Yet Another Hat Game

Maura B. Paterson

Department of Economics, Mathematics and Statistics
Birkbeck, University of London
Malet Street, London WC1E 7HX, UK

m.paterson@bbk.ac.uk

Douglas R. Stinson[*]

David R. Cheriton School of Computer Science
University of Waterloo
Waterloo Ontario, N2L 3G1, Canada

dstinson@uwaterloo.ca

Abstract

Several different "hat games" have recently received a fair amount of attention. Typically, in a hat game, one or more players are required to correctly guess their hat colour when given some information about other players' hat colours. Some versions of these games have been motivated by research in complexity theory and have ties to well-known research problems in coding theory, and some variations have led to interesting new research.

In this paper, we review **Ebert's Hat Game**, which garnered a considerable amount of publicity in the late 90's and early 00's, and the **Hats-on-a-line Game**. Then we introduce a new hat game which is a "hybrid" of these two games and provide an optimal strategy for playing the new game. The optimal strategy is quite simple, but the proof involves an interesting combinatorial argument.

1   Introduction

In this introduction, we review two popular hat games and mention some related work. In Section 2, we introduce our new game and give a complete solution for it. In Section 3, we make some brief comments.

Table 1: Analysis of Ebert's hat game for three players

configuration			guesses			outcome
brown	brown	brown	gray	gray	gray	lose
brown	brown	gray			**gray**	win
brown	gray	brown		**gray**		win
brown	gray	gray	**brown**			win
gray	brown	brown	**gray**			win
gray	brown	gray		**brown**		win
gray	gray	brown			**brown**	win
gray	gray	gray	brown	brown	brown	lose

1.1   Ebert's Hat Game

The following hat game was posed in a 1998 computer science PhD thesis by Todd Ebert
[6] (also see [7]). This game garnered a considerable amount of publicity in the late 90's
and early 00's and was written up in the *New York Times* [10]. There are three players in
the game: Alice, Bob, and Charlie. The three players enter a room and a gray or brown
hat is placed on each person's head. The colour of each hat is determined by a coin toss,
with the outcome of one coin toss having no effect on the others.

Each person can see the other players' hats but not his or her own hat. No communi-
cation of any sort is allowed, except for an initial strategy session before the game begins.
Once they have had a chance to look at the other hats, the players must simultaneously
guess the colour of their own hats, or pass. So each player's response is one of "gray",
"brown" or "pass". The group shares a hypothetical $1,000,000 prize if at least one player
guesses correctly and no players guess incorrectly.

It is not hard to devise a strategy that will win 50% of the time. For example, Alice
could guess "gray" while Bob and Charlie pass. Is it possible to do better? Clearly, any
guess has only a 50% chance of being correct. If more than one player guesses, then the
probabilities are reduced: the probability that two guesses are correct is 25%, and the
probability that three guesses are correct is 12.5%. Hence, it seems at first glance that it
is impossible to win more than 50% of the time.

However, suppose each player uses the following rule: If he observes two hats of the
same colour (i.e., gray – gray or brown – brown), then he guesses the opposite colour.
Otherwise, when two hats of different colours are observed, he passes. To analyse the
probability of winning when using this strategy, we consider all possible distributions of
hats. There are $2 \times 2 \times 2 = 8$ cases to consider. In each case, we can figure out if
the players win or lose. The probability of winning is equal to the number of winning
configurations divided by eight. In the following Table 1, we provide an analysis of all
eight cases. Boldface type is used to indicate correct votes.

The group wins in six out of eight cases, so their probability of winning is $6/8 =
3/4 = 75\%$. Observe that each individual guess is correct with a 50% probability. Among

the eight cases, there are six correct guesses and six incorrect guesses. The six correct guesses occurred in six different cases, while the six incorrect guesses were squeezed into two cases. This is why the probability of winning is much higher than 50%, even though each guess has only a 50% chance of being correct!

Here is another way to describe the optimal 3-player strategy:

- specify brown-brown-brown and gray-gray-gray as *bad configurations*.

- If a player's hat colour could result in a bad configuration, then that player guesses the opposite colour.

- If a player's hat colour could not result in a bad configuration, then that player passes.

Strategies for more players are based on this idea of specifying certain appropriately chosen bad configurations and then using a similar strategy as in the 3-player game. The bad configurations are obtained using *Hamming codes*, which are perfect single error correcting codes. For every integer $m \geqslant 2$, there is a Hamming code of length $n = 2^m - 1$ containing $2^{2^m - m - 1} = 2^{n-m}$ codewords.

In a Hamming code, every non-codeword can be changed into exactly one codeword by changing one entry. (This property allows the Hamming code to correct any single error that occurs during transmission.) If the configuration of hats is not a codeword, then there is a unique position i such that changing entry i creates a codeword. Player i will therefore guess correctly and every other player will pass. If the configuration of hats is a codeword, then everyone will guess incorrectly. Thus the group wins if and only if the configuration of hats is not a codeword.

Since there are 2^{n-m} codewords and 2^n configurations in total, the success probability is $1 - 2^{-m} = 1 - 1/(n+1)$. It can be proven fairly easily that this success probability is optimal, and can be attained only when a perfect 1-error correcting code exists. More generally, any strategy for this hat game on an arbitrary number n of players is "equivalent" to a *covering code* of length n, and thus optimal strategies (for any number of players) are known if and only if optimal covering codes are known (see [9] for additional information).

1.2   Hats-on-a-line

Another popular hat game has n players standing in a line. Hats of two colours (gray and brown) are distributed randomly to each player. Each player P_i ($1 \leqslant i \leqslant n$) can only see the hats worn by players $P_{i+1}, \ldots, P_n$ (i.e., the players "ahead of" P_i in the line). Each player is required to guess their hat colour, and they guess in the order $P_1, \ldots, P_n$. The objective is to maximise the number of correct guesses [3, 2].

Clearly the first player's guess will be correct with probability 50%, no matter what her strategy is. However, a simple strategy can be devised in which players $P_2, \ldots, P_n$ always guess correctly by making use of information gleaned from prior guesses.

As before, suppose that 0 corresponds to gray and 1 corresponds to brown. Let c_i denote the colour of player P_i's hat, $1 \leqslant i \leqslant n$. Here is the strategy:

- P_1 knows the values $c_2, \ldots, c_n$ (she can see the hats belonging to $P_2, \ldots, P_n$). P_1 provides as her guess the value

$$g_1 = \sum_{i=2}^{n} c_i \bmod 2.$$

- P_2 hears the value g_1 provided by P_1 and P_2 knows the values $c_3, \ldots, c_n$. Therefore P_2 can compute

$$c_2 = g_1 - \sum_{i=3}^{n} c_i \bmod 2.$$

P_2's guess is c_2, which is correct.

- For any player P_j with $j \geqslant 2$, P_j hears the values $g_1, c_2, \ldots, c_{j-1}$ provided by $P_1, \ldots, P_{j-1}$ respectively, and P_j knows the values $c_{j+1}, \ldots, c_n$. Therefore P_j can compute

$$c_j = g_1 - \sum_{i \in \{2,\ldots,n\} \setminus \{j\}} c_i \bmod 2.$$

P_j's guess is c_j, which is correct.

It is not hard to see that the same strategy can be applied for an arbitrary number of colours, q, where $q > 1$. The colours are named $0, \ldots, q-1$ and all computations are performed modulo q. If this is done, then P_1 has probability $1/q$ of guessing correctly, and the remaining $n - 1$ players will always guess correctly. Clearly this is optimal.

1.3   Related Work

A few years prior to the introduction of Ebert's Hat Game, in 1994, a similar game was described by Aspnes, Beigel, Furst and Rudich [1]. In their version of the game, players are not allowed to pass, and the objective is for a majority of the players to guess correctly. For the three-player game, it is easy to describe a strategy that will succeed with probability 3/4, just as in Ebert's game:

- Alice votes the opposite of Bob's hat colour;

- Bob votes the opposite of Charlie's hat colour; and

- Charlie votes the opposite of Alice's hat colour.

This game is analysed in Table 2, where the outcomes for all the possible configurations are listed.

It is also possible to devise a strategy for the majority hats game that uses Hamming codes. We basically follow the presentation from [4]. The idea, which is due to Berlekamp,

Table 2: Analysis of the majority hat game for three players

configuration			guesses			outcome
brown	brown	brown	gray	gray	gray	lose
brown	brown	gray	gray	**brown**	**gray**	win
brown	gray	brown	**brown**	**gray**	gray	win
brown	gray	gray	**brown**	brown	**gray**	win
gray	brown	brown	**gray**	gray	**brown**	win
gray	brown	gray	**gray**	**brown**	brown	win
gray	gray	brown	brown	**gray**	**brown**	win
gray	gray	gray	brown	brown	brown	lose

is to associate a strategy for n players with an orientation of the edges of the n-dimensional cube $\{0,1\}^n$. Each player's view corresponds in a natural way to an edge of the cube, and that player's guess will be determined by the head of the edge, as specified by the orientation.

If n is a power of 2 minus 1, then there is Hamming code of length n. Direct all the edges of the cube incident with a codeword away from the codeword. The remaining edges form an eulerian graph on the vertices that are not codewords; these edges can be directed according to any eulerian circuit.

The number of correct guesses for a given configuration is equal to the indegree of the corresponding vertex. From this observation, it is not difficult to see that any codeword is a losing configuration for this strategy — in fact, every guess will be incorrect. If the configuration of hats is not a codeword, then there will be $(n+1)/2$ correct guesses and $(n-1)/2$ incorrect guesses. So the success probability is $1 - 1/(n+1)$, as in the Ebert hat game, and this can again be shown to be optimal.

Many other variations of the hat game have been proposed. We complete this section by briefly mentioning some of them.

- Hats could be distributed according to a non-uniform probability distribution ([8]).

- Usually, it is stipulated that each player gets a single guess as to his or her hat colour; however, allowing players to have multiple guesses has also been considered ([1]).

- When sequential responses are used, it may be the case that players can hear all the previous responses (we call this *complete auditory information*), or only some of them, as in [2].

- Some games seek to guarantee that a certain minimum number of correct guesses are made, regardless of the configuration of hats, e.g., in an adversarial setting ([1, 11]).

- In [5], a directed graph, termed a "sight graph", is used to specify the hats that each player can see. Note that the visual information in Ebert's game corresponds to a sight graph that is a complete directed graph, while the Hats-on-a-line Game corresponds to the transitive closure of a directed path.

In general, players' strategies can be deterministic or nondeterministic (randomized). In the situation where hat distribution is done randomly, it suffices to consider only deterministic strategies. However, in an adversarial setting, an optimal strategy may require randomization.

2 A New Hats-on-a-line Game

When the second author gave a talk to high school students about Ebert's Hat Game, one student asked about sequential voting. It is attractive to consider sequential voting especially in the context of the Hats-on-a-line Game, but in that game the objective is different than in Ebert's game. A natural "hybrid" game would allow sequential voting, but retain the same objective as in Ebert's game. So we consider the following new hats-on-a-line game specified as follows:

- hats of $q > 1$ colours are distributed randomly;

- visual information is restricted to the hats-on-a-line scenario;

- sequential voting occurs in the order $P_1, \ldots, P_n$ with abstentions allowed; and

- the objective is that at least one player guesses correctly and no player guesses incorrectly.

We'll call this game the **New Hats-on-a-line Game**.

First, we observe that it is sufficient to consider strategies where exactly one player makes a guess. If the first player to guess is incorrect, then any subsequent guesses are irrelevant because the players have already lost the game. On the other hand, if the first player to guess is correct, then the players will win if all the later players pass.

We consider the simple strategy presented in Table 3, which we term the **Gray Strategy**. The Gray Strategy can be applied for any number of colours (assuming that gray is one of the colours, of course!).

It is easy to analyse the success probability of the Gray Strategy:

Theorem 2.1. *The success probability of the Gray Strategy for the New Hats-on-a-line Game with q hat colours and n players is $1 - ((q-1)/q)^n$.*

Proof. The probability that P_1 sees no gray hat is $((q-1)/q)^{n-1}$. In this case, her guess of "gray" is correct with probability $1/q$. If P_1 passes, then there is at least one gray hat among the remaining $n-1$ players. Let $j = \max\{i : P_i \text{ has a gray hat}\}$. Then players

> Assume that gray is one of the hat colours. For each player P_i ($1 \leqslant i \leqslant n$), when it is player P_i's turn, if he can see at least one gray hat, he passes; otherwise, he guesses "gray".

$P_1, \ldots, P_{j-1}$ will pass and player P_j will correctly guess "gray". So the group wins if player P_1 passes. Overall, the probability of winning is

$$\frac{1}{q} \times \left(\frac{q-1}{q}\right)^{n-1} + 1 \times \left(1 - \left(\frac{q-1}{q}\right)^{n-1}\right) = 1 - \left(\frac{q-1}{q}\right)^n .$$

$\square$

The main purpose of this section is to show that the Gray Strategy is an optimal strategy. (By the term "optimal", we mean that the strategy has the maximum possible probability of success, where the maximum is computed over all possible strategies allowed by the game.) We'll do two simple special cases before proceeding to the general proof. (The proof of the general case is independent of these two proofs, but the proofs of the special cases are still of interest due to their simplicity.)

We first show that the Gray Strategy is optimal if $q = 2$. In this proof and all other proofs in this section, we can restrict our attention without loss of generality to deterministic strategies.

Theorem 2.2. *The maximum success probability for any strategy for the New Hats-on-a-line Game with two hat colours and n players is $1 - 2^{-n}$.*

Proof. The proof is by induction on n. For $n = 1$, the result is trivial, as any guess by P_1 is correct with probability $1/2$. So we can assume $n > 1$.

Suppose there are c configurations of $n - 1$ hats for which player P_1 guesses a colour. We consider two cases:

case 1: c $\geqslant$ 1

> There are c cases where P_1's guess is correct with probability $1/2$. Therefore the probability of an incorrect guess by P_1 is
>
> $$\frac{1}{2} \times \frac{c}{2^{n-1}} \geqslant \frac{1}{2^n}.$$

case 2: c $= 0$

> Since player P_1 always passes, the game reduces to an $(n-1)$-player game, in which the probability of winning is at most $1 - 2^{-n+1}$, by induction.

Considering both cases, we see that the probability of winning is at most $\max\{1-2^{-n}, 1-2^{-n+1}\} = 1 - 2^{-n}$. $\qquad\square$

We observe that the above proof holds even when every player has complete visual information, as the restricted visual information in the hats-on-a-line model is not used in the proof.

We next prove optimality for the two-player game for an arbitrary number of hat colours, as follows.

Theorem 2.3. *The maximum success probability for any strategy for the New Hats-on-a-line Game with q hat colours and two players is*

$$1 - \left(\frac{q-1}{q}\right)^2 = \frac{2q-1}{q^2}.$$

Proof. Suppose that player P_1 guesses her hat colour for r out of the q possible colours for P_2's hat that she might see. Any guess she makes is correct with probability $1/q$.

We distinguish two cases:

case 1: r = q

If $r = q$, then the overall success probability is $1/q$.

case 2: r < q

In this case, player P_1 passes with probability $(q-r)/q$. Given that P_1 passes, P_2 knows that his hat is one of $q-r$ equally possible colours, so his guess will be correct with probability $1/(q-r)$. Therefore the overall success probability is

$$\frac{1}{q} \times \frac{r}{q} + \frac{1}{q-r} \times \frac{q-r}{q} = \frac{r}{q^2} + \frac{1}{q}.$$

To maximise this quantity, we take $r = q - 1$. This yields a success probability of $(2q-1)/q^2$.

Case 2 yields the optimal strategy because $(2q-1)/q^2 > 1/q$ when $q > 1$. $\qquad\square$

2.1 The Main Theorem

Based on the partial results proven above, it is tempting to conjecture that the maximum success strategy is $1 - ((q-1)/q)^n$, for any integers $n > 1$ and $q > 1$. In fact, we will prove that this is always the case.

The proof is done in two steps. A strategy is defined to be *restricted* if any guess made by any player other than the first player is always correct (furthermore, as already stated, it is not permitted for all players to pass). First, we show that any optimal strategy must be a restricted strategy. Then we prove optimality of the Gray Strategy by considering only restricted strategies.

In all of our proofs, we denote the colour of P_i's hat by c_i, $1 \leqslant i \leqslant n$. The n-tuple $(c_1, \ldots, c_n)$ is the *configuration* of hats.

Lemma 2.4. *Any optimal strategy for the New Hats-on-a-line Game is a restricted strategy.*

Proof. Suppose $\mathcal{S}$ is an optimal strategy for the New Hats-on-a-line Game that is not restricted. If player P_1 passes, then the outcome of the game is determined by the $(n-1)$-tuple $(c_2, \ldots, c_n)$, which is known to P_1. Since P_1 knows the strategies of all the players, she can determine exactly which $(n-1)$-tuples will lead to incorrect guesses by a later player. Denote this set of $(n-1)$-tuples by F. Because $\mathcal{S}$ is not restricted, it follows that $F \neq \emptyset$.

We create a new strategy $\mathcal{S}'$ by modifying $\mathcal{S}$ as follows:

1. If $(c_2, \ldots, c_n) \in F$, then P_1 guesses an arbitrary colour (e.g., P_1 could guess "gray").

2. If $(c_2, \ldots, c_n) \notin F$, then proceed as in $\mathcal{S}$.

It is easy to see that $\mathcal{S}'$ is a restricted strategy. The strategies $\mathcal{S}$ and $\mathcal{S}'$ differ only in what happens for configurations $(c_1, \ldots, c_n)$ where $(c_2, \ldots, c_n) \in F$. When $(c_2, \ldots, c_n) \in F$, $\mathcal{S}'$ will guess correctly with probability $1/q$. On the other hand, $\mathcal{S}$ always results in an incorrect guess when $(c_2, \ldots, c_n) \in F$. Because $|F| \geqslant 1$, the success probability of $\mathcal{S}'$ is greater than the success probability of $\mathcal{S}$. This contradicts the optimality of $\mathcal{S}$ and the desired result follows. $\qquad\square$

Now we proceed to the second part of the proof.

Lemma 2.5. *The maximum success probability for any restricted strategy for the New Hats-on-a-line Game with q hat colours and n players is $1 - ((q-1)/q)^n$.*

Proof. Suppose an optimal restricted strategy $\mathcal{S}$ is being used. Let A denote the set of $(n-1)$-tuples $(c_2, \ldots, c_n)$ for which P_1 guesses; let B denote the set of $(n-1)$-tuples for which P_1 passes and P_2 guesses (correctly); and let C denote the set of $(n-1)$-tuples for which P_1 and P_2 both pass. Clearly every $(n-1)$-tuple is in exactly one of A, B, or C, so

$$|A| + |B| + |C| = q^{n-1}. \tag{1}$$

Now construct A' (B', C', resp.) from A (B, C, resp.) by deleting the first co-ordinate (i.e., the value c_2) from each $(n-1)$-tuple. A', B' and C' are treated as multisets. We make some simple observations:

(i) $B' \cap C' = \emptyset$. This is because P_2's strategy is determined by the $(n-2)$-tuple $(c_3, \ldots, c_n)$.

(ii) For each $(c_3, \ldots, c_n) \in B'$, there are precisely $q-1$ occurrences of $(c_3, \ldots, c_n) \in A'$. This follows because player P_2 can be guaranteed to guess correctly only when his hat colour is determined uniquely.

(iii) $A' \cap C' = \emptyset$. This follows from the optimality of the strategy $\mathcal{S}$. (The existence of an $(n-1)$-tuple $(c_2, \ldots, c_n) \in A$ such that $(c_3, \ldots, c_n) \in C'$ contradicts the optimality of $\mathcal{S}$: P_1 should pass, for this configuration will eventually lead to a correct guess by a later player.)

We now define a restricted strategy $\mathcal{S}'$ for the $(n-1)$-player game with players $P_2, \ldots, P_n$ (here P_2 is the "first" player). The strategy is obtained by modifying $\mathcal{S}$, as follows:

1. P_2 guesses (arbitrarily) if $(c_3, \ldots, c_n) \in A' \cup B'$ and P_2 passes if $(c_3, \ldots, c_n) \in C'$. (This is well-defined in view of the three preceding observations.)

2. $P_3, \ldots, P_n$ proceed exactly as in strategy $\mathcal{S}$.

Since the set of $(n-2)$-tuples for which P_2 passes is the same in both of strategies $\mathcal{S}$ and $\mathcal{S}'$, it follows that $P_3, \ldots, P_n$ only make correct guesses in $\mathcal{S}'$, and therefore $\mathcal{S}'$ is restricted.

Let β_n denote the maximum number of $(n-1)$-tuples for which the first player passes in an optimal restricted strategy. We will prove that

$$\beta_n \leqslant q^{n-1} - (q-1)^{n-1}. \tag{2}$$

This is true for $n = 2$, since $\beta_2 \leqslant 1$.

Now we proceed by induction on n. We will use a few equations and inequalities. First, from (ii), it is clear that

$$|A| \geqslant (q-1)|B|. \tag{3}$$

Next, because $\mathcal{S}'$ is a restricted strategy for $n-1$ players, we have

$$|C| \leqslant q\beta_{n-1}. \tag{4}$$

Finally, from the optimality of $\mathcal{S}$, it must be the case that

$$|B| + |C| = \beta_n. \tag{5}$$

Applying (1), (3), (4) and (5), we have

$$\begin{aligned}
\beta_n &= |B| + |C| \\
&= q^{n-1} - |A| \\
&\leqslant q^{n-1} - (q-1)|B| \\
&= q^{n-1} - (q-1)(\beta_n - |C|) \\
&\leqslant q^{n-1} - (q-1)\beta_n + q(q-1)\beta_{n-1},
\end{aligned}$$

from which we obtain

$$\beta_n \leqslant q^{n-2} + (q-1)\beta_{n-1}.$$

Applying the induction assumption, we see that

$$\beta_n \leqslant q^{n-2} + (q-1)(q^{n-2} - (q-1)^{n-2}) = q^{n-1} - (q-1)^{n-1},$$

showing that (2) is true.

Finally, using (2), the success probability of $\mathcal{S}$ is computed to be

$$\Pr[P_1 \text{ passes}] + \frac{1}{q} \times \Pr[P_1 \text{ guesses}]$$

$$= \Pr[P_1 \text{ passes}] + \frac{1}{q} \times (1 - \Pr[P_1 \text{ passes}])$$

$$= \frac{1}{q} + \Pr[P_1 \text{ passes}] \times \left(1 - \frac{1}{q}\right)$$

$$\leqslant \frac{1}{q} + \frac{\beta_n}{q^{n-1}} \times \left(1 - \frac{1}{q}\right)$$

$$\leqslant \frac{1}{q} + \left(\frac{q^{n-1} - (q-1)^{n-1}}{q^{n-1}}\right) \times \left(1 - \frac{1}{q}\right)$$

$$= 1 - \left(\frac{q-1}{q}\right)^n .$$

$\square$

Summarizing, we have proven our main theorem.

Theorem 2.6. *The Gray Strategy for the New Hats-on-a-line Game with q hat colours and n players is optimal.*

Proof. This is an immediate consequence of Theorem 2.1 and Lemmas 2.4 and 2.5. $\square$

3   Comments

It is interesting to compare Ebert's Hat Game, the Hats-on-a-line Game and the New Hats-on-a-line Game. The optimal solutions to Ebert's game are easily shown to be equivalent to covering codes. There are many open problems concerning these combinatorial structures, so the optimal solution to Ebert's game is not known in general. The optimal solution to the Hats-on-a-line Game is a simple arithmetic strategy, and it is obvious that the strategy is optimal. We have introduced the New Hats-on-a-line Game as a hybrid of the two preceding games. The optimal strategy is very simple, but the proof of optimality is rather delicate combinatorial proof by induction. This game does not seem to have any connection to combinatorial structures such as covering codes. The analysis of these three games utilize different techniques. At the present time, there does not appear to be any kind of unified approach that is appropriate for understanding these games and/or other types of hat games.

References

[1] J. Aspnes, R. Beigel, M. Furst and S. Rudich, *The Expressive Power of Voting Polynomials*, Combinatorica **14** (1994), 135–148.

[2] Sarang Aravamuthan and Sachin Lodha, *Covering Codes for Hats-on-a-line*, The Electronic Journal of Combinatorics **13** (2006), #R21.

[3] Tom Bohman, Oleg Pikhurko, Alan Frieze and Danny Sleator, *Puzzle 15: Hat Problems*, The Puzzle Toad. `http://www.cs.cmu.edu/puzzle/puzzle15.html`

[4] J. P. Buhler, *Hat Tricks*, Mathematical Intelligencer, **24(4)** (2002), 44–49.

[5] S. Butler, M. T. Hajiaghayi, R. D. Kleinberg and T. Leighton, *Hat guessing games*, SIAM Review **51** (2009), 399–413.

[6] Todd Ebert, *Applications of Recursive Operators to Randomness and Complexity*, PhD Thesis, University of California Santa Barbara, 1998.

[7] T. Ebert and H. Vollmer, *On the Autoreducibility of Random Sequences*, Proc. 25th International Symposium on Mathematical Foundations of Computer Science, Lecture Notes in Computer Science, **1893** (2000), 333–342.

[8] W. Guo, S. Kasala, M. Bhaskara Rao and B. Tucker, *The Hat Problem and Some Variations*, in "Advances in Distribution Theory, Order Statistics, and Inference", Springer, 2006, pp. 459–479.

[9] Hendrik W. Lenstra and Gadiel Seroussi, *On Hats and Other Covers (extended summary)*. `http://arxiv.org/pdf/cs/0509045`

[10] Sara Robinson, *Why Mathematicians Now Care About Their Hat Color*, New York Times, April 10, 2001.
`http://www.nytimes.com/2001/04/10/science/10MATH.html`

[11] Peter Winkler, *Games People Don't Play*, in "Puzzlers' Tribute: A Feast for the Mind", A. K. Peters, 2002, pp. 301–313.