphs

Let $m_1, \ldots, m_r$ be positive integers, $D = \{d_1, \ldots d_k\}$ a set of integers $d_i,\ 1 \leqslant d_i \leqslant r$. The *Hamming graph* $H = Ham(m_1, \ldots, m_r; D)$ has as its vertex set the abelian group

$$\Gamma \ = \ Z_{m_1} \otimes \cdots \otimes Z_{m_r} \ . \tag{14}$$

The *Hamming distance* of vertices $x = (x_i) \in \Gamma$ and $y = (y_i) \in \Gamma$ is

$$d(x, y) \ = \ |\{i :\ 1 \leqslant i \leqslant r,\ x_i \neq y_i\}| \ .$$

Vertices x and y are adjacent in H, if $d(x, y) \in D$. We show $H = Cay(\Gamma, S)$ with $S \in B(\Gamma)$. Then Theorem 8 implies that H is integral.

The *weight* of $x = (x_i) \in \Gamma$ is

$$w(x) \ = \ |\{i :\ 1 \leqslant i \leqslant r,\ x_i \neq 0\}| \ .$$

We achieve $H = Cay(\Gamma, S)$ by

$$S \ = \ S_1 \cup \ldots \cup S_k \ , \quad S_j \ = \ \{x \in \Gamma :\ w(x) = d_j\} \ \text{ for } \ 1 \leqslant j \leqslant k \ .$$

It remains to show $S_j \in B(\Gamma)$ for every $j,\ 1 \leqslant j \leqslant k$, or generally

$$S(d) \ = \ \{x \in \Gamma :\ w(x) = d\} \ \in \ B(\Gamma) \ \text{ for every } \ d,\ 1 \leqslant d \leqslant r \ .$$

Define the *support* of $x = (x_i) \in \Gamma$ by $supp(x) = \{i :\ 1 \leqslant i \leqslant r,\ x_i \neq 0\}$.
Let $\{A_q :\ 1 \leqslant q \leqslant \binom{r}{d}\}$ be the family of all d-element subsets of $\{1, \ldots, r\}$ and

$$B_q \ = \ \{x \in S(d) :\ supp(x) = A_q\} \ \text{ for } \ 1 \leqslant q \leqslant \binom{r}{d} \ . \tag{15}$$

Then we have

$$S(d) \; = \bigcup_{1 \leqslant q \leqslant \binom{r}{d}} B_q \; .$$

So it is sufficient to show $B_q \in B(\Gamma)$ for every $q,\; 1 \leqslant q \leqslant \binom{r}{d}$.

If for $A \subseteq \{1, \ldots, r\}$ we define

$$P(A) \; = \; \{x = (x_i) \in \Gamma : \; x_i = 0 \;\; \text{for every} \;\; i \notin A\}$$

then $P(A)$ is a subgroup of Γ, which by (14) is isomorphic to $\bigotimes_{i \in A} Z_{m_i}$. By (15) we now conclude

$$B_q \; = \; P(A_q) \setminus \bigcup_{A \subsetneq A_q} P(A) \; \in \; B(\Gamma) \; .$$

Thus we arrive at the following result.

Proposition 14. *Every Hamming graph $Ham(m_1, \ldots, m_r; D)$ is an integral Cayley graph.*

4.2 Sudoku graphs

For an integer $n \geqslant 2$ an n-*Sudoku* is an arrangement of $n \times n$ square blocks each consisting of $n \times n$ cells. In Figure 1 we display an example for the commonly used format given by $n = 3$. Each cell has to be filled with a number (color) ranging from 1 to n^2 such that every block, row or column contains all of the colors $1, \ldots, n^2$. For a Sudoku puzzle certain colored cells are stipulated (in Figure 1 in bold type). The aim is to color the remaining cells according to the above conditions. This puzzle may be considered as the task to complete a partial proper coloring of the underlying graph $Sud(n)$ to a proper coloring of this graph.

9	1	4	2	5	6	**3**	7	**8**
7	6	5	**3**	1	**8**	2	9	4
3	8	2	7	**9**	4	**6**	5	**1**
1	**2**	6	9	8	**7**	5	**4**	3
5	4	**7**	6	3	**2**	**1**	8	**9**
8	**9**	3	**1**	**4**	5	7	**6**	2
6	5	**1**	8	**2**	9	**4**	3	7
4	3	9	**5**	7	**1**	8	2	6
2	7	**8**	4	**6**	3	9	1	**5**

Figure 1

The *Sudoku graph* $Sud(n)$ has as its vertices the n^2 cells of an n-Sudoku. Vertices (cells) are adjacent, if they are in the same block, row or column. Based on the representation of $Sud(n)$ as a certain product (NEPS) of complete graphs it has been shown in [13] that $Sud(n)$ is integral. Its eigenvalues (multiplicities in brackets) in descending order are:

$$3n^2 - 2n - 1 \; [1], \;\; 2n^2 - 2n - 1 \; [2(n-1)], \;\; n^2 - n - 1 \; [2n(n-1)],$$
$$n^2 - 2n - 1 \; [(n-1)^2], \;\; -1 \; [n^2(n-1)^2], \;\; -1 - n \; [2n(n-1)^2] \; .$$

Here we show $Sud(n) = Cay(\Gamma, S)$ for an abelian group Γ and $S \in B(\Gamma)$. So $Sud(n)$ is an integral Cayley graph according to Theorem 8. The above eigenvalues could also be determined by Lovász's method, Lemma 7.

We represent the vertices (cells) of $Sud(n)$ by the elements

$$x = (x_1, x_2, x_3, x_4) \in \Gamma = Z_n^4, \quad Z_n = \{0, 1, \ldots, n-1\} \ .$$

For a given cell the first pair (x_1, x_2) of coordinates localizes the block of the cell. The second pair (x_3, x_4) describes the position of the cell within its block. According to the different types of edges in $Sud(n)$ the set S is partitioned into three subsets, $S = S_1 \cup S_2 \cup S_3$,

$$\begin{array}{rcl} S_1 & = & \{(0,0,x_3,x_4) : \ x_3, x_4 \in Z_n, \ (x_3,x_4) \neq (0,0)\}, \\ S_2 & = & \{(0,x_2,0,x_4) : \ x_2, x_4 \in Z_n, \ x_2 \neq 0\}, \\ S_3 & = & \{(x_1,0,x_3,0) : \ x_1, x_3 \in Z_n, \ x_1 \neq 0\} \ . \end{array}$$

Edges within a block are provided by S_1. The remaining edges within a row or within a column are provided by S_2 and S_3. Thus we achieve $Sud(n) = Cay(\Gamma, S)$. Let $Z_1 = \{0\}$.

$$\begin{array}{l} S_1 = Z_1 \otimes Z_1 \otimes Z_n \otimes Z_n \setminus Z_1 \otimes Z_1 \otimes Z_1 \otimes Z_1 \ \text{ implies } \ S_1 \in B(\Gamma) \ . \\ S_2 = Z_1 \otimes Z_n \otimes Z_1 \otimes Z_n \setminus Z_1 \otimes Z_1 \otimes Z_1 \otimes Z_n \ \text{ implies } \ S_2 \in B(\Gamma) \ . \\ S_3 = Z_n \otimes Z_1 \otimes Z_n \otimes Z_1 \setminus Z_1 \otimes Z_1 \otimes Z_n \otimes Z_1 \ \text{ implies } \ S_3 \in B(\Gamma) \ . \end{array}$$

Therefore, $S \in B(\Gamma)$ and $Sud(n) = Cay(\Gamma, S)$ is an integral Cayley graph by Theorem 8.

In a variant of Sudoku, *positional Sudoku*, discussed by Elsholtz and Mütze [7] the cells have to satisfy an additional condition. Distinct cells in the same position of their respective blocks have to be colored differently. The underlying *positional Sudoku graph* $SudP(n)$ gets additional edges in comparison to $Sud(n)$. In the Cayley graph representation these edges are established by $S_4 = \{(x_1, x_2, 0, 0) : \ x_1, x_2 \in Z_n, \ x_1 \neq 0, \ x_2 \neq 0\}$.

$$S_4 \ = \ Z_n \otimes Z_n \otimes Z_1 \otimes Z_1 \setminus (Z_1 \otimes Z_n \otimes Z_1 \otimes Z_1 \ \cup \ Z_n \otimes Z_1 \otimes Z_1 \otimes Z_1) \ .$$

This implies $S_4 \in B(\Gamma)$ and $\tilde{S} = S_1 \cup S_2 \cup S_3 \cup S_4 \in B(\Gamma)$. Therefore, $SudP(n) = Cay(\Gamma, \tilde{S})$ is an integral Cayley graph by Theorem 8. Its eigenvalues can be determined by Lovász's method, Lemma 7. We list them in descending order (multiplicities in brackets).

$$\begin{array}{l} 4n(n-1) \ [1], \ 2n^2 - 3n \ [4(n-1)], \ n(n-2) \ [4(n-1)^2], \\ 0 \ [(n-1)^4], \ \ -n \ [4(n-1)^3] \ \ -2n \ [2(n-1)^2] \end{array}$$

We summarize the main results of this subsection.

Proposition 15. *Every Sudoku graph $Sud(n)$ and every positional Sudoku graph $SudP(n)$ is an integral Cayley graph.*

4.3 Pandiagonal Latin square graphs

A *Latin square* is an $n \times n$-matrix with entries from $\{1, \ldots, n\}$ such that every number $1, \ldots, n$ appears exactly once in every row and in every column. For a *pandiagonal* Latin square two additional conditions have to be satisfied. Every number $1, \ldots, n$ has to appear exactly once in the main diagonal and its broken parallels as well as in the secondary diagonal and its broken parallels. Hedayat [10] proved that an $n \times n$-pandiagonal Latin square exists, if and only if $n \equiv \pm 1$ modulo 6. Figure 2 presents a 7×7-pandiagonal Latin square.

1	2	3	4	5	6	7
6	7	1	2	3	4	5
4	5	6	7	1	2	3
2	3	4	5	6	7	1
7	1	2	3	4	5	6
5	6	7	1	2	3	4
3	4	5	6	7	1	2

Figure 2

For $n \geqslant 2$ the *pandiagonal Latin square graph* $PLSG(n)$ has as its vertex set the n^2 positions of an $n \times n$-matrix. Distinct vertices (positions) are adjacent, if they are in the same row, in the same column, in the same (broken) parallel to the main diagonal or in the same (broken) parallel to the secondary diagonal. The graph $PLSG(n)$ is defined for every integer $n \geqslant 2$. The existence of an $n \times n$-pandiagonal Latin square is equivalent to chromatic number $\chi(PLSG(n)) = n$. Here we show that $PLSG(n) = Cay(\Gamma, S)$ is an integral Cayley graph.

Naturally, we describe the vertex set of $PLSG(n)$ by $\Gamma = Z_n \otimes Z_n$. The set S is partitioned into four parts, $S = S_1 \cup S_2 \cup S_3 \cup S_4$, according to the four types of edges in $PLSG(n)$. The sets

$$S_1 = \{(0, x_2) : \ x_2 \in Z_n, \ x_2 \neq 0\}, \quad S_2 = \{(x_1, 0) : \ x_1 \in Z_n, \ x_1 \neq 0\}$$

provide the edges between positions in the same row or column. With the notation $Z_1 = \{0\}$ we see

$$S_1 = Z_1 \otimes Z_n \setminus Z_1 \otimes Z_1 \in B(\Gamma), \quad S_2 = Z_n \otimes Z_1 \setminus Z_1 \otimes Z_1 \in B(\Gamma) \ .$$

The set $S_3 = \{(x_1, x_1) : \ x_1 \in Z_n, \ x_1 \neq 0\}$ provides the edges between positions in the main diagonal or in one of its broken parallels. As $U_1 = \{(x_1, x_1) : \ x_1 \in Z_n\}$ is a subgroup of Γ, we conclude $S_3 = U_1 \setminus Z_1 \otimes Z_1 \in B(\Gamma)$. The set

$$S_4 = \{(x_1, -x_1) : \ x_1 \in Z_n, \ x_1 \neq 0, \ x_1 \neq \frac{n}{2} \text{ for even } n\}$$

provides the remaining edges between positions in the secondary diagonal or one of its broken parallels. If we define

$U_2 = \{(x_1, -x_1) : x_1 \in Z_n\}, \ U_3 = \{(0, 0), (\frac{n}{2}, \frac{n}{2})\}$ for even n and $U_3 = \{(0, 0)\}$ for odd n, then U_2 and U_3 are subgroups of Γ and $S_4 = U_2 \backslash U_3 \in B(\Gamma)$. Now we arrive at $PLSG(n) = Cay(\Gamma, S)$, $S = S_1 \cup S_2 \cup S_3 \cup S_4 \in B(\Gamma)$. Theorem 8 implies that $PLSG(n)$ is an integral Cayley graph.

Proposition 16. *Every pandiagonal Latin square graph $PLSG(n)$ is an integral Cayley graph.*

We have determined the eigenvalues of $PLSG(n)$ by Lovász's method, Lemma 7. The fact that there are exactly three different eigenvalues for odd $n \geqslant 5$ reflects that $PLSG(n)$ is strongly regular in this case (see [11]). We list the eigenvalues in descending order (multiplicities in brackets).

$$\text{Case } n \text{ odd:} \quad 4n-4\ [1],\ n-4\ [4n-4],\ -4\ [n^2-4n+3]\ .$$

$$\begin{aligned}\text{Case } n \text{ even:} \quad & 4n-5\ [1],\ 2n-5\ [1],\ n-3\ [n]\ , \\ & n-5\ [3n-6],\ -3\ [\tfrac{n^2}{2}-n],\ -5\ [\tfrac{n^2}{2}-3n+4]\ .\end{aligned}$$

5 Problems and remarks

1. We think that the converse of Theorem 8 is not only true for cyclic groups (Theorem 10), but for abelian groups in general. By a computer search we found no counterexample to this conjecture among abelian groups up to order 71.

2. There are nonabelian groups, e.g. all dihedral groups D_n, $|D_n| = 2n \geqslant 8$, for which Theorem 8 does not hold.

3. Up to order 12 we have found three nonabelian Cayley integral groups: the symmetric group S_3 of order 6, the group Q_8 of quaternions as a group of order 8, and the semidirect product (see Cohn [6]) of Z_3 and Z_4 as a group of order 12. Determine all nonabelian Cayley integral groups.

4. Let the finite abelian group Γ be represented as the direct product of cyclic groups, $\Gamma = Z_{m_1} \otimes \cdots \otimes Z_{m_r}$. Describe an effective method to decide for a subset $S \subseteq \Gamma$, if S belongs to the Boolean algebra $B(\Gamma)$ generated by the subgroups of Γ.

5. From the existence theorem of Hedayat we know that $\chi(PLSG(n)) = n$, if $n \equiv \pm 1$ modulo 6. Determine the chromatic number $\chi(PLSG(n))$ in the remaining cases.

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