Winning Positions in Simplicial Nim

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Submitted: Jun 9, 2009; Accepted: May 27, 2010; Published: Jun 7, 2010
Mathematics Subject Classifications: 91A05, 91A43, 91A44, 91A46

Abstract

Simplicial Nim, introduced by Ehrenborg and Steingrímsson, is a generalization of the classical two-player game of Nim. The heaps are placed on the vertices of a simplicial complex and a player's move may affect any number of piles provided that the corresponding vertices form a face of the complex. In this paper, we present properties of a complex that are equivalent to the $\mathcal{P}$-positions (winning positions for the second player) being closed under addition. We provide examples of such complexes and answer a number of open questions posed by Ehrenborg and Steingrímsson.

1 Introduction

Simplicial Nim, as defined by Ehrenborg and Steingrímsson in [2], is a generalization of the classical game of Nim. It is a combinatorial game for two players who move alternately and, as usual, the last player able to make a move is the winner. Moreover, like Nim, Simplicial Nim is played with a number of piles of chips and a legal move consists of removing a positive number of chips.

A *simplicial complex* Δ on a finite set V is defined to be a collection of subsets of V such that $\{v\} \in \Delta$ for every $v \in V$, and $B \in \Delta$ whenever $A \in \Delta$ and $B \subseteq A$. The elements of V and Δ are called *vertices* (or *points*) and *faces* respectively. A face that is maximal with respect to inclusion is called a *facet*.

To play Simplicial Nim, begin with a simplicial complex Δ and place a pile of chips on each vertex of V. On his turn, a player may remove chips from any nonempty set of piles provided that the vertices corresponding to the affected piles form a face of Δ. Note that a player may remove *any* number of chips from each of the piles on which he chooses to play and that at least one chip must be removed. Observe also that the classical game of Nim is the particular case of Simplicial Nim in which the facets of Δ are precisely the vertices of V.

Example 1. We illustrate the rules of Simplical Nim with a sample playing of the game. Let Δ be the simplicial complex on $V = \{1, 2, 3, 4\}$ with facets $\{1, 2, 3\}$, $\{1, 4\}$, and $\{3, 4\}$. Let the vector $(5, 2, 2, 10)$ represent the pile sizes so that there are 5 chips on vertex 1, and so on. Alice, moving first, decides to use the face $\{1, 4\}$ and elects to remove 3 chips from vertex 1 and 6 chips from vertex 4. After this move, the pile sizes are $(2, 2, 2, 4)$. Bob responds, using the face $\{4\}$, and moves to $(2, 2, 2, 1)$. Now Alice, using the face $\{1, 2, 3\}$, may move to $(1, 0, 1, 1)$. Notice that Bob cannot remove all the chips (and, therefore, win the game) on his next move since $\{1, 3, 4\}$ is not a face. In fact, since Bob must remove at least one chip, he must leave Alice with a position from which she can win the game on her next turn.

In their study [2] of Simplicial Nim, Ehrenborg and Steingrímsson pose a number of tantalizing, open questions about the game. The purpose of this paper is to address some of these questions.

In particular, Ehrenborg and Steingrímsson display a class of simplicial complexes (the pointed circuit complexes to be defined in Section 3) for which the set of $\mathcal{P}$-positions is closed under addition, and ask whether it is possible to classify all such complexes. The majority of this paper is devoted to the study of this question. In Section 3, we find a number of conditions on a simplicial complex, each of which is equivalent to the $\mathcal{P}$-positions being closed under addition.

In Section 4, we consider *graph complexes* in which the facets are the edges of a graph. We determine precisely which graph complexes have the property that the $\mathcal{P}$-positions are closed under sums.

Section 5 is devoted to finding additional examples of complexes in which the $\mathcal{P}$-positions are closed under sums. In this section, we obtain a general result and some natural examples.

Finally, in Section 6, we provide counterexamples which disprove some conjectures presented in [2].

2 Terminology and Preliminaries

Let Δ be a simplicial complex on the finite set V. We shall represent a position in the game of Simplicial Nim by a vector $\mathbf{n} = (n_v)_{v \in V}$ of nonnegative integers where, for each $v \in V$, n_v is the number of chips in the pile on vertex v. As in [2], for each $v \in V$, let $\mathbf{e}(v)$ be the unit vector whose vth component is 1 and all other components are 0. For a subset S of V, let $\mathbf{e}(S) = \sum_{v \in S} \mathbf{e}(v)$. We will use the following standard notation: for positions $\mathbf{m}$ and $\mathbf{n}$, we write $\mathbf{m} \leqslant \mathbf{n}$ if and only if $m_v \leqslant n_v$ for all $v \in V$.

We assume that the reader is familiar with the standard terminology of combinatorial game theory, as described in [1]. Simplicial Nim is an example of an *impartial game* which means that, at any point in the game, the set of possible moves does not depend on whose turn it is. The positions of an impartial game may be partitioned uniquely into two classes, often denoted $\mathcal{N}$ and $\mathcal{P}$. The $\mathcal{P}$-positions are those from which the player who has just moved (the previous player) has a winning strategy; from an $\mathcal{N}$-position, the next player to play may win. The following theorem, which we state without proof, is well-known and a cornerstone in the theory of impartial combinatorial games.

Theorem 2. *A set $\mathcal{T}$ of positions in an impartial combinatorial game is identical to the set of $\mathcal{P}$-positions if and only if the following three conditions hold.*

1. *Any position from which there is no legal move is in $\mathcal{T}$.*
2. *From any position not in $\mathcal{T}$, there exists a move to a position in $\mathcal{T}$.*
3. *There does not exist a move between any pair of positions in $\mathcal{T}$.*

3 Complexes whose $\mathcal{P}$-positions are Closed under Addition

In [2], Ehrenborg and Steingrímsson showed that the $\mathcal{P}$-positions of a pointed circuit complex (to be defined below) are closed under vector addition. The authors asked (Question 9.3 in [2]) if it is possible to classify all simplicial complexes whose set of $\mathcal{P}$-positions is closed under addition. The purpose of this section is to investigate this question. Our main result is Theorem 12 in which we present a number of properties of a simplicial complex equivalent to its $\mathcal{P}$-positions being closed under addition. These properties are then used in subsequent sections to find actual examples of such complexes.

A minimal (with respect to inclusion) non-face of a simplicial complex is called a *circuit*. In other words, a circuit is a subset C of the vertex set V such that every subset properly contained in C is a face. For instance, in the simplicial complex of Example 1, the circuits are $\{2, 4\}$ and $\{1, 3, 4\}$. Notice that any subset of V which is not a face contains a circuit.

We will require the following definitions which are found in [2]. Let v be a vertex of Δ and C a circuit. If $v \in C$ and v is in no other circuit of Δ then C is said to be *pointed*, or more precisely, *pointed by* v. Furthermore, Δ is a *pointed circuit complex* if every circuit of Δ is pointed.

The circuits play a fundamental role in the analysis of Simplicial Nim. For example, the following lemma from [2], shows that any positive integer multiple of the characteristic vector of a circuit is a $\mathcal{P}$-position.

Lemma 3. *If C is a circuit of the simplicial complex Δ then $n \cdot \mathbf{e}(C)$ is a $\mathcal{P}$-position for all $n \in \mathbb{N}$.*

It will be necessary to refer to the set of all nonnegative integer linear combinations of the characteristic vectors of circuits (or simply, nonnegative combinations of circuits) so we make the following definition.

Definition 4. For the simplicial complex Δ, let $\mathcal{C}$ denote the collection of circuits of Δ. Let $\mathcal{S}$ denote the set of all positions of the form

$$\sum_{C \in \mathcal{C}} a_C \cdot \mathbf{e}(C)$$

where a_C is a nonnegative integer for each circuit C.

Ehrenborg and Steingrímsson [2] showed that if Δ is pointed then $\mathcal{S}$ is precisely the set of $\mathcal{P}$-positions from which it then follows that the $\mathcal{P}$-positions of such a complex are closed under addition. The following lemma and the ideas in its proof are similar to Theorem 3.2 in [2].

We first make the following definition. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a nonnegative vector, i.e. $v_i \geqslant 0$ for all $1 \leqslant i \leqslant n$. We define the *support* of $\mathbf{v}$ to be

$$\operatorname{supp}(\mathbf{v}) = \{i \mid v_i > 0\}.$$

Lemma 5. *From any position not in $\mathcal{S}$, there is a move to a position in $\mathcal{S}$.*

Proof. Suppose that the position $\mathbf{n}$ is not in $\mathcal{S}$. Now $\mathbf{n} \neq \mathbf{0}$ (since $\mathbf{0} \in S$) and if there is a move from $\mathbf{n}$ to $\mathbf{0}$ then we are done. We assume then that $\mathbf{n}$ is not an immediate win. Therefore, $\operatorname{supp}(\mathbf{n})$ contains a circuit so the set $\mathcal{T} = \{\mathbf{x} \in \mathcal{S} \mid \mathbf{x} \leqslant \mathbf{n}\}$ is not empty. Let $\mathbf{m} \in \mathcal{T}$ be a maximal element of $\mathcal{T}$ in the sense that if $\mathbf{v} \in \mathcal{T}$ and $\mathbf{v} \geqslant \mathbf{m}$ then $\mathbf{v} = \mathbf{m}$. Now let $F = \{v \in V \mid \mathbf{m}_v < \mathbf{n}_v\}$ and notice that since $\mathbf{n} \notin \mathcal{S}$, F is not empty. If F contains the circuit C then $\mathbf{m} + \mathbf{e}(C) \leqslant \mathbf{n}$ which contradicts the maximality of $\mathbf{m}$. Therefore, F does not contain a circuit and so F must be a face. There is a move then, using the face F, from $\mathbf{n}$ to $\mathbf{m}$. $\square$

Each of the following four propositions (Propositions 6, 7, 9, and 11) establishes the equivalence of a pair of properties of a simplicial complex. Taken together, these propositions form the main result of this section, namely Theorem 12.

Proposition 6. *The $\mathcal{P}$-positions of Δ are closed under addition if and only if the set of $\mathcal{P}$-positions of Δ is equal to the set $\mathcal{S}$.*

Proof. Suppose that the $\mathcal{P}$-positions are exactly those positions in $\mathcal{S}$. Clearly, the set $\mathcal{S}$ is closed under addition and so also then is the set of $\mathcal{P}$-positions.

Conversely, suppose that the $\mathcal{P}$-positions are closed under addition. It follows that since $\mathbf{e}(C)$ is a $\mathcal{P}$-position for each circuit C (by Lemma 3), every position of the form $\sum_{C\in\mathcal{C}} a_C \cdot \mathbf{e}(C)$ where a_C is a nonnegative integer for each circuit C, is a $\mathcal{P}$-position. Thus the set $\mathcal{S}$ is contained in the set of $\mathcal{P}$-positions. Finally, suppose that the position $\mathbf{n}$ is not in $\mathcal{S}$. By Lemma 5, there is a move from $\mathbf{n}$ to a position $\mathbf{m} \in \mathcal{S} \subseteq \mathcal{P}$ so $\mathbf{n}$ cannot be a $\mathcal{P}$-position. $\square$

Proposition 7. *The set of $\mathcal{P}$-positions of Δ is equal to the set $\mathcal{S}$ if and only if there does not exist a move between any pair of positions in $\mathcal{S}$.*

Proof. In the game of Simplicial Nim, $\mathbf{0}$ is the only terminal position and this position is in $\mathcal{S}$. Moreover, by Lemma 5, from any position not in $\mathcal{S}$, there is a move to a position in $\mathcal{S}$. Therefore, the set $\mathcal{S}$ satisfies the first two conditions in Theorem 2.

The set $\mathcal{S}$, then, is precisely the set of $\mathcal{P}$-positions if and only if $\mathcal{S}$ satisfies the third condition in Theorem 2, namely that there does not exist a move between any pair of positions in $\mathcal{S}$. $\square$

Definition 8. The simplicial complex Δ has *property P* if and only if there does not exist a nonempty face F such that

$$\operatorname{supp}\left(\sum_{C\in\mathcal{C}} b_C \cdot \mathbf{e}(C)\right) = F$$

where b_C is an integer for each circuit C. (Note that we allow the possibility that $b_C < 0$ for some circuit C; in fact, this must happen in order for the above equation to hold.)

In other words, a complex does not have property P if and only if there is a nonnegative vector (not $\mathbf{0}$), whose positive components are supported by a face, that may be written as an integer combination of circuits.

Proposition 9. *The simplicial complex Δ has property P if and only if there does not exist a move between any pair of positions in $\mathcal{S}$.*

Proof. Suppose that there is a move from a position $\mathbf{n} \in \mathcal{S}$ to a position $\mathbf{m} \in \mathcal{S}$. Then there is a face F such that $\mathbf{n} - \sum_{v\in F} c_v \mathbf{e}(\{v\}) = \mathbf{m}$ where c_v is a positive integer for all $v \in F$. Since $\mathbf{n}$ and $\mathbf{m}$ are in $\mathcal{S}$, for each circuit C, there are nonnegative integers a_C and b_C such that $\mathbf{n} = \sum_{C\in\mathcal{C}} a_C \mathbf{e}(C)$ and $\mathbf{m} = \sum_{C\in\mathcal{C}} b_C \mathbf{e}(C)$. But now

$$\sum_{C\in\mathcal{C}} (a_C - b_C)\mathbf{e}(C) = \sum_{v\in F} c_v \mathbf{e}(\{v\})$$

so Δ does not have property P.

Conversely, suppose that Δ does not have property P so that there is a nonempty face F and a set of positive integers $\{c_v \mid v \in F\}$ such that

$$\sum_{v\in F} c_v \mathbf{e}(\{v\}) = \sum_{C\in\mathcal{C}} a_C \mathbf{e}(C)$$

where a_C is an integer for each circuit C. Now not all the a_C are equal to zero, and of the nonzero a_C, clearly not all can be negative, nor may all be positive, lest F contain a circuit which is impossible. Therefore, some of the nonzero coefficients a_C are positive and some are negative. Set $\mathbf{n} = \sum_{a_C>0} a_C \mathbf{e}(C)$ and $\mathbf{m} = \sum_{a_C<0} a_C \mathbf{e}(C)$ so that the above equation becomes

$$\mathbf{n} - \sum_{v\in F} c_v \mathbf{e}(\{v\}) = \mathbf{m}.$$

Thus there is a move from $\mathbf{n} \in \mathcal{S}$ using face F to $\mathbf{m} \in \mathcal{S}$. $\square$

Definition 10. Let $C_1, C_2, \ldots, C_p$ be the circuits of the simplicial complex Δ and let $V = \{1, 2, \ldots, n\}$ be the vertices. Define $\mathbf{A} = (a_{ij})$ to be the $n \times p$ point-circuit incidence matrix, that is,

$$a_{ij} = \begin{cases} 1 & \text{if } i \in C_j \\ 0 & \text{otherwise.} \end{cases}$$

We say that Δ has *property Q* if, for any integer vector $\mathbf{x}$ such that $\mathbf{Ax} \geqslant \mathbf{0}$, there exists a vector $\mathbf{y}$ with nonnegative integer entries such that $\mathbf{Ay} = \mathbf{Ax}$.

In other words, Δ has property Q if and only if any nonnegative vector that is expressible as an integer combination of circuits may, in fact, be written as a *nonnegative* integer combination of circuits.

Proposition 11. *For a simplicial complex Δ, property P is equivalent to property Q.*

Proof. Suppose that Δ has property P. We wish to show that, for every integer vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{f} = (f_1, f_2, \ldots, f_n) \geqslant \mathbf{0}$, there exists a nonnegative integer vector $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = \mathbf{f}$. The proof is by induction on $\sum_i f_i$.

First, if $\sum f_i = 0$ then $\mathbf{f} = \mathbf{0}$ and we take $\mathbf{y} = \mathbf{0}$.

Now suppose that $\mathbf{f} \geqslant \mathbf{0}$ has some positive components. Since Δ has property P, $\operatorname{supp}(\mathbf{f})$ is not a face. Therefore, $\operatorname{supp}(\mathbf{f})$ contains a circuit C and so $\mathbf{A}\mathbf{x} - (\mathbf{e}(C))^t = \mathbf{f} - (\mathbf{e}(C))^t \geqslant \mathbf{0}$. By induction, there is a nonnegative integer vector $\mathbf{y}'$ such that $\mathbf{A}\mathbf{y}' = \mathbf{f} - (\mathbf{e}(C))^t$. Now taking $\mathbf{y}$ to be the vector obtained from $\mathbf{y}'$ by increasing by 1 the coordinate corresponding to the circuit C, we have $\mathbf{A}\mathbf{y} = \mathbf{f} = \mathbf{A}\mathbf{x}$.

Conversely, if Δ does not have property P then there is a nonempty face F and nonnegative integer vector $\mathbf{f}$ such that $\sum_{C \in \mathcal{C}} b_C \cdot \mathbf{e}(C) = \mathbf{f}$ where b_C is an integer for each circuit C and $\operatorname{supp}(\mathbf{f}) = F$. Suppose that there is a nonnegative integer vector $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = \mathbf{f}$. But now for any circuit C for which the corresponding component of $\mathbf{y}$ is positive, we have $C \subseteq F$ which is impossible. Therefore, Δ does not have property Q. $\square$

Theorem 12. *Let Δ be a simplicial complex. The following are equivalent:*

(i) The $\mathcal{P}$-positions of Δ are closed under addition.

(ii) The $\mathcal{P}$-positions of Δ are precisely those positions in the set $\mathcal{S}$.

(iii) There does not exist a move between any pair of positions in $\mathcal{S}$.

(iv) Δ has property P.

(v) Δ has property Q.

Proof. That (i) and (ii) are equivalent is the result of Proposition 6, for (ii) and (iii) is Proposition 7, for (iii) and (iv) is Proposition 9, and finally, (iv) and (v) are equivalent by Proposition 11. $\square$

4 Graph Complexes

In this section, we consider a simplicial complex derived in a natural way from a graph. Our main result is Theorem 19 which characterizes those graphs whose corresponding complexes have the property that the $\mathcal{P}$-positions are closed under addition. The terminology from graph theory used in this section is standard; we refer the reader to the text by West [3] for definitions.

Definition 13. Let G be a simple graph. The *graph complex over* G, denoted $\Delta = \Delta(G)$, is the simplicial complex whose facets consist of the edges and isolated vertices of G.

We begin by identifying the circuits of $\Delta(G)$.

Lemma 14. *The circuits of the graph complex Δ on the graph G consist of the non-edges of G and the triangles of G.*

Proof. If $\{x, y\}$ is a non-edge of G then $\{x, y\}$ is a circuit of Δ since it is a minimal non-face. Moreover, if $\{x, y, z\}$ is a triangle then it is also a circuit since it is not a face but its subsets $\{x, y\}$, $\{y, z\}$, and $\{x, z\}$ are all faces.

Now suppose that $S \subset V$ is a circuit of Δ. Clearly then $|S| > 1$. If $S = \{x, y\}$ then S must be a non-edge of G. If $S = \{x, y, z\}$ then $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ must all be faces of Δ and, therefore, edges of G. Thus $\{x, y, z\}$ is a triangle. Finally, if $|S| \geqslant 4$ then S cannot be a circuit since any of its subsets of size 3 would not be a face. $\square$

The next two lemmas give two sufficient conditions for a graph complex not to have property P.

Lemma 15. *If G has an independent set of size 3 then Δ does not have property P.*

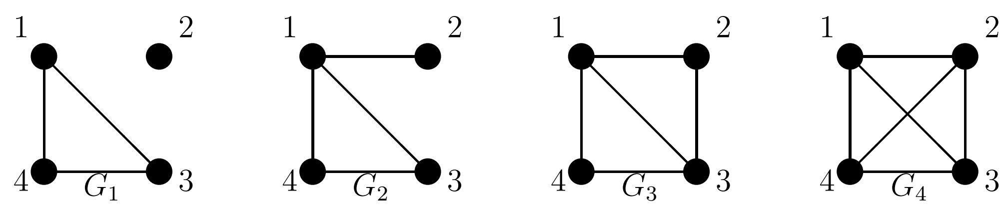
Proof. Suppose that $\{x, y, z\} \subset V(G)$ is an independent set. By Lemma 14 then, $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ are all circuits of Δ. But now

$$\mathbf{e}(\{x, y\}) + \mathbf{e}(\{x, z\}) - \mathbf{e}(\{y, z\}) = 2 \cdot \mathbf{e}(\{x\})$$

and $\{x\}$ is a face so Δ does not have property P. $\square$

Lemma 16. *If G has at least 4 vertices and contains a triangle then Δ does not have property P.*

Proof. Pictured below are the 4 nonisomorphic graphs on 4 vertices that contain a triangle.



In G_1, for example, we have

$$\mathbf{e}(\{1, 2\}) + \mathbf{e}(\{2, 3\}) + \mathbf{e}(\{2, 4\}) - \mathbf{e}(\{1, 3, 4\}) = 3 \cdot \mathbf{e}(\{2\})$$

and so, by Definition 8, the complex over G_1 does not have property P. In similar fashion, it may be shown that the graph complexes over G_2, G_3, and G_4 do not have property P.

Finally, any graph G satisfying the hypotheses of the lemma must contain one of G_1, G_2, G_3, or G_4 as an induced subgraph so $\Delta = \Delta(G)$ does not have property P. $\square$

Lemma 17. *If G contains a cycle then Δ has property P if and only if G is isomorphic to C_3 or C_4.*

Proof. We first note that the graph complexes over C_3 and C_4 are pointed and, therefore, have property P.

Suppose that G contains a cycle. If the girth of G is at least 6 then G contains an induced k-cycle with $k \geqslant 6$. But such a cycle has an independent set of size 3 so Δ does not have property P by Lemma 15.

If the girth of G is exactly 5 then G contains an induced 5-cycle C. Suppose that the vertices of C, traversing the cycle, are 1, 2, 3, 4, and 5. Now $\{1,3\}$, $\{2,5\}$, and $\{3,5\}$ are circuits and $\{1,2\}$ is a face and

$$\mathbf{e}(\{1,3\}) + \mathbf{e}(\{2,5\}) - \mathbf{e}(\{3,5\}) = \mathbf{e}(\{1,2\})$$

so Δ does not have property P.

Now suppose that the girth of G is 4. If G has exactly 4 vertices then G is isomorphic to C_4 so Δ has property P as noted above. If G has more than 4 vertices then it contains a cycle C of length 4 and a vertex x which is not on C. If x is adjacent to fewer than 2 vertices of C then x together with some 2 vertices of C forms an independent set of size 3 so Δ does not have property P. If x is adjacent to exactly 2 vertices, y and z, of C then y and z are not adjacent lest x, y, and z be the vertices of a triangle in G which is not possible. But now the other two vertices of C, together with x, form an independent set of size 3 so Δ does not have property P. Finally, if x is adjacent to 3 or more vertices of C then some two of these vertices are adjacent and, together with x, form a triangle in G, an impossibility.

Lastly, suppose that the girth of G is 3. If G contains exactly 3 vertices then G is isomorphic to C_3 so Δ has property P. If G has 4 or more vertices then Δ does not have property P by Lemma 16. $\square$

Lemma 18. *The graph complex over P_4 does not have property P.*

Proof. Let the vertices of P_4, traversing the path, be 1, 2, 3, and 4. Now

$$\mathbf{e}(\{1,3\}) + \mathbf{e}(\{2,4\}) - \mathbf{e}(\{1,4\}) = \mathbf{e}(\{2,3\}).$$

so the graph complex over P_4 does not have property P. $\square$

We are now ready to prove the main result of this section.

Theorem 19. *There are precisely 8 graph complexes which are closed under addition, or equivalently, have property P. These complexes are those corresponding to the following graphs:*

$$P_1, \quad P_2, \quad P_3, \quad P_1 + P_1, \quad P_1 + P_2, \quad P_2 + P_2, \quad C_3, \quad C_4.$$

Proof. The graph complexes corresponding to P_1, P_2, P_3, $P_1 + P_1$, and $P_1 + P_2$ may be seen to be pointed and, therefore, have property P. Moreover, as noted in the proof of Lemma 17, the graph complexes of C_3 and C_4 are also pointed and so have property P.

The only remaining graph in the list is $P_2 + P_2$. Let $\{1,2\}$ and $\{3,4\}$ be the edges of this graph. Clearly, Simplicial Nim on this graph complex is equivalent to ordinary Nim with two heaps, one of size the sum of the heaps on vertices 1 and 2, and the other of size the sum of the heaps on vertices 3 and 4. Therefore, the $\mathcal{P}$-positions are precisely those in which the sum of the heaps on vertices 1 and 2 is equal to the sum of the heaps on vertices 3 and 4. In other words the set of $\mathcal{P}$-positions may be written as

$$(a+b, c+d, a+c, b+d) = a(1,0,1,0) + b(1,0,0,1) + c(0,1,1,0) + d(0,1,0,1)$$

which is the set of all nonnegative integer combinations of the circuits $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, and $\{2,4\}$. Therefore, this complex has property P by Theorem 12.

Conversely, suppose that the simplicial complex Δ corresponding to the graph G has property P. If G contains a cycle then G must be isomorphic to either C_3 or C_4 by Lemma 17. On the other hand, supposing that G has no cycles means that every component of G is a tree. Now, by Lemma 15, G does not contain an independent set of size 3, so each component of G must be a path having no more than 4 vertices. Paths of exactly 4 vertices are excluded by Lemma 18 so every component of G is isomorphic to either P_1, P_2, or P_3. Finally, since the maximum size of an independent set in G is 2, the only possibilities for G are P_1, P_2, P_3, P_1+P_1, P_1+P_2, or P_2+P_2. □

5 More Complexes whose $\mathcal{P}$-positions are Closed under Addition

The purpose of this section is to find further examples of complexes in which the $\mathcal{P}$-positions are closed under addition. The main result is Theorem 21 in which we derive a necessary condition for a complex to have property Q. We then apply the theorem to some natural examples.

In [2], Ehrenborg and Steingrímsson showed that a position in a pointed circuit complex is a $\mathcal{P}$-position if and only if it is a nonnegative integer combination of circuits. We present the following equivalent result.

Theorem 20. *If Δ is a pointed circuit complex then Δ has property Q.*

Proof. Let the circuits of Δ be $C_1, C_2, \ldots, C_p$ and suppose, for $i = 1, 2, \ldots, p$, that the circuit C_i is pointed by the vertex v_i.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_p)$ be an integer vector such that $\mathbf{Ax} \geqslant \mathbf{0}$ where $\mathbf{A}$ is the point-circuit incidence matrix defined in Section 3. Comparing the ith coordinate on both sides of this inequality yields $x_i \geqslant 0$ since v_i is in circuit C_i and no other. Thus $\mathbf{Ax} \geqslant \mathbf{0}$ implies that $\mathbf{x} \geqslant \mathbf{0}$ so Δ has property Q. □

We require the following standard definition: subsets $S_1, S_2, \ldots, S_k$ of the set X are said to *partition* X if $S_1 \cup S_2 \cup \cdots \cup S_k = X$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$.

Theorem 21. *Suppose that $C_1, C_2, \ldots, C_r$ and $D_1, D_2, \ldots, D_s$ are the circuits of the simplicial complex Δ and that $\{C_1, C_2, \ldots, C_r\}$ and $\{D_1, D_2, \ldots, D_s\}$ both partition V. Moreover, suppose that for every pair of indices (i, j), there is a vertex v such that C_i and D_j are the only circuits of Δ that contain v. Then Δ has property Q.*

Proof. Suppose that there is an integer vector $\mathbf{x} = (a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s)$ such that $\mathbf{Ax} \geqslant \mathbf{0}$. We wish to find a nonnegative integer vector $\mathbf{y}$ such that $\mathbf{Ay} = \mathbf{Ax}$. If $\mathbf{x} \geqslant \mathbf{0}$ then we simply take $\mathbf{y} = \mathbf{x}$. Therefore, we will suppose that $\mathbf{x}$ has some negative components.

Now consider the pair of indices (i, j). There is a vertex v such that C_i and D_j are the only circuits that contain v so comparing the vth coordinate on both sides of the inequality $\mathbf{Ax} \geqslant \mathbf{0}$ yields $a_i + b_j \geqslant 0$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$.

Let $S = \{k \mid x_k < 0\}$ be the set of coordinates in which $\mathbf{x} = (x_1, x_2, \dots, x_{r+s})$ is negative. Now S cannot intersect both $\{1, 2, \dots, r\}$ and $\{r+1, r+2, \dots, r+s\}$ nontrivially, lest some inequality of the form $a_i + b_j \geqslant 0$ be violated. Therefore, either $S \subseteq \{1, 2, \dots, r\}$ or $S \subseteq \{r+1, r+2, \dots, r+s\}$ but not both. Suppose, without loss of generality, that the former alternative holds and let

$$\mathbf{z} = (\underbrace{1, 1, \dots, 1}_{r}, \underbrace{-1, -1, \dots, -1}_{s}).$$

Since each vertex lies in a unique C_i and a unique D_j, we have $\mathbf{Az} = \mathbf{0}$. Let $a_p = \min_i\{a_i\}$ be the most negative component of $\mathbf{x}$ and set $\mathbf{y} = \mathbf{x} + |a_p|\mathbf{z}$. For all $1 \leqslant j \leqslant s$, we have $a_p + b_j \geqslant 0$ so $b_j \geqslant |a_p|$ and thus $\mathbf{y} \geqslant \mathbf{0}$. Finally, since $\mathbf{Az} = \mathbf{0}$, we have $\mathbf{Ay} = \mathbf{Ax}$ and so Δ has property Q. $\square$

Remark 22. The following example shows that the second assumption in Theorem 21 may not be omitted.

Let Δ be the simplicial complex on $V = \{1, 2, \dots, 6\}$ having facets $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{1, 6\}$, $\{2, 4\}$, and $\{3, 5\}$. Then the circuits of Δ are $C_1 = \{1, 4\}$, $C_2 = \{2, 5\}$, $C_3 = \{3, 6\}$, $D_1 = \{1, 5\}$, $D_2 = \{2, 6\}$, and $D_3 = \{3, 4\}$.

We see that $\{C_1, C_2, C_3\}$ and $\{D_1, D_2, D_3\}$ both partition V but, considering C_1 and D_2 for example, shows that the second hypothesis does not hold. Finally, since

$$\mathbf{e}(C_1) + \mathbf{e}(C_3) - \mathbf{e}(D_3) = \mathbf{e}(\{1, 6\})$$

Δ does not have property P and so does not have property Q either.

Example 23. The *cycle complex* $C_{n,k}$ was defined in [2]; it has vertex set $\{1, 2, \dots, n\}$ and facets $\{i, i+1, \dots, i+k-1\}$ where $i = 1, 2, \dots, n$, and the addition is done modulo n. In other words, a move on $C_{n,k}$ may affect the piles on any k consecutive vertices of the cycle.

For example, the cycle complex $C_{6,3}$ has vertex set $\{1, 2, \dots, 6\}$ and facets $\{1, 2, 3\}$, $\{2, 3, 4\}$, …, $\{5, 6, 1\}$, $\{6, 1, 2\}$. The circuits are $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$ and satisfy the hypothesis of Theorem 21 so the complex $C_{6,3}$ has property Q.

Therefore, the $\mathcal{P}$-positions of $C_{6,3}$ are closed under addition and they have the form

$$\begin{aligned} & a \cdot \mathbf{e}(\{1,4\}) + b \cdot \mathbf{e}(\{2,5\}) + c \cdot \mathbf{e}(\{3,6\}) + d \cdot \mathbf{e}(\{1,3,5\}) + e \cdot \mathbf{e}(\{2,4,6\}) \\ = \; & (a+d, b+e, c+d, a+e, b+d, c+e) \end{aligned}$$

where a, b, c, d, e are nonnegative integers.

Example 24. Consider the following instance of Simplicial Nim in which the vertices of the complex are arranged into a rectangular grid. A move may affect any subset of the piles provided that the underlying subset of vertices does not contain any row or any column. In this case, the circuits are precisely the subsets of vertices corresponding to the rows and columns. Thus, Theorem 21 applies so this complex has property Q.

For example, suppose that the game is played on a 3×3 grid with the vertices labelled as shown below.

1	2	3
4	5	6
7	8	9

A valid move, for instance, would be to play on the piles on vertices 2, 3, 4, 5, and 9. The circuits are $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{1,4,7\}$, $\{2,5,8\}$, and $\{3,6,9\}$ so by Theorem 21, the $\mathcal{P}$-positions are closed under addition and they have the form

$$(a+d, a+e, a+f, b+d, b+e, b+f, c+d, c+e, c+f)$$

where a, b, c, d, e, f are nonnegative integers.

6 Questions and Conjectures

In [2], Ehrenborg and Steingrímsson pose a number of interesting questions and conjectures about Simplicial Nim. In this section, we settle some of these questions in the negative by providing counterexamples.

The first question we consider is Conjecture 8.1 in [2].

Conjecture 25. Suppose that $\mathbf{n}$ is a $\mathcal{P}$-position of the cycle complex $C_{n,k}$. If $\mathbf{1} = (1, 1, \ldots, 1)$ is a $\mathcal{P}$-position of $C_{n,k}$ then $\mathbf{n} + \mathbf{1}$ is also a $\mathcal{P}$-position of $C_{n,k}$.

This conjecture is false and the cycle complex $C_{8,2}$ provides a counterexample. Now $\mathbf{n} = 2\cdot\mathbf{e}(\{6,8\}) = (0,0,0,0,0,2,0,2)$ is a $\mathcal{P}$-position. However, $\mathbf{n}+\mathbf{1} = (1,1,1,1,1,3,1,3)$ is not a $\mathcal{P}$-position since from it there is a move to $(1,1,1,1,1,3,0,1)$ which may be shown to be a $\mathcal{P}$-position.

We require the following definition from [2].

Definition 26. Let Δ be a simplicial complex on the vertex set V. The position $\mathbf{n}$ on Δ is called *genuine* if $n_v > 0$ each $v \in V$.

In [2], the converse of Conjecture 25 is shown to be false. However, the authors ask whether the converse holds for genuine $\mathcal{P}$-positions.

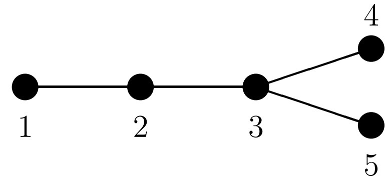
Question 27. If $\mathbf{n}$ is a $\mathcal{P}$-position on the cycle complex $C_{n,k}$ and $\mathbf{m}$ is a genuine position with $\mathbf{n} = \mathbf{m} + \mathbf{1}$, is then $\mathbf{m}$ a $\mathcal{P}$-position?

The answer to this question is also no. Once again, in the complex $C_{8,2}$, $\mathbf{n} = (2,3,2,2,3,3,3,3)$ is a $\mathcal{P}$-position. However, $\mathbf{m} = \mathbf{n} - \mathbf{1} = (1,2,1,1,2,2,2,2)$ is not a $\mathcal{P}$-position since there is a move from $\mathbf{m}$ to $(1,2,1,1,2,2,0,2)$ which may be verified to be a $\mathcal{P}$-position.

The next question is Question 9.2 in [2].

Question 28. Assume that $\mathbf{1} = (1,1,\ldots,1)$ is a $\mathcal{P}$-position of the complex Δ. Is then the position $n \cdot \mathbf{1} = (n,n,\ldots,n)$ also a $\mathcal{P}$-position of Δ?

The answer is no and the complex on the graph pictured below provides a counterexample.



The table below lists all the $\mathcal{P}$-positions in which all heaps sizes are at most 3. (For the sake of brevity, positions of the form $n \cdot \mathbf{e}(C)$ where C is a circuit have been omitted from the table.) Note that $\mathbf{1}$ and $2 \cdot \mathbf{1}$ are both $\mathcal{P}$-positions but $3 \cdot \mathbf{1}$ is not since from $3 \cdot \mathbf{1}$ there is a move to the $\mathcal{P}$-position $(3,2,1,3,3)$.

(0, 1, 0, 2, 3)	(1, 2, 0, 1, 2)	(2, 3, 2, 3, 3)
(0, 2, 0, 1, 3)	(1, 3, 1, 3, 3)	(3, 0, 0, 1, 2)
(0, 3, 0, 1, 2)	(2, 0, 0, 1, 3)	(3, 0, 1, 0, 2)
(1, 0, 0, 2, 3)	(2, 0, 1, 0, 1)	(3, 0, 1, 1, 1)
(1, 0, 1, 2, 2)	(2, 1, 0, 0, 3)	(3, 0, 2, 0, 1)
(1, 1, 0, 0, 2)	(2, 1, 0, 1, 2)	(3, 1, 1, 1, 3)
(1, 1, 0, 1, 3)	(2, 1, 2, 1, 1)	(3, 1, 2, 2, 2)
(1, 1, 1, 1, 1)	(2, 2, 1, 2, 3)	(3, 2, 1, 3, 3)
(1, 2, 0, 0, 3)	(2, 2, 2, 2, 2)	(3, 2, 2, 2, 3)

Finally in this section, we consider the following question which is Question 9.1 in [2].

Question 29. Assume that $\mathbf{n}$ is a $\mathcal{P}$-position of the complex Δ. Is then the position $2 \cdot \mathbf{n}$ also a $\mathcal{P}$-position of Δ? Is the converse true?

The answer to this question is no and the graph complex considered above provides a counterexample. The table lists $\mathbf{n} = (2,1,2,1,1)$ as a $\mathcal{P}$-position but it may be shown that $2 \cdot \mathbf{n} = (4,2,4,2,2)$ is not since there is a move from $2 \cdot \mathbf{n}$ to $(4,1,3,2,2)$ which is a $\mathcal{P}$-position.

We have not been able to settle the converse question. Notice, however, if Question 29 has an affirmative answer for the complex Δ then the converse also holds for Δ. For, if the converse is false, then there is a position $\mathbf{x}$ such that $2 \cdot \mathbf{x}$ is a $\mathcal{P}$-position yet $\mathbf{x}$ is not. In this case then, there is a move from $\mathbf{x}$ to a $\mathcal{P}$-position $\mathbf{y}$ and so, using the same face, one can move from the $\mathcal{P}$-position $2 \cdot \mathbf{x}$ to $2 \cdot \mathbf{y}$. Thus $2 \cdot \mathbf{y}$ cannot also be a $\mathcal{P}$-position so the original statement cannot hold.

Acknowledgements

The author wishes to thank the anonymous referee for many helpful suggestions.

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