Double-critical graphs and complete minors

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Abstract

A connected k-chromatic graph G is double-critical if for all edges uv of G the graph $G - u - v$ is $(k-2)$-colourable. The only known double-critical k-chromatic graph is the complete k-graph K_k. The conjecture that there are no other double-critical graphs is a special case of a conjecture from 1966, due to Erdős and Lovász. The conjecture has been verified for k at most 5. We prove for $k = 6$ and $k = 7$ that any non-complete double-critical k-chromatic graph is 6-connected and contains a complete k-graph as a minor.

1 Introduction

A long-standing conjecture, due to Erdős and Lovász [5], states that the complete graphs are the only double-critical graphs. We refer to this conjecture as the *Double-Critical Graph Conjecture*. A more elaborate statement of the conjecture is given in Section 2, where several other fundamental concepts used in the present paper are defined. The Double-Critical Graph Conjecture is easily seen to be true for double-critical k-chromatic graphs with k at most 4. Mozhan [16] and Stiebitz [19, 20] independently proved the conjecture to hold for $k = 5$, but it still remains open for all integers k greater than 5. The Double-Critical Graph Conjecture is a special case of a more general conjecture, the so-called Erdős-Lovász Tihany Conjecture [5], which states that for any graph G with $\chi(G) > \omega(G)$ and any two integers $a, b \geqslant 2$ with $a + b = \chi(G) + 1$, there is a partition (A, B) of the vertex set $V(G)$ such that $\chi(G[A]) \geqslant a$ and $\chi(G[B]) \geqslant b$. The Erdős-Lovász Tihany Conjecture holds for every pair $(a, b) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$ (see [3, 16, 19, 20]). Kostochka and Stiebitz [13] proved it to be true for line graphs of multigraphs, while Balogh et al. [1] proved it to be true for quasi-line graphs and for graphs with independence number 2.

In addition, Stiebitz (private communication) has proved a weakening of the Erdős-Lovász Tihany conjecture, namely that for any graph G with $\chi(G) > \omega(G)$ and any two integers $a, b \geqslant 2$ with $a + b = \chi(G) + 1$, there are two disjoint subsets A and B of the vertex set $V(G)$ such that $\delta(G[A]) \geqslant a - 1$ and $\delta(G[B]) \geqslant b - 1$. (Note that for this conclusion to hold it is not enough to assume that $G \not\supseteq K_{a+b-1}$ and $\delta(G) \geqslant a+b-2$, that is, the Erdős-Lovász Tihany conjecture does not hold in general for the so-called colouring number. The 6-cycle with all shortest diagonals added is a counterexample with $a = 2$ and $b = 4$.) For $a = 2$, the truth of this weaker version of the Erdős-Lovász Tihany conjecture follows easily from Theorem 3.1 of the present paper.

Given the difficulty in settling the Double-Critical Graph Conjecture we pose the following weaker conjecture:

Conjecture 1.1. *Every double-critical k-chromatic graph is contractible to the complete k-graph.*

Conjecture 1.1 is a weaker version of Hadwiger's Conjecture [9], which states that every k-chromatic graph is contractible to the complete k-graph. Hadwiger's Conjecture is one of the most fundamental conjectures of Graph Theory, much effort has gone into settling it, but it remains open for $k \geqslant 7$. For more information on Hadwiger's Conjecture and related problems we refer the reader to [11, 22].

In this paper we mainly devote attention to the *double-critical* 7-chromatic graphs. It seems that relatively little is known about 7-chromatic graphs. Jakobsen [10] proved that every 7-chromatic graph has a K_7 with two edges missing as a minor. It is apparently not known whether every 7-chromatic graph is contractible to K_7 with one edge missing. Kawarabayashi and Toft [12] proved that every 7-chromatic graph is contractible to K_7 or $K_{4,4}$.

The main result of this paper is that any double-critical 6- or 7-chromatic graph is contractible to the complete graph on six or seven vertices, respectively. These results are proved in Sections 6 and 7 using results of Győri [8] and Mader [15], but not the Four Colour Theorem. Krusenstjerna-Hafstrøm and Toft [14] proved that any double-critical k-chromatic non-complete graph is 5-connected and $(k+1)$-edge-connected. In Section 5, we extend that result by proving that any double-critical k-chromatic non-complete graph is 6-connected. In Section 3, we exhibit a number of basic properties of double-critical non-complete graphs. In particular, we observe that the minimum degree of any double-critical non-complete k-chromatic graph G is at least $k + 1$ and that no two vertices of degree $k + 1$ are adjacent in G, cf. Proposition 3.9 and Theorem 3.1. Gallai [7] also used the concept of decomposable graphs in the study of critical graphs. In Section 4, we use double-critical decomposable graphs to study the maximum ratio between the number of double-critical edges in a non-complete critical graph and the size of the graph, in particular, we prove that, for every non-complete 4-critical graph G, this ratio is at most $1/2$ and the maximum is attained if and only if G is a wheel. Finally, in Section 8, we study two variations of the concept of double-criticalness, which we have termed double-edge-criticalness and mixed-double-criticalness. It turns out to be straightforward to show that the only double-edge-critical graphs and mixed-double-critical graphs are the

complete graphs.

2 Notation

All graphs considered in this paper are simple and finite. We let $n(G)$ and $m(G)$ denote the order and size of a graph G, respectively. The path, the cycle and the complete graph on n vertices is denoted P_n, C_n and K_n, respectively. The *length* of a path or a cycle is its number of edges. The set of integers $\{1, 2, \ldots, k\}$ will be denoted $[k]$. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. A *k-colouring* of a graph G is a function φ from the vertex set $V(G)$ of G into a set $\mathcal{C}$ of cardinality k so that $\varphi(u) \neq \varphi(v)$ for every edge $uv \in E(G)$, and a graph is *k-colourable* if it has a k-colouring. The elements of the set $\mathcal{C}$ are referred to as colours, and a vertex $v \in V(G)$ is said to be assigned the colour $\varphi(v)$ by φ. The set of vertices S assigned the same colour $c \in \mathcal{C}$ is said to constitute the colour class c. The minimum integer k for which a graph G is k-colourable is called its *chromatic number* of G and it is denoted $\chi(G)$. An *independent set* S of G is a set such that the induced graph $G[S]$ is edge-empty. The maximum integer k for which there exists an independent set S of G of cardinality k is the *independence number* of G and is denoted $\alpha(G)$. A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and/or vertices and contracting edges. An *H-minor* of G is a minor of G isomorphic to H. Given a graph G and a subset U of $V(G)$ such that the induced graph $G[U]$ is connected, the graph obtained from G by contracting U into one vertex is denoted G/U, and the vertex of G/U corresponding to the set U of G is denoted v_U. Let $\delta(G)$ denote the minimum degree of G. For a vertex v of a graph G, the (open) neighbourhood of v in G is denoted $N_G(v)$ while $N_G[v]$ denotes the closed neighbourhood $N_G(v) \cup \{v\}$. Given two subsets X and Y of $V(G)$, we denote by $E[X, Y]$ the set of edges of G with one end-vertex in X and the other end-vertex in Y, and by $e(X, Y)$ their number. If $X = Y$, then we simply write $E(X)$ and $e(X)$ for $E[X, X]$ and $e(X, X)$, respectively. The induced graph $G[N(v)]$ is refered to as the neighbourhood graph of v (w.r.t. G) and it is denoted G_v. The independence number $\alpha(G_v)$ is denoted α_v. The degree of a vertex v in G is denoted $\deg_G(v)$ or $\deg(v)$. A graph G is called *vertex-critical* or, simply, *critical* if $\chi(G - v) < \chi(G)$ for every vertex $v \in V(G)$. A connected graph G is called *double-critical* if

$$\chi(G - x - y) \leqslant \chi(G) - 2 \text{ for all edges } xy \in E(G) \tag{1}$$

Of course, $\chi(G - x - y)$ can never be strictly less than $\chi(G) - 2$, so we could require $\chi(G - x - y) = \chi(G) - 2$ in (1). It is also clear that any double-critical graph is vertex-critical. The concept of vertex-critical graphs was first introduced by Dirac [4] and have since been studied extensively, see, for instance, [11]. As noted by Dirac [4], every critical k-chromatic graph G has minimum degree $\delta(G) \geqslant k - 1$. An edge $xy \in E(G)$ such that $\chi(G - x - y) = \chi(G) - 2$ is referred to as a *double-critical edge*. For graph-theoretic terminology not explained in this paper, we refer the reader to [2].

3 Basic properties of double-critical graphs

In this section we let G denote a non-complete double-critical k-chromatic graph. Thus, by the aforementioned results, k is at least 6.

Proposition 3.1. *The graph G does not contain a complete $(k-1)$-graph as a subgraph.*

Proof. Suppose G contains K_{k-1} as a subgraph. Since G is k-chromatic and double-critical, it follows that $G - V(K_{k-1})$ is edge-empty, but not vertex-empty. Since G is also vertex-critical, $\delta(G) \geqslant k-1$, and therefore every $v \in V(G - K_{k-1})$ is adjacent to every vertex of $V(K_{k-1})$ in G, in particular, G contains K_k as a subgraph. Since G is vertex-critical, $G = K_k$, a contradiction. □

Proposition 3.2. *If H is a connected subgraph of G with $n(H) \geqslant 2$, then the graph $G/V(H)$ obtained from G by contracting H is $(k-1)$-colourable.*

Proof. The graph H contains at least one edge uv, and the graph $G-u-v$ is $(k-2)$-colourable, which, in particular, implies that the graph $G-H$ is $(k-2)$-colourable. Now, any $(k-2)$-colouring of $G-H$ may be extended to a $(k-1)$-colouring of $G/V(H)$ by assigning a new colour to the vertex $v_{V(H)}$. □

Given any edge $xy \in E(G)$, define

$$\begin{aligned} A(xy) &:= N(x)\backslash N[y] \\ B(xy) &:= N(x) \cap N(y) \\ C(xy) &:= N(y)\backslash N[x] \\ D(xy) &:= V(G)\backslash(N(x) \cup N(y)) \\ &= V(G)\backslash\left(A(xy) \cup B(xy) \cup C(xy) \cup \{x,y\}\right) \end{aligned}$$

We refer to $B(xy)$ as the *common neighbourhood* of x and y (in G).

In the proof of Proposition 3.3 we use what has become known as generalized Kempe chains, cf. [17, 21]. Given a k-colouring φ of a graph H, a vertex $x \in H$ and a permutation π of the colours $1, 2, \ldots, k$. Let N_1 denote the set of neighbours of x of colour $\pi(\varphi(x))$, let N_2 denote the set of neighbours of N_1 of colour $\pi(\pi(\varphi(x)))$, let N_3 denote the set of neighbours of N_2 of colour $\pi^3(\varphi(x))$, etc. We call $N(x,\varphi,\pi) = \{x\} \cup N_1 \cup N_2 \cup \cdots$ *a generalized Kempe chain from x w.r.t. φ and π*. Changing the colour $\varphi(y)$ for all vertices $y \in N(x,\varphi,\pi)$ from $\varphi(y)$ to $\pi(\varphi(y))$ gives a new k-colouring of H.

Proposition 3.3. *For all edges $xy \in E(G)$, $(k-2)$-colourings of $G-x-y$ and any non-empty sequence $j_1, j_2, \ldots, j_i$ of i different colours from $[k-2]$, there is a path of order $i+2$ starting at x, ending at y and with the t'th vertex after x having colour j_t for all $t \in [i]$. In particular, xy is contained in at least $(k-2)!/(k-2-i)!$ cycles of length $i+2$.*

Proof. Let xy denote an arbitrary edge of G and let φ denote a $(k-2)$-colouring of $G-x-y$ which uses the colours of $[k-2]$. The function φ is extended to a proper $(k-1)$-colouring of $G-xy$ by defining $\varphi(x) = \varphi(y) = k-1$. Let π denote the cyclic permutation

$(k-1, j_1, j_2, \ldots, j_i)$. If the generalized Kempe chain $N(x, \varphi, \pi)$ does not contain the vertex y, then by reassigning colours on the vertices of $N(x, \varphi, \pi)$ as described above, a $(k-1)$-colouring ψ of $G - xy$ with $\psi(x) \neq k-1 = \psi(y)$ is obtained, contradicting the fact that G is k-chromatic. Thus, the generalized Kempe chain $N(x, \varphi, \pi)$ must contain the vertex y. Since x and y are the only vertices which are assigned the colour $k-1$ by φ, it follows that the induced graph $G[N(x, \varphi, \pi)]$ contains an (x, y)-path of order $i+2$ with vertices coloured consecutively $k-1, j_1, j_2, \ldots, j_i, k-1$. The last claim of the proposition follows from the fact there are $(k-2)!/(k-2-i)!$ ways of selecting and ordering i elements from the set $[k-2]$. □

Note that the number of cycles of a given length obtained in Proposition 3.3 is exactly the number of such cycles in the complete k-graph. Moreover, Proposition 3.3 immediately implies the following result.

Corollary 3.1. *For all edges $xy \in E(G)$ and $(k-2)$-colourings of $G - x - y$, the set $B(xy)$ of common neighbours of x and y in G contains vertices from every colour class $i \in [k-2]$, in particular, $|B(xy)| \geqslant k-2$, and xy is contained in at least $k-2$ triangles.*

Proposition 3.4. *For all vertices $x \in V(G)$, the minimum degree in the induced graph of the neighbourhood of x in G is at least $k-2$, that is, $\delta(G_x) \geqslant k-2$.*

Proof. According to Corollary 3.1, $|B(xy)| \geqslant k-2$ for any vertex $y \in N(x)$, which implies that y has at least $k-2$ neighbours in G_x. □

Proposition 3.5. *For any vertex $x \in V(G)$, there exists a vertex $y \in N(x)$ such that the set $A(xy)$ is not empty.*

Proof. Let x denote any vertex of G, and let z in $N(x)$. The common neighbourhood $B(xz)$ contains at least $k-2$ vertices, and so, since K_{k-1} is not a subgraph of G, not every pair of vertices of $B(xy)$ are adjacent, say $y, y' \in B(xz)$ are non-adjacent. Now $y' \in A(xy)$, in particular, $A(xy)$ is not empty. □

Proposition 3.6. *There exists at least one edge $xy \in E(G)$ such that the set $D(xy)$ is not empty.*

Proof. According to Proposition 3.5, there exists at least one edge $uv \in E(G)$ such that $A(uv)$ is not empty. Fix a vertex $a \in A(uv)$. This vertex a cannot be adjacent to every vertex of $B(uv)$, since that, according to Corollary 3.1, would leave no colour available for a in a $(k-2)$-colouring of $G - u - v$. Suppose a is not adjacent to $z \in B(uv)$. Now $a \in D(vz)$, in particular, $D(vz)$ is not empty. □

Proposition 3.7. *If $A(xy)$ is not empty for some $xy \in E(G)$, then $\delta(G[A(xy)]) \geqslant 1$, that is, $G[A(xy)]$ contains no isolated vertices. By symmetry, $\delta(G[C(xy)]) \geqslant 1$, if $C(xy)$ is not empty.*

Proof. Suppose $G[A(xy)]$ contains some isolated vertex, say a. Now, since G is double-critical, $|B(xa)| \geqslant k-2$, and, since a is isolated in $A(xy)$, the common neighbours of x and a must lie in $B(xy)$, in particular, any $(k-2)$-colouring of $G-a-x$ must assign all colours of the set $[k-2]$ to common neighbours of a and x in $B(xy)$. But this leaves no colour in the set $[k-2]$ available for y, which contradicts the fact that $G-a-x$ is $(k-2)$-colourable. This contradiction implies that $G[A(xy)]$ contains no isolated vertices. □

Proposition 3.8. *If some vertex $y \in N(x)$ is not adjacent to some vertex $z \in N(x)\backslash\{y\}$, then there exists another vertex $w \in N(x)\backslash\{y,z\}$, which is also not adjacent to y. Equivalently, no vertex of the complement $\overline{G_x}$ has degree 1 in $\overline{G_x}$.*

Proof. The statement follows directly from Proposition 3.7. If $y \in N(x)$ is not adjacent to $z \in N(x)\backslash\{y\}$, then $z \in A(xy)$ and, since $G[A(xy)]$ contains no isolated vertices, the set $A(xy)\backslash\{z\}$ cannot be empty. □

Proposition 3.9. *Every vertex of G has at least $k+1$ neighbours.*

Proof. According to Proposition 3.5, for any vertex $x \in V(G)$, there exists a vertex $y \in N(x)$ such that $A(xy) \neq \emptyset$, and, according to Proposition 3.7, $\delta(G[A(xy)]) \geqslant 1$, in particular, $|A(xy)| \geqslant 2$. Since $N(x)$ is the union of the disjoint sets $A(xy)$, $B(xy)$ and $\{y\}$, we obtain

$$\deg_G(x) = |N(x)| \geqslant |A(xy)| + |B(xy)| + 1 \geqslant 2 + (k-2) + 1 = k+1$$

where we used the fact that $|B(xy)| \geqslant k-2$, according to Corollary 3.1. □

Proposition 3.10. *For any vertex $x \in V(G)$,*

$$\deg_G(x) - \alpha_x \geqslant |B(xy)| + 1 \geqslant k-1 \tag{2}$$

where $y \in N(x)$ is any vertex contained in an independent set in $N[x]$ of size α_x. Moreover, $\alpha_x \geqslant 2$.

Proof. Let S denote an independent set in $N(x)$ of size α_x. Obviously, $\alpha_x \geqslant 2$, otherwise G would contain a K_k. Choose some vertex $y \in S$. Now the non-empty set $S\backslash\{y\}$ is a subset of $A(xy)$, and, according to Proposition 3.7, $\delta(G[A(xy)]) \geqslant 1$. Let a_1 and a_2 denote two neighbouring vertices of $A(xy)$. The independet set S of G_x contains at most one of the vertices a_1 and a_2, say $a_1 \notin S$. Therefore S is a subset of $\{y\} \cup A(xy)\backslash\{a_1\}$, and so we obtain

$$\alpha_x \leqslant |A(xy)| = |N(x)| - |B(xy)| - 1 \leqslant \deg_G(x) - (k-2) - 1$$

from which (2) follows. □

Proposition 3.11. *For any vertex x not adjacent to all other vertices of G, $\chi(G_x) \leqslant k-3$.*

Proof. Since G is connected there must be some vertex, say z, in $V(G)\backslash N[x]$, which is adjacent to some vertex, say y, in $N(x)$. Now, clearly, z is a vertex of $C(xy)$, in particular, $C(xy)$ is not empty, which, according to Proposition 3.7, implies that $C(xy)$ contains at least one edge, say $e = zv$. Since G is double-critical, it follows that $\chi(G-z-v) \leqslant k-2$, in particular, the subgraph $G[N[x]]$ of $G - z - v$ is $(k-2)$-colourable, and so G_x is $(k-3)$-colourable. □

Proposition 3.12. *If $\deg_G(x) = k+1$, then the complement $\overline{G_x}$ consists of isolated vertices (possibly none) and cycles (at least one), where the length of the cycles are at least five.*

Proof. Given $\deg_G(x) = k+1$, suppose that some vertex $y \in G_x$ has three edges missing in G_x, say yz_1, yz_2, yz_3. Now $B(xy)$ is a subset of $N(x)\backslash\{y, z_1, z_2, z_3\}$. However, $|N(x)\backslash\{y, z_1, z_2, z_3\}| = (k+1) - 4$, which implies $|B(xy)| \leqslant k-3$, contrary to Corollary 3.1. Thus no vertex of G_x is missing more than two edges. According to Proposition 3.7, if a vertex of G_x is missing one edge, then it is missing at least two edges. Thus, it follows that $\overline{G_x}$ consists of isolated vertices and cycles. If $\overline{G_x}$ consists of only isolated vertices, then G_x would be a complete graph, and G would contain a complete $(k+1)$-graph, contrary to our assumptions. Thus, $\overline{G_x}$ contains at least one cycle C. Let s denote a vertex of C, and let r and t denote the two distinct vertices of $A(xs)$. Now $G-x-s$ is $(k-2)$-colourable and, according to Corollary 3.1, each of the $k-2$ colours is assigned to at least one vertex of the common neighbourhood $B(xs)$. Thus, both r and t must have at least one non-neighbour in $B(xs)$, and, since r and t are adjacent, it follows that r and t must have distinct non-neighbours, say q and u, in $B(xs)$. Now, q, r, s, t and u induce a path of length four in $\overline{G_x}$ and so the cycle C containing P has length at least five. □

Theorem 3.1. *No two vertices of degree $k+1$ are adjacent in G.*

Proof. Firstly, suppose x and y are two adjacent vertices of degree $k+1$ in G. Suppose that the one of the sets $A(xy)$ and $C(xy)$ is empty, say $A(xy) = \emptyset$. Then $|B(xy)| = k$ and $C(xy) = \emptyset$. Obviously, $\alpha_x \geqslant 2$, and it follows from Proposition 3.10 that α_x is equal to two. Let φ denote a $(k-2)$-colouring of $G - x - y$. Now $|B(xy)| = k$, $\alpha_x = 2$ and the fact that φ applies each colour $c \in [k-2]$ to at least one vertex of $B(xy)$ implies that exactly two colours $i, j \in [k-2]$ are applied twice among the vertices of $B(xy)$, say $\varphi(u_1) = \varphi(u_2) = k-3$ and $\varphi(v_1) = \varphi(v_2) = k-2$, where u_1, u_2, v_1 and v_2 denotes four distinct vertices of $B(xy)$. Now each of the colours $1, \ldots, k-4$ appears exactly once in the colouring of the vertices of $W := B(xy)\backslash\{u_1, u_2, v_1, v_2\}$, say $W = \{w_1, \ldots, w_{k-4}\}$ and $\varphi(w_i) = i$ for each $i \in [k-4]$. Now it follows from Proposition 3.3 that there exists a path xw_iw_jy for each pair of distinct colours $i, j \in [k-4]$. Therefore $G[W] = K_{k-4}$. If one of the vertices u_1, u_2, v_1 or v_2, say u_1, is adjacent to every vertex of W, then $G[W \cup \{u_1, x, y\}] = K_{k-1}$, which contradicts Proposition 3.1. Hence each of the vertices u_1, u_2, v_1 and v_2 is missing at least one neighbour in W. It follows from Proposition 3.12, that the complement $\overline{G[B(xy)]}$ consists of isolated vertices and cycles of length at least five. Now it is easy to see that $\overline{G[B(xy)]}$ contains exactly one cycle, and we may w.l.o.g. assume

that $u_1w_1v_1v_2w_2u_2$ are the vertices of that cycle. Now $G[\{u_1, v_1\} \cup W \backslash \{w_1\}] = K_{k-1}$, and we have again obtained a contradiction.

Secondly, suppose that one of the sets $A(xy)$ and $C(xy)$ is not empty, say $A(xy) \neq \emptyset$. Since, according to Corollary 3.1, the common neighbourhood $B(xy)$ contains at least $k-2$ vertices, it follows from Proposition 3.7 that $|A(xy)| = 2$ and so $|B(xy)| = k-2$, which implies $|C(xy)| = 2$. Suppose $A(xy) = \{a_1, a_2\}$, $C(xy) = \{c_1, c_2\}$, and let C_A denote the cycle of the complement $\overline{G_x}$ which contains the vertices a_1, y and a_2, say $C_A = a_1 y a_2 u_1 \dots u_i$, where $u_1, \dots, u_i \in B(xy)$ and $i \geqslant 2$. Similarly, let C_C denote the cycle of the complement $\overline{G_y}$ which contains the vertices c_1, x and c_2, say $C_A = c_1 x c_2 v_1 \dots v_j$, where $v_1, \dots, v_j \in B(xy)$ and $j \geqslant 2$. Since both $\overline{G_x}$ and $\overline{G_y}$ consists of only isolated vertices (possibly none) and cycles, it follows that we must have $(u_1, \dots, u_i) = (v_1, \dots, v_j)$ or $(u_1, \dots, u_i) = (v_j, \dots, v_j)$. We assume w.l.o.g. that the former holds.

Let φ denote some $(k-2)$-colouring of $G - x - y$ using the colours of $[k-2]$, and suppose w.l.o.g. $\phi(a_1) = k-2$ and $\varphi(a_2) = k-3$. Again, the structure of $\overline{G_x}$ and $\overline{G_y}$ implies $\varphi(u_1) = k-3$ and $\varphi(u_i) = k-2$, which also implies $\varphi(c_1) = k-2$ and $\varphi(c_2) = k-3$.

Let $U = B(xy) \backslash \{u_1, u_i\}$. Now U has size $k-4$ and precisely one vertex of U is assigned the colour i for each $i \in [k-4]$. Since no other vertices of $(N(x) \cup N(y)) \backslash U$ is assigned a colour from the set $[k-4]$, it follows from Proposition 3.3 that for each pair of distinct colours $s, t \in [k-4]$ there exists a path $xu^s u^t y$ where u^s and u^t are vertices of U assigned the colours s and t, respectively. This implies $G[U] = K_{k-4}$. No vertex of G_x has more than two edges missing in G_x and so, in particular, each of the adjacent vertices a_1 and a_2 are adjacent to every vertex of U. Now $G[U \cup \{a_1, a_2, x\}] = K_{k-1}$, which contradicts Proposition 3.1. Thus, no two vertices of degree $k+1$ are adjacent in G. □

4 Decomposable graphs and the ratio of double-critical edges in graphs

A graph G is called *decomposable* if it consists of two disjoint non-empty subgraphs G_1 and G_2 together with all edges joining a vertex of G_1 and a vertex of G_2.

Proposition 4.1. *Let G be a graph decomposable into G_1 and G_2. Then G is double-critical if and only if G_1 and G_2 are both double-critical.*

Proof. Let G be double-critical. Then $\chi(G) = \chi(G_1) + \chi(G_2)$. Moreover, for $xy \in E(G_1)$ we have

$$\chi(G) - 2 = \chi(G - x - y) = \chi(G_1 - x - y) + \chi(G_2)$$

which implies $\chi(G_1 - x - y) = \chi(G_1) - 2$. Hence G_1 is double-critical, and similarly G_2 is.

Conversely, assume that G_1 and G_2 are both double-critical. Then for $xy \in E(G_1)$ we have

$$\chi(G - x - y) = \chi(G_1 - x - y) + \chi(G_2) = \chi(G_1) - 2 + \chi(G_2) = \chi(G) - 2$$

For $xy \in E(G_2)$ we have similarly that $\chi(G - x - y) = \chi(G) - 2$. For $x \in V(G_1)$ and $y \in V(G_2)$ we have

$$\chi(G - x - y) = \chi(G_1 - x) + \chi(G_2 - y) = \chi(G_1) - 1 + \chi(G_2) - 1 = \chi(G) - 2$$

Hence G is double-critical. □

Gallai proved the theorem that a k-critical graph with at most $2k-2$ vertices is always decomposable [6]. It follows easily from Gallai's Theorem, Proposition 4.1 and the fact that no double-critical non-complete graph with $\chi \leqslant 5$ exist, that a double-critical 6-chromatic graph $G \neq K_6$ has at least 11 vertices. In fact, such a graph must have at least 12 vertices. Suppose $|V(G)| = 11$. Then G cannot be decomposable by Proposition 4.1; moreover, no vertex of a k-critical graph can have a vertex of degree $|V(G)| - 2$; hence $\Delta(G) = 8$ by Theorem 3.1, say $\deg(x) = 8$. Let y and z denote the two vertices of $G - N[x]$. The vertices y and z have to be adjacent. Hence $\chi(G - y - z) = 4$ and $\chi(G_x) = 3$, which implies $\chi(G) = 5$, a contradiction.

It also follows from Gallai's theorem and our results on double-critical 6- and 7-chromatic graphs that any double-critical 8-chromatic graph without K_8 as a minor, if it exists, must have at least 15 vertices.

In the second part of the proof of Proposition 4.1, to prove that an edge xy with $x \in V(G_1)$ and $y \in V(G_2)$ is double-critical in G, we only need that x is critical in G_1 and y is critical in G_2. Hence it is easy to find examples of critical graphs with many double-critical edges. Take for example two disjoint odd cycles of equal length $\geqslant 5$ and join them completely by edges. The result is a family of 6-critical graphs in which the proportion of double-critical edges is as high as we want, say more than 99.99 percent of all edges may be double-critical. In general, for any integer $k \geqslant 6$, let $H_{k,\ell}$ denote the graph constructed by taking the complete $(k-6)$-graph and two copies of an odd cycle C_ℓ with $\ell \geqslant 5$ and joining these three graphs completely. Then the non-complete graph $H_{k,\ell}$ is k-critical, and the ratio of double-critical edges to the size of $H_{k,\ell}$ can be made arbitrarily close to 1 by choosing the integer ℓ sufficiently large. These observations perhaps indicate the difficulty in proving the Double-Critical Graph Conjecture: it is not enough to use just a few double-critical edges in a proof of the conjecture.

Taking an odd cycle C_ℓ ($\ell \geqslant 5$)and the complete 2-graph and joining them completely, we obtain a non-complete 5-critical graph with at least 2/3 of all edges being double-critical. Maybe these graphs are best possible:

Conjecture 4.1. *If G denotes a 5-critical non-complete graph, then G contains at most $c := (2 + \frac{1}{3n(G)-5})\frac{m(G)}{3}$ double-critical edges. Moreover, G contains precisely c double-critical edges if and only if G is decomposable into two graphs G_1 and G_2, where G_1 is the complete 2-graph and G_2 is an odd cycle of length $\geqslant 5$.*

The conjecture, if true, would be an interesting extension of a theorem by Mozhan [16] and Stiebitz [20] which states that there is at least one non-double-critical edge. Computer tests using the list of vertex-critical graphs made available by Royle [18] indicate that Conjecture 4.1 holds for graphs of order less than 12. Moreover, the analogous statement

holds for 4-critical graphs, cf. Theorem 4.1 below. In the proof of Theorem 4.1 we apply the following lemma, which is of interest in its own right.

Lemma 4.1. *No non-complete 4-critical graph contains two non-incident double-critical edges.*

Proof of Lemma 4.1. Suppose G contains two non-incident double-critical edges xy and vw. Since $\chi(G - \{v, w, x, y\}) = 2$, each component of $G - \{v, w, x, y\}$ is a bipartite graph. Let A_i and B_i ($i \in [j]$) denote the partition sets of each bipartite component of $G - \{v, w, x, y\}$. (For each $i \in [j]$, at least one of the sets A_i and B_i are non-empty.) Since G is critical, it follows that no clique of G is a cut set of G [2, Th. 14.7], in particular, both $G - x - y$ and $G - v - w$ are connected graphs. Hence, in $G - v - w$, there is at least one edge between a vertex of $\{x, y\}$ and a vertex of $A_i \cup B_i$ for each $i \in [j]$. Similarly, for v and w in $G - x - y$. If, say x is adjacent to a vertex $a_1 \in A_i$, then y cannot be adjacent to a vertex $a_2 \in A_i$, since then there would be a an even length (a_1, a_2)-path P in the induced graph $G[A_i \cup B_i]$ and so the induced graph $G[V(P) \cup \{x, y\}]$ would contain an odd cycle, which contradicts the fact that the supergraph $G - v - w$ of $G[V(P) \cup \{x, y\}]$ is bipartite. Similarly, if x is adjacent to a vertex of A_i, then x cannot be adjacent to a vertex of B_i. Similar observations hold for v and w. Let $A := A_1 \cup \cdots \cup A_j$ and $B := B_1 \cup \cdots \cup B_j$. We may w.l.o.g. assume that the neighbours of x in $G - v - w - y$ are in the set A and the neighbours of y in $G - v - w - x$ are in B. In the following we distinguish between two cases.

(i) First, suppose that, in $G - x - y$, one of the vertices v and w is adjacent to only vertices of $A \cup \{v, w\}$, while the other is adjacent to only vertices of $B \cup \{v, w\}$. By symmetry, we may assume that v in $G - x - y$ is adjacent to only vertices of $A \cup \{w\}$, while w in $G - x - y$ is adjacent to only vertices of $B \cup \{v\}$. In this case we assign the colour 1 to the vertices of $A \cup \{w\}$, the colour 2 to the vertices of $B \cup \{v\}$.

Suppose that one of the edges xv or yw is not in G. By symmetry, it suffices to consider the case that xv is not in G. In this case we assign the colour 2 to the vertex x and the colour 3 to y. Since x is not adjacent to any vertices of $B_1 \cup \cdots \cup B_j$, we obtain a 3-colouring of G, which contradicts the assumption that G is 4-chromatic.

Thus, both of the edges xv and yw are present in G. Suppose that xw or yv are missing from G. Again, by symmetry, it suffices to consider the case where yv is missing from G. Now assign the colour 2 to the vertex x and the colour 3 to the vertex y and a new colour to the vertex v. Again, we have a 3-colouring of G, a contradiction. Thus each of the edges xw and yv are in G, and so the vertices x, y, v and w induce a complete 4-graph in G. However, no 4-critical graph $\neq K_4$ contains K_4 as a subgraph, and so we have a contradiction.

(ii) Suppose (i) is not the case. Then we may choose the notation such that there exist some integer $\ell \in \{2, \ldots, j\}$ such that for every integer $s \in \{1, \ldots, \ell\}$ the vertex v is not adjacent to a vertex of B_s and the vertex w is not adjacent to a vertex of A_s; and for every integer $t \in \{\ell, \ldots, j\}$ the vertex v is not adjacent to a vertex of A_t and the vertex w is not adjacent to a vertex of B_t.

Since $G \not\subseteq K_4$, we may by symmetry assume that $xv \notin E(G)$. Now colour the vertices v, x and all vertices of B_s $(s = 1, \ldots, \ell - 1)$ with colour 1; colour the vertex w, all vertices of A_s $(s = 1, \ldots, \ell - 1)$ and all vertices of B_t $(t = \ell, \ldots, j)$ with colour 2; and colour the vertex y and all the vertices of A_t $(t = \ell, \ldots, j)$ with colour 3. The result is a 3-colouring of G. This contradicts G being 4-chromatic. Hence G does not contain two non-incident double-critical edges.

□

Theorem 4.1. *If G denotes a 4-critical non-complete graph, then G contains at most $m(G)/2$ double-critical edges. Moreover, G contains precisely $m(G)/2$ double-critical edges if and only if G contains a vertex v of degree $n(G) - 1$ such that the graph $G - v$ is an odd cycle of length $\geqslant 5$.*

Proof. Let G denote a 4-critical non-complete graph. According to Lemma 4.1, G contains no two non-incident double-critical edges, that is, every two double-critical edges of G are incident. Then, either the double-critical edges of G all share a common end-vertex or they induce a triangle. In the later case G contains strictly less that $m(G)/2$ double-critical edges, since $n(G) \geqslant 5$ and $m(G) \geqslant 3n(G)/2 > 6$. In the former case, let v denote the common endvertex of the double-critical edges.

Now, the number of double-critical edges is at most $\deg(v)$, which is at most $n(G) - 1$. Since G is 4-critical, it follows that $G - v$ is connected and 3-chromatic. Hence $G - v$ is connected and contains an odd cycle, which implies $m(G - v) \geqslant n(G - v)$. Hence $m(G) = \deg(v) + m(G - v) \geqslant \deg(v) + n(G) - 1 \geqslant 2\deg(v)$, which implies the desired inequality. If the inequality is, in fact, an equality, then $\deg(v) = n(G) - 1$ and G is decomposable with $G - v$ an odd cycle of length $\geqslant 5$. The reverse implication is just a simple calculation. □

5 Connectivity of double-critical graphs

Proposition 5.1. *Suppose G is a non-complete double-critical k-chromatic graph with $k \geqslant 6$. Then no minimal separating set of G can be partitioned into two disjoint sets A and B such that the induced graphs $G[A]$ and $G[B]$ are edge-empty and complete, respectively.*

Proof. Suppose that some minimal separating set S of G can be partitioned into disjoint sets A and B such that $G[A]$ and $G[B]$ are edge-empty and complete, respectively. We may assume that A is non-empty. Let H_1 denote a component of $G - S$, and let $H_2 := G - (S \cup V(H_1))$. Since A is not empty, there is at least one vertex $x \in A$, and, by the minimality of the separating set S, this vertex x has neighbours in both $V(H_1)$ and $V(H_2)$, say x is adjacent to $y_1 \in V(H_1)$ and $y_2 \in V(H_2)$. Since G is double-critical, the graph $G - x - y_2$ is $(k-2)$-colourable, in particular, there exists a $(k-2)$-colouring φ_1 of the subgraph $G_1 := G[V(H_1) \cup B]$. Similarly, there exists a $(k-2)$-colouring φ_2 of $G_2 := G[V(H_2) \cup B]$. The two graphs have precisely the vertices of B in common,

and the vertices of B induce a complete graph in both G_1 and G_2. Thus, both φ_1 and φ_2 use exactly $|B|$ colours to colour the vertices of B, assigning each vertex a unique colour. By permuting the colours assigned by, say φ_2, to the vertices of B, we may assume $\varphi_1(b) = \varphi_2(b)$ for every vertex $b \in B$. Now φ_1 and φ_2 can be combined into a $(k-2)$-colouring φ of $G - A$. This colouring φ may be extended to a $(k-1)$-colouring of G by assigning every vertex of the independent set A the some new colour. This contradicts the fact that G is k-chromatic, and so no minimal separating set S as assumed can exist. □

Krusenstjerna-Hafstrøm and Toft [14] states that any double-critical k-chromatic non-complete graph is 5-connected and $(k+1)$-edge-connected. In the following we prove that any double-critical k-chromatic non-complete graph is 6-connected.

Theorem 5.1. *Every double-critical k-chromatic non-complete graph is 6-connected.*

Proof. Suppose G is a double-critical k-chromatic non-complete graph. Then, by the results mentioned in Section 1, k is at least 6. Recall, that any double-critical graph, by definition, is connected. Thus, since G is not complete, there exists some subset $U \subseteq V(G)$ such that $G - U$ is disconnected. Let S denote a minimal separating set of G. We show $|S| \geqslant 6$. If $|S| \leqslant 3$, then S can be partitioned into two disjoint subset A and B such that the induced graphs $G[A]$ and $G[B]$ are edge-empty and complete, respectively, and, thus, we have a contradiction by Proposition 5.1. Suppose $|S| \geqslant 4$, and let H_1 and H_2 denote disjoint non-empty subgraphs of $G - S$ such that $G - S = H_1 \cup H_2$.

If $|S| \leqslant 5$, then each vertex v of $V(H_1)$ has at most five neighbours in S and so v must have at least two neighbours in $V(H_1)$, since $\delta(G) \geqslant k+1 \geqslant 7$. In particular, there is at least one edge u_1u_2 in H_1, and so $G - u_1 - u_2$ is $(k-2)$-colourable. This implies that the subgraph $G_2 := G - H_1$ of $G - u_1 - u_2$ is $(k-2)$-colourable. Let φ_2 denote a $(k-2)$-colouring of G_2. A similar argument shows that $G_1 := G - H_2$ is $(k-2)$-colourable. Let φ_1 denote a $(k-2)$-colouring of G_1. If φ_1 or φ_2 applies just one colour to the vertices of S, then S is an independent set of G, which contradicts Proposition 5.1. Thus, we may assume that both φ_1 and φ_2 applies at least two colours to the vertices of S. Let $|\varphi_i(S)|$ denote the number of colours applied by φ_i $(i = 1, 2)$ to the vertices of S. By symmetry, we may assume $|\varphi_1(S)| \geqslant |\varphi_2(S)| \geqslant 2$.

Moreover, if $|\varphi_1(S)| = |\varphi_2(S)| = |S|$, then, clearly, the colours applied by say φ_1 may be permuted such that $\varphi_1(s) = \varphi_2(s)$ for every $s \in S$ and so φ_1 and φ_2 may be combined into a $(k-2)$-coloring of G, a contradiction. Thus, $|\varphi_1(S)| = |S|$ implies $|\varphi_2(S)| < |S|$.

In general, we redefine the $(k-2)$-colourings φ_1 and φ_2 into $(k-1)$-colourings of G_1 and G_2, respectively, such that, after a suitable permutation of the colours of say φ_1, $\varphi_1(s) = \varphi_2(s)$ for every vertex $s \in S$. Hereafter a proper $(k-1)$-colouring of G may be defined as $\varphi(v) = \varphi_1(v)$ for every $v \in V(G_1)$ and $\varphi(v) = \varphi_2(v)$ for every $v \in V(G) \backslash V(G_1)$, which contradicts the fact that G is k-chromatic. In the following cases we only state the appropriate redefinition of φ_1 and φ_2.

Suppose that $|S| = 4$, say $S = \{v_1, v_2, v_3, v_4\}$. We consider several cases depending on the values of $|\varphi_1(S)|$ and $|\varphi_2(S)|$. If $|\varphi_i(S)| = 2$ for some $i \in \{1, 2\}$, then φ_i must apply both colours twice on vertices of S (by Proposition 5.1).

(1) Suppose that $|\varphi_1(S)| = 4$.

(1.1) Suppose that $|\varphi_2(S)| = 3$. In this case φ_2 uses the same colour at two vertices of S, say $\varphi_2(v_1) = \varphi_2(v_2)$. We simply redefine φ_2 such that $\varphi_2(v_1) = k-1$. Now both φ_1 and φ_2 applies four distinct colours to the vertices of S and so they may be combined into a $(k-1)$-colouring of G, a contradiction.

(1.2) Suppose that $|\varphi_2(S)| = 2$, say $\varphi_2(v_1) = \varphi_2(v_2)$ and $\varphi_2(v_3) = \varphi_2(v_4)$. This implies $v_1v_2 \notin E(G)$, and so φ_1 may be redefined such that $\varphi_1(v_1) = \varphi_1(v_2) = k-1$. Moreover, φ_2 is redefined such that $\varphi_2(v_4) = k-1$.

(2) Suppose that $|\varphi_1(S)| = 3$, say $\varphi_1(v_1) = 1$, $\varphi_1(v_2) = 2$ and $\varphi_1(v_3) = \varphi_1(v_4) = 3$.

(2.1) Suppose that $|\varphi_2(S)| = 3$, say $\varphi_2(x) = \varphi_2(y)$ for two distinct vertices $x, y \in S$. Redefine φ_1 and φ_2 such that $\varphi_1(v_4) = k-1$ and $\varphi_2(x) = k-1$.

(2.2) Suppose that $|\varphi_2(S)| = 2$. If $\varphi_2(v_1) = \varphi_2(v_2)$ and $\varphi_2(v_3) = \varphi_2(v_4)$, then the desired $(k-1)$-colourings are obtained by redefining φ_2 such that $\varphi_2(v_2) = k-1$. If $\varphi_2(v_2) = \varphi_2(v_3)$ and $\varphi_2(v_4) = \varphi_2(v_1)$, then the desired $(k-1)$-colourings are obtained by redefining φ_2 such that $\varphi_2(v_3) = \varphi_2(v_4) = k-1$.

(3) Suppose that $|\varphi_1(S)| = 2$. This implies $|\varphi_2(S)| = 2$. We may, w.l.o.g., assume $\varphi_1(v_1) = \varphi_1(v_2)$ and $\varphi_1(v_3) = \varphi_1(v_4)$, in particular, $v_1v_2 \notin E(G)$. If $\varphi_2(v_1) = \varphi_2(v_2)$ and $\varphi_2(v_3) = \varphi_2(v_4)$, then, obviously, φ_1 and φ_2 may be combined into a $(k-2)$-colouring of G, a contradiction. Thus, we may assume that $\varphi_2(v_2) = \varphi_2(v_3)$ and $\varphi_2(v_4) = \varphi_2(v_1)$. In this case we redefine both φ_1 and φ_2 such that $\varphi_1(v_4) = k-1$, and, since $v_1v_2 \notin E(G)$, $\varphi_2(v_1) = \varphi_2(v_2) = k-1$.

This completes the case $|S| = 4$. Suppose $|S| = 5$, say $S = \{v_1, v_2, v_3, v_4, v_5\}$. According to Proposition 5.1, neither φ_1 nor φ_2 uses the same colour for more than three vertices. Suppose that one of the colourings φ_1 or φ_2, say φ_2, applies the same colour to three vertices of S, say $\varphi_2(v_3) = \varphi_2(v_4) = \varphi_2(v_5)$. Now $\{v_3, v_4, v_5\}$ is an independent set. If (i) $\varphi_1(v_1) = \varphi_1(v_2)$ and $\varphi_2(v_1) = \varphi_2(v_2)$ or (ii) $\varphi_1(v_1) \neq \varphi_1(v_2)$ and $\varphi_2(v_1) \neq \varphi_2(v_2)$, then we redefine φ_1 such that $\varphi_1(v_3) = \varphi_1(v_4) = \varphi_1(v_5) = k-1$, and so φ_1 and φ_2 may, after a suitable permutation of the colours of say φ_1, be combined into a $(k-1)$-colouring of G. Otherwise, if $\varphi_1(v_1) \neq \varphi_1(v_2)$ and $\varphi_2(v_1) = \varphi_2(v_2)$, then we redefine both φ_1 and φ_2 such that $\varphi_1(v_3) = \varphi_1(v_4) = \varphi_1(v_5) = k-1$ and $\varphi_2(v_2) = k-1$. If $\varphi_1(v_1) = \varphi_1(v_2)$ and $\varphi_2(v_1) \neq \varphi_2(v_2)$, then we redefine both φ_1 and φ_2 such that $\varphi_1(v_3) = \varphi_1(v_4) = \varphi_1(v_5) = k-1$ and $\varphi_2(v_1) = \varphi_2(v_2) = k-1$. In both cases φ_1 and φ_2 may be combined into a $(k-1)$-colouring of G Thus, we may assume that neither φ_1 nor φ_2 applies the same colour to three or more vertices of S, in particular, $|\varphi_i(S)| \geqslant 3$ for both $i \in \{1, 2\}$. Again, we may assume $|\varphi_1(S)| \geqslant |\varphi_2(S)|$.

(a) Suppose that $|\varphi_1(S)| = 5$.

(a.1) Suppose that $|\varphi_2(S)| = 4$ with say $\varphi_2(v_4) = \varphi_2(v_5)$. In this case $v_4v_5 \notin E(G)$ and so we redefine φ_1 such that $\varphi_1(v_4) = \varphi_1(v_5) = k-1$.

(a.2) Suppose that $|\varphi_2(S)| = 3$. Since φ_2 cannot assign the same colour to three or more vertices of S, we may assume $\varphi_2(v_2) = \varphi_2(v_3)$ and $\varphi_2(v_4) = \varphi_2(v_5)$. In this case $v_4v_5 \notin E(G)$, and so we redefine φ_1 and φ_2 such that $\varphi_1(v_4) = \varphi_1(v_5) = k-1$ and $\varphi_2(v_3) = k-1$.

(b) Suppose $|\varphi_1(S)| = 4$, say $\varphi_1(v_4) = \varphi_1(v_5)$.

(b.1) Suppose $|\varphi_2(S)| = 4$ with $\varphi_2(x) = \varphi_2(y)$ for two distinct vertices $x, y \in S$. In this case we redefine φ_1 and φ_2 such that $\varphi_1(v_5) = k-1$ and $\varphi_2(y) = k-1$.

(b.2) Suppose $|\varphi_2(S)| = 3$. In this case we distinguish between two subcases depending on the number of colours φ_2 applies to the vertices of the set $\{v_1, v_2, v_3\}$. As noted earlier, we must have $|\varphi_2(\{v_1, v_2, v_3\})| \geqslant 2$. If $|\varphi_2(\{v_1, v_2, v_3\})| = 3$, then we redefine φ_2 such that $\varphi_2(v_4) = \varphi_2(v_5) = k-1$. Otherwise, if $|\varphi_2(\{v_1, v_2, v_3\})| = 2$ with say $\varphi_2(v_2) = \varphi_2(v_3)$. Now $v_2v_3, v_4v_5 \notin E(G)$ and so we redefine φ_1 and φ_2 such that $\varphi_1(v_2) = \varphi_1(v_3) = k-1$ and $\varphi_2(v_4) = \varphi_2(v_5) = k-1$.

(c) Suppose that $|\varphi_1(S)| = 3$, say $\varphi_1(v_2) = \varphi_1(v_3)$ and $\varphi_1(v_4) = \varphi_1(v_5)$. In this case we must have $|\varphi_2(S)| = 3$. As noted earlier, φ_2 does not assign the same colour to three vertices of S, and so we may assume φ_2 applies the colours $1, 2$ and 3 to the vertices of S and that only one vertex of S is assigned the colour 1 while two pairs of vertices of given the colours 2 and 3, respectively. We distinguish between four subcases depending on which vertex of S is assigned the colour 1 by φ_2 and and the number of colours φ_2 applies to the vertices of the two sets $\{v_2, v_3\}$ and $\{v_4, v_5\}$. We may assume $|\varphi_2(\{v_2, v_3\})| \geqslant |\varphi_2(\{v_4, v_5\})|$.

(c.1) If $|\varphi_2(\{v_2, v_3\})| = |\varphi_2(\{v_4, v_5\})| = 1$, then, clearly, φ_1 and φ_2 may be combined into a $(k-2)$-colouring of G, a contradiction.

(c.2) Suppose $|\varphi_2(\{v_2, v_3\})| = 2$, $|\varphi_2(\{v_4, v_5\})| = 2$ and $\varphi_2(v_1) = 1$. Suppose that φ_2 assigns the colour 2 to the two distinct vertices $x, y \in S\backslash\{v_1\}$. Now we redefine φ_1 and φ_2 such that $\varphi_1(x) = \varphi_1(y) = k-1$ and $\varphi_2(z) = k-1$ for some vertex $z \in S\backslash\{v_1, x, y\}$.

(c.3) Suppose $|\varphi_2(\{v_2, v_3\})| = 2$, $|\varphi_2(\{v_4, v_5\})| = 2$ and $\varphi_2(v_1) \neq 1$, say $\varphi_2(v_5) = 1$. In this case there is a vertex $x \in \{v_2, v_3\}$ such that $\varphi_2(x) = \varphi_2(v_4)$. Now we redefine φ_1 and φ_2 such that $\varphi_1(x) = \varphi_1(v_4) = k-1$ and $\varphi_2(v_1) = k-1$.

(c.4) If $|\varphi_2(\{v_2, v_3\})| = 2$ and $|\varphi_2(\{v_4, v_5\})| = 1$, then we redefine the mapping φ_2 such that $\varphi_2(v_2) = \varphi_2(v_3) = k-1$.

□

6 Double-critical 6-chromatic graphs

In this section we prove, without use of the Four Colour Theorem, that any double-critical 6-chromatic graph is contractible to K_6.

Theorem 6.1. *Every double-critical* 6*-chromatic graph* G *contains* K_6 *as a minor.*

Proof. If G is a the complete 6-graph, then we are done. Hence we may assume that G is not the complete 6-graph. Now, according to Proposition 3.9, $\delta(G) \geqslant 7$. Firstly, suppose that $\delta(G) \geqslant 8$. Then $m(G) = \frac{1}{2}\sum_{v\in V(G)} \deg(v) \geqslant 4n(G) > 4n(G) - 9$. Győri [8] and Mader [15] proved that any graph H with $n(H) \geqslant 6$ and $m(H) \geqslant 4n(H) - 9$ is contractible to K_6, which implies the desired result. Secondly, suppose that G contains a vertex, say x, of degree 7. Let y_i ($i \in [7]$) denote the neighbours of x. Now, according to Proposition 3.12, the complement of the induced subgraph G_x consists of isolated vertices and cycles (at least one) of length at least five. Since $n(G_x) = 7$, the complement $\overline{G_x}$ must contain exactly one cycle C_ℓ. We consider three cases depending on the length of C_ℓ. Suppose $C_\ell = \{y_1, y_2, \ldots, y_\ell\}$. If $\ell = 5$, then $\{y_1, y_3, y_6, y_7\}$ induces a K_4, and so $\{y_1, y_3, y_6, y_7, x\}$ induces a K_5, which contradicts Proposition 3.1. If $\ell = 6$, then $\{y_1, y_3, y_5, y_7, x\}$ induces a K_5; again, a contradiction. Finally, if $\ell = 7$, then by contracting the edges y_2y_5 and y_4y_7 of G_x into two distinct vertices a complete 5-graph is obtained, as is readily verified. Since, by definition, x is adjacent to every vertex of $V(G_x)$, it follows that G is contractible to K_6. □

The proof of Theorem 6.1 implies the following result.

Corollary 6.1. *Every double-critical* 6*-chromatic graph* G *with* $\delta(G) = 7$ *has the property that for every vertex* $x \in V(G)$ *with* $\deg(x) = 7$*, the complement* $\overline{G_x}$ *is a* 7*-cycle.*

7 Double-critical 7-chromatic graphs

Let G denote a double-critical non-complete 7-chromatic graph. Recall, that given a vertex $x \in V(G)$, we let G_x denote the induced graph $G[N(x)]$ and $\alpha_x := \alpha(G_x)$. The following corollary is a direct consequence of Proposition 3.11.

Corollary 7.1. *For any vertex* x *of* G *not joined to all other vertices,* $\chi(G_x) \leqslant 4$*.*

Proposition 7.1. *For any vertex* x *of* G *of degree* 9*,* $\alpha_x = 3$*.*

Proof. It follows from Proposition 3.10, that α_x is at most 3. Since $\chi(G_x)\cdot\alpha_x \geqslant n(G_x) = 9$, it follows from Corollary 7.1, that $\alpha_x \geqslant 9/\chi(G_x) \geqslant 9/4$, which implies $\alpha_x \geqslant 3$. Thus, $\alpha_x = 3$. □

Proposition 7.2. *If* x *is a vertex of degree* 9 *in* G*, then the complement* $\overline{G_x}$ *does not contain a* K_4^- *as a subgraph.*

Proof. Let x denote a vertex of degree 9 in G. By Proposition 3.4, the minimum degree in G_x is at least $k-2=5$. Suppose that the vertices y_1, y_2, z_1, z_2 are the vertices of a subgraph K_4^- in $\overline{G_x}$, that is, a 4-cycle with a diagonal edge y_1y_2. The graph $G-x-y_1$ is 5-colourable, and, according to Corollary 3.1, every one of the five colours occurs in $B(xy_1)$. None of the vertices y_2, z_1 or z_2 are in $B(xy_1)$, that is, $B(xy_1) \subseteq V(G_x)\backslash\{y_1, y_2, z_1, z_2\}$. Now the vertex y_2 is not adjacent to every vertex of $B(xy_1)$, since that would leave none of the five colours available for properly colouring y_2. Thus, in G_x the vertex y_2 has at least four non-neighbours (y_1, z_1, z_2 and, at least, one vertex from $B(xy_1)$). Since $n(G_x)=9$, we find that y_2 has at most $8-4$ neighbours in $N[x]$, and we have a contradiction. □

Proposition 7.3. *For any vertex x of degree 9 in G, any vertex of an $\alpha(G_x)$-set has degree 5 in the neighbourhood graph G_x.*

Proof. Let x denote vertex of G of degree 9, and let $W=\{w_1, w_2, w_3\}$ denote any independent set in G_x. This vertices of W all have degree at most 6 in G_x and, by Proposition 3.4, at least 5. Suppose that, say, $w_1 \in W$ has degree 6. Now $B(xw_2)$ is a subset of $N(w_1; G_x)$, $G-x-w_2$ is 5-colourable, and, according to Corollary 3.1, every one of the five colours occurs in $B(xy_1)$. This, however, leaves none of the five colours available for w_1, and we have a contradiction. It follows that any vertex of an independent set of three vertices in G_x have degree 5 in G_x. □

Proposition 7.4. *If G has a vertex x of degree 9, then*

(i) *the vertices of any α_x-set $W=\{w_1, w_2, w_3\}$ all have degree 5 in G_x,*

(ii) *the vertices of $V(G_x)$ have degree 5, 6 or 8 in G_x,*

(iii) *every vertex w_i ($i=1,2,3$) has exactly one private non-neighbour w.r.t. W in G_x, that is, there exist three distinct vertices in G_x-W, which we denote by y_1, y_2 and y_3, such that each w_i ($i=1,2,3$) is adjacent to every vertex of $G_x-(W\cup y_i)$, and*

(iv) *each vertex y_i has a neighbour and non-neighbour in $V(G_x)\backslash(W\cup\{y_1,y_2,y_3\})$ (see Figure 1).*

In the following, let $W:=\{w_1,w_2,w_3\}$, $Y:=\{y_1,y_2,y_3\}$ and $Z:=V(G_x)\backslash(W\cup Y)$. Note that the above corollary does not claim that each vertex y_i has a private non-neighbour in Z w.r.t. to Y.

Proof. Claim (i) follows from Proposition 7.1 and Proposition 7.3. According to Proposition 3.4, $\delta(G_x)\geqslant 5$, and, obviously, $\Delta(G_x)\leqslant 8$, since $n(G_x)=9$. If some vertex $y\in G_x$ has degree strictly less than 8, then, according to Proposition 3.8, it has at least two non-neighbours in G_x, that is, $\deg(y,G_x)\leqslant 8-2$. This establishes (ii). As for the claim (iii), each vertex w_i ($i=1,2,3$) has exactly five neighbours in $V(G_x)\backslash W$, which is a set of six vertices, and so w_i has exactly one non-neigbour in $V(G_x)\backslash W$. Suppose say w_1 and w_2 have a common non-neighbour in $V(G_x)\backslash W$, say u. Now the vertices w_1, w_2, w_3 and u induce a K_4 or K_4^- in the complement $\overline{G_x}$, which contradicts Propositions 7.2.

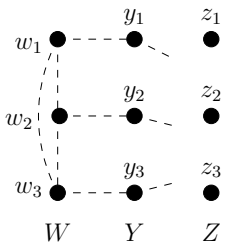


Figure 1: The graph G_x as described in Proposition 7.4. The dashed curves indicate missing edges. The missing edges from W to $Y \cup Z$ are exactly as indicated in the figure, while there may be more missing edges in $E(G_x - W)$ than indicated. The dashed curves starting at vertices of y_i $(i = 1, 2, 3)$ and not ending at a vertex represent a missing edges between y_i and a vertex of Z.

Hence, (iii) follows. Now for claim (iv). The fact that each vertex y_i in Y has at least one neighbour in Z follows (ii) and the fact that y_i is not adjacent to w_i. It remains to show that y_i has at least one non-neighbour in Z. The graph $G - x - w_1$ is 5-colourable, in particular, there exists a 5-colouring c of $G_x - w_1$, which, according to Corollary 3.1, assigns every colour from [5] to at least one vertex of $B(xw_1)$. In this case $B(xw_1)$ consists of precisely the vertices y_2, y_3, z_1, z_2 and z_3. We may assume $\varphi(y_2) = 1$, $\varphi(y_3) = 2$, $\varphi(z_1) = 3$, $\varphi(z_2) = 4$ and $\varphi(z_3) = 5$. Since w_2 is adjacent to every vertex of $Z \cup Y \backslash \{y_2\}$, the only colour available for w_2 is the colour assign to y_2, that is, $\varphi(w_2) = \varphi(y_2) = 1$. Similarly, $\varphi(w_3) = \varphi(y_3) = 2$. Both the vertices w_2 and w_3 are adjacent to y_1 and so the colour assigned to y_1 cannot be one of the colours 1 or 2, that is, $\varphi(y_1) \in \{3, 4, 5\}$. This implies, since $\varphi(z_1) = 3$, $\varphi(z_2) = 4$ and $\varphi(z_3) = 5$, that y_1 cannot be adjacent to all three vertices z_1, z_2 and z_3. Thus, (iv) is established. □

Corollary 7.2. *If G has a vertex x of degree 9, then there are at least two edges between vertices of Y.*

Proof. If $m(G[Y]) \leqslant 1$, then it follows from (iiic) and (iv) of Proposition 7.4, that some vertex $y_i \in Y$ has at most four neighbours in G_x. But this contradicts (b) of the same proposition. Thus, $m(G[Y]) \geqslant 2$. □

Lemma 7.1. *If x is a vertex of G with minimum degree 9 and the neighbourhood graph G_x is isomorphic to the graph F of Figure 2, then G is contractible to K_7.*

Proof. According to Corollary 7.1, $\chi(G[N[x]]) \leqslant 5$, and so $N[x] \neq V(G)$. Let H denote some component in $G - N[x]$. There are several ways of contracting G_x to K_6^-. For instance, by contracting the three edges w_1y_3, w_2y_1 and w_3y_2 into three distinct vertices a K_6^- is obtained, where the vertices z_1 and z_3 remain non-adjacent. Thus, if there were a z_1-z_3-path $P(z_1, z_3)$ with internal vertices completely contained in the set $V(G)\backslash N[x]$, then, by contracting the edges of $P(z_1, z_3)$, we would have a neighbourhood graph of x, which were contractible to K_6. Similarly, there exists contractions of G_x such that if

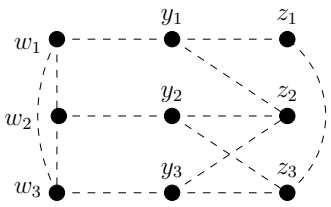


Figure 2: The graph F. The dashed lines between vertices indicate missing edges. Any edge which is not explicity indicated missing is present in F.

only there were a w_1-y_1-path $P(w_1, y_1)$, w_2-y_2-path $P(w_2, y_1)$ or w_3-y_3-path $P(w_3, y_3)$ with internal vertices completely contained in the set $V(G)\backslash N[x]$, then such a path could be contracted such that the neighbourhood graph of x would be contractible to K_6. Assume that none of the above mentioned paths $P(z_1, z_3)$, $P(w_1, y_1)$, $P(w_2, y_1)$ and $P(w_3, y_3)$ exist. In particular, for each pair of vertices (z_1, z_3), (w_1, y_1), (w_2, y_2) and (w_3, y_3) at most one vertex is adjacent to a vertex of $V(H)$, since if both, say z_1 and z_3 were adjacent to, say $u \in V(H)$ and $v \in V(H)$, respectively, then there would be a z_1-z_3-path with internal vertices completely contained in the set $V(G)\backslash N[x]$, contradicting our assumption. Now it follows that in G there can be at most five vertices of $V(G_x)$ adjacent to vertices of $V(H)$. By removing from G the vertices of $V(G_x)$, which are adjacent to vertices of $V(H)$, the graph splits into at least two distinct components with x in one component and the vertices of $V(H)$ in another component. This contradicts Theorem 5.1, which states that G is 6-connected, and so the proof is complete. □

Theorem 7.1. *Every double-critical 7-chromatic graph G contains K_7 as a minor.*

Proof. If G is a complete 7-graph, then we are done. Hence, we may assume that G is not a complete 7-graph, and so, according to Proposition 3.9, $\delta(G) \geqslant 8$. If $\delta(G) \geqslant 10$, then $m(G) \geqslant 5n(G) > 5n - 14$, and it follows from a theorem of Mader [15] that G contains K_7 as a minor. Let x denote a vertex of minimum degree. Suppose $\delta(G) = 8$. Now, according to Proposition 3.12, the complement $\overline{G_x}$ consists of isolated vertices and cycles (at least one), each having length at least five. Since $n(G_x) = 9$, it follows that $\overline{G_x}$ contains exactly one cycle C_ℓ of length at least 5.

(i) If $\ell = 5$, then $G[y_1, y_3, y_6, y_7, y_8, x]$ is the complete 6-graph, a contradiction.

(ii) If $\ell = 6$, then $G[y_1, y_3, y_5, y_7, y_8, x]$ is the complete 6-graph, a contradiction.

(iii) If $\ell = 7$, then by contracting the edges y_1y_4 and y_2y_6 of G_x into two distinct vertices a complete 6-graph is obtained, and so $G \geqslant K_7$.

(iv) If $\ell = 8$, then by contracting the edges y_1y_5 and y_3y_7 of G_x into two distinct vertices a complete 6-graph is obtained, and so $G \geqslant K_7$.

Now, suppose $\delta(G) = 9$. By Proposition 7.4, there is an α_x-set $W = \{w_1, w_2, w_3\}$ of three distinct vertices such that there is a set $Y = \{y_1, y_2, y_3\} \subseteq V(G)\backslash W$ of three distinct

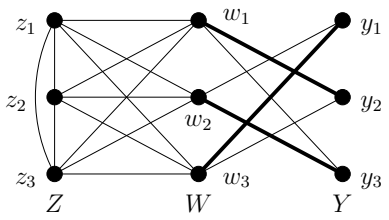


Figure 3: In Case 1.2.3, the graph G_x contains the graph depicted above as a subgraph. The thick curves indicate the edges to be contracted. By contracting the three edges of G_x as indicated above, a K_6 minor is obtained.

vertices such that $N(w_i, G_x) = V(G_x)\backslash(W \cup y_i)$ (see Figure 1). Let $Z = \{z_1, z_2, z_3\}$ denote the three remaining vertices of $G_x - (W \cup Y)$. We shall investigate the structure of G_x and consider several cases. Thus, $e(W) = 0$, and, as follows from Corollary 7.2, $e(Y) \geqslant 2$.

Suppose $e(Z) = 3$. By contracting the edges w_1y_2, w_2y_3 and w_3y_1 of G_x into three distinct vertices a complete 6-graph is obtained (see Figure 3). Thus, $G \geqslant K_7$. In the following we shall be assuming $e(Z) \leqslant 2$.

Secondly, suppose $e(Z) = 0$. Now Z is an α_x-set and it follows from Proposition 7.4, that G_x possess the structure as indicated in Figure 4.

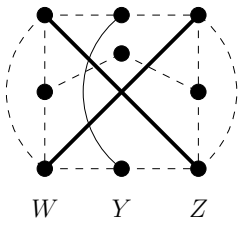


Figure 4: The graph G_x contains the graph depicted above as a subgraph. The dashed curves represent edges missing in G_x. Except for the edges of $E(Y)$, any two pair of edge which are not explicity shown as non-adjacent are adjacent. The edge-set $E(Y)$ contains at least two edges. By symmetry, we assume $y_1y_3 \in E(Y)$. By contracting two edges represented by thick curves, it becomes clear that G_x contains K_6 as a minor.

By contracting the edges w_1z_3 and w_3z_1 of G_x into two distinct vertices w_1' and w_3', we find that the vertices $w_1', w_2, w_3', y_1, y_3$ and z_2 induce a complete 6-graph, and we are done. Thus, in the following we shall be assuming $e(Z) \geqslant 1$. Moreover, we shall distinguish between several cases depending on the number of edges in $E(Y)$ and $E(Z)$. So far we have established $e(Y) \geqslant 2$ and $2 \geqslant e(Z) \geqslant 1$. We shall often use the fact that $\deg(u, G_x) \in \{5, 6, 8\}$ for every vertex $u \in G_x$, in particular, each vertex of G_x can have at most three non-neighbours in G_x (excluding itself).

(1) Suppose $e(Y) = 3$.

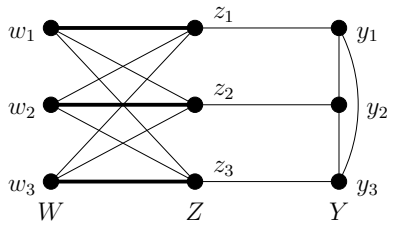


Figure 5: The graph G_x contains the graph depicted above as a subgraph. The thick curves indicate the edges to be contracted. By contracting two edges of G_x as indicated above, it becomes obvious that G_x contains K_6 as a minor.

(1.1) If, in addition, there is a matching M of Y into Z, say $M = \{y_1z_1, y_2z_2, y_3z_3\}$, then contracting the edges w_iz_i $(i = 1, 2, 3)$ into three distinct vertices results in a complete 6-graph, and we are done (see Figure 5).

(1.2) Suppose that there is no matching of Y into Z. Now it follows from Hall's Theorem [2, Th. 16.4] that there exists some non-empty set $S \subseteq Y$ such that $e(S, Z) < |S|$ (recall, that $e(S, Z)$ denotes the number of edges with one end-vertex in S and the other end-vertex in Z). According to Proposition 7.4, $e(S, Z) \geqslant 1$ for any non-empty $S \subseteq Y$.

(1.2.1) Suppose that $e(Y, Z) = 1$, say $E(Y, Z) = \{z_1\}$. Now y_1, y_2 and y_3 are all non-neighbours of z_2 and z_3, and so both z_2 and z_3 must be adjacent to each other and to z_1, that is, $e(Z) = 3$, contradicting our assumption that $e(Z) \leqslant 2$.

(1.2.2) Suppose that $e(Y, Z) = 2$, say $E(Y, Z) = \{z_1, z_2\}$. Now y_1, y_2 and y_3 are three non-neighbours of z_3, and so z_3 must be adjacent to both z_2 and z_3. Since $e(Z) \leqslant 2$, it must be the case that z_1 and z_2 are non-neighbours. Since no vertex of G_x has precisely one non-neighbour, both z_1 and z_2 must have at least one non-neighbour in Y. By symmetry, we may assume that y_1 is a non-neighbour of z_1. Now w_1, z_1 and z_3 are three non-neighbours of y_1, and so y_1 cannot be a non-neighbour of z_2. It follows that y_2 or y_3 must be a non-neighbour of z_2. By symmetry, we may assume $y_2z_2 \notin E(G)$. Now there may be no more edges missing in G_x, however, we assume that there are more edges missing, and show that G_x remains contractible to K_6. Each of the vertices y_1 and y_2 has three non-neighbours specified, while y_3 already has two non-neighbours specified. Thus, the only possible hitherto undetermined missing edge must be either y_3z_1 or y_3z_2 (not both, since that would imply y_3 to have at least four non-neighbours). By symmetry, we may assume $y_3z_2 \notin E(G)$. Now it is clear that G_x is isomorphic to the graph depicted in Figure 6, and so it follows from Lemma 7.1 that G is contractible to K_7.

(1.2.3) Suppose that $e(Y, Z) = 3$. Now, since there is no matching of Y into Z there must be some non-empty proper subset S of Y such that $|S| \leqslant 2$ and $e(S, Z) <$

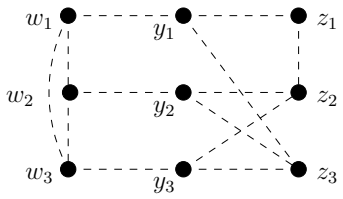


Figure 6: In Case 1.2.2, the graph G_x is isomorphic to the graph depicted above. Any edge which is not explicity indicated missing is present.

$|S|$. Recall, $e(S,Z) \geqslant 1$ for any non-empty subset S of Y, and so it must be the case that $|S| = 2$ and $e(S,Z) = 1$, say $S = \{y_1, y_2\}$ and $E(S,Z) = \{z_1\}$. The assumption $e(Y,Z) = 3$ implies that y_3 is adjacent to both z_2 and z_3. According to Proposition 7.4 (iv), each vertex of Y has a non-neighbour in Z, and so it must be the case that y_3 is not adjacent to z_1. Now, since z_1 has one non-neighbour in $V(G_x)\backslash\{z_1\}$, Proposition 3.8 (b) implies that it must have at least one other non-neighbour in $V(G_x) - z_1$. The only possible non-neighbours of z_1 in $V(G_x)\backslash\{z_1, y_3\}$ are z_2 and z_3, and, by symmetry, we may assume that z_1 and z_2 are not adjacent. Thus, z_2 is adjacent to neither z_1, y_1 nor y_2 and so z_2 must be adjacent to every vertex of $V(G_x)\backslash\{z_1, z_2, y_1, y_2\}$, in particular, z_2 is adjacent to z_3. Thus, G_x contains the graph depicted in Figure 7 as a subgraph. Now, by contracting the edges w_1z_1, w_2y_1 and w_3y_2 of G_x into three distinct vertices a complete 6-graph is obtained.

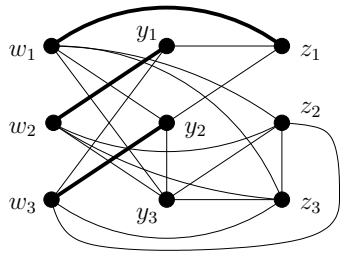


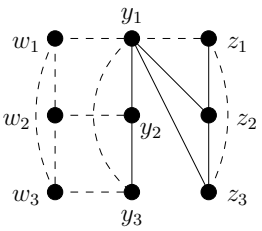
Figure 7: The graph G_x contains the graph depicted above as a subgraph. The thick curves indicate the edges to be contracted. By contracting three edges of G_x as indicated above, it becomes obvious that G_x contains K_6 as a minor.

(2) Suppose $e(Y) = 2$, say $y_1y_2, y_2y_3 \in E(G)$.

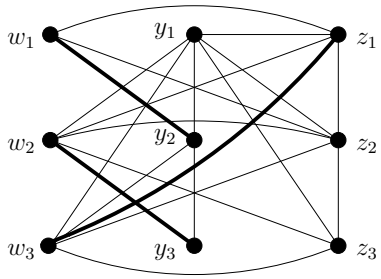
(2.1) Suppose that $e(Z) = 2$, say $z_1z_2, z_2z_3 \in E(G)$.

(2.1.1) Suppose that at least one of the edges y_1z_1 or y_3z_3 are not in $E(G)$, say $y_1z_1 \notin E(G)$. The vertex y_1 has three non-neighbours in G_x, namely w_1, y_3 and z_1. Thus, y_1 must be adjacent to both z_2 and z_3. We have determined the edges

of $E(W)$, $E(Y)$ and $E(Z)$, and the edges joining vertices of W with vertices of $Y \cup Z$. Moreover, G_x contains at least two edges joining vertices of Y with vertices of Z, as indicated in Figure 8 (a). It follows that G_x contains the graph depicted in Figure 8 (b) as a subgraph. By contracting the edges w_1y_2, w_2y_3 and w_3z_1 of G_x into three distinct vertices a complete 6-graph is obtained, and so $G \geqslant K_7$.



(a) The graph G_x is completely determined, except for possible some edges between Y and Z.

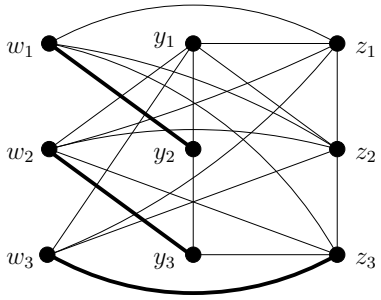


(b) The graph depicted above is a subgraph of G_x.

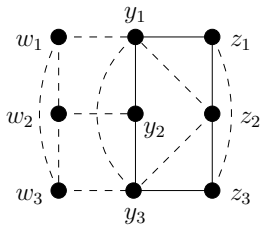
Figure 8: Illustration for Case 2.1.1.

(2.1.2) Suppose that both y_1z_1 and y_3z_3 are in $E(G)$.

(2.1.2.1) Suppose that y_1z_2 or y_3z_2 is in $E(G)$, say $y_1z_2 \in E(G)$. In this case G_x contains the graph depicted in Figure 9 (a) as a subgraph, and so by contracting the edges w_1y_2, w_2y_3 and w_3z_3 into three distinct vertices a complete 6-graph is obtained.



(a) In Case 2.1.2.1, G_x contains the graph depicted above as a subgraph.



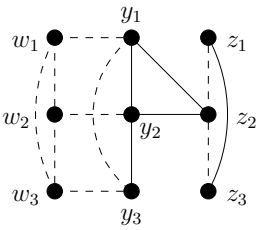
(b) In Case 2.1.2.2, G_x is at least missing the edges as indicated in the above graph.

Figure 9: Illustration for Case 2.1.2.

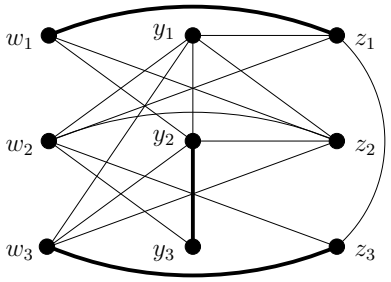
(2.1.2.2) Suppose that neither y_1z_2 nor y_3z_2 is in $E(G)$. Now $S := \{y_1, z_2, y_3\}$ is an independent set of G_x and so, according to Proposition 7.4 (iii), the vertex z_2 has a private non-neighbour in $V(G_x) - S$ w.r.t. S, and, as is easily seen from Figure 9 (b), the only possible non-neighbour of z_2 in $V(G_x)$ is y_2. The vertices z_1 and z_3 are not adjacent, and so, according to Proposition 7.4 (ii), each of them must have a second non-neighbour. Since y_1 and y_3 already have three non-neighbours specified, it follows that the only possible non-neighbour of z_1 and z_3 is y_2, but if neither z_1 nor z_3 are adjacent to y_2, then y_2 would have at least four non-neighbours in G_x, a contradiction.

(2.2) Suppose that $e(Z) = 1$, say $E(Z) = \{z_1z_3\}$.

(2.2.1) Suppose that $y_2z_2 \in E(G)$. Now at least one of the edges y_1z_2 and y_3z_2 is in $E(G)$, since otherwise z_2 would have at least four non-neighbour. By symmetry, we may assume $y_1z_2 \in E(G)$. At least one of the edges y_1z_1 and y_1z_3 must be in $E(G)$, since y_1 cannot have more than three non-neighbours. By symmetry, we may assume $y_1z_1 \in E(G)$ (see Figure 10 (a)). By contracting the edges w_1z_1, w_3z_3 and y_2y_3 of G_x into three distinct vertices we obtain a complete 6-graph (see Figure 10 (b)), and, thus, $G \geqslant K_7$.



(a) The graph G_x is completely determined, except for some edges between Y and Z.



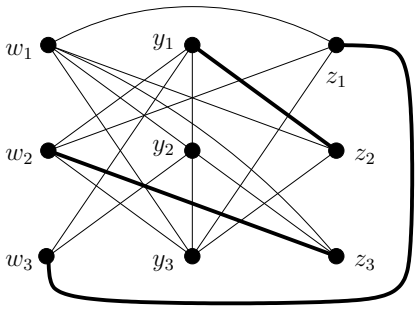
(b) The above graph is a subgraph of G_x.

Figure 10: Illustration for Case 2.2.1.

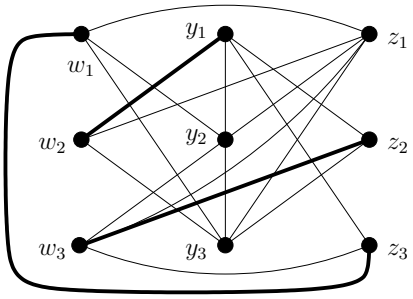
(2.2.2) Suppose that $y_2z_2 \notin E(G)$. Each of the vertices z_1 and z_3 has exactly one non-neighbour in Z, namely z_2, and so each must have at least one non-neighbour in Y. If neither z_1 nor z_3 were adjacent to y_2, then y_2 would have at least four non-neighbours in G_x. Thus, at least one of z_1 and z_3 is not adjacent to y_1 or y_3. By symmetry, we may assume that $y_1z_1 \notin E(G)$. Now we need to determine the non-neighbour of y_3 in Y.

(2.2.2.1) Suppose that $y_2z_3 \in E(G)$. Since y_1 already has three non-neighbours, it must be the case that y_3 is a non-neighbour of z_3 in Y. There may also be an edge joining y_2 and z_1, but in any caseG_x contains the graph depicted in Figure 11 (a)

as a subgraph. Thus, by contracting the edges w_2z_1, w_3z_1 and y_1z_2 into three distinct vertices, we find that $K_6 \leqslant G_x$.



(a) In Case 2.2.2.1, G_x contains the graph depicted above as a subgraph.



(b) In Case 2.2.2.2, G_x contains the graph depicted above as a subgraph.

Figure 11: Illustration for Case 2.2.2.

(2.2.2.2) Suppose that $y_2z_3 \notin E(G)$. In this case we find that $S := \{y_2, z_2, z_3\}$ is a maximum independent set in G_x and so, according to Proposition 7.4 (iii), each of the vertices of S has a private non-neighbour in $V(G_x) - S$ w.r.t. S. The vertices w_1, y_3 and z_1 are all non-neighbours of y_1, and so z_3 cannot be a non-neighbour of y_1. It follows that the non-neighbour of z_3 in $V(G_x) - S$ must be y_3. Now each of the vertices of Y has three non-neighbours, and so there can be no further edges missing from G_x, that is, G_x contains the graph depicted in Figure 11 (b) as a subgraph.

This, finally, completes the case $\delta(G) = 9$, and so the proof is complete. □

Obviously, if every k-chromatic graph for some fixed integer k is contractible to the complete k-graph, then every ℓ-chromatic graph with $\ell \geqslant k$ is contractible to the complete k-graph. The corresponding result for *double-critical* graphs is not obviously true. However, for $k \leqslant 7$, it follows from the aforementioned results and Corollary 7.3 that every double-critical ℓ-chromatic graph with $\ell \geqslant k$ is contractible to the complete k-graph.

Corollary 7.3. *Every double-critical k-chromatic graph with $k \geqslant 7$ contains K_7 as a minor.*

Proof. Let G denote an arbitray double-critical k-chromatic graph with $k \geqslant 7$. If G is complete, then we are done. If $k = 7$, then the desired result follows from Theorem 7.1. If $k \geqslant 9$, then, according to Proposition 3.9, $\delta(G) \geqslant 10$ and so the desired result follows from a theorem of Győri [8] and Mader [15]. Suppose $k = 8$ and that G is non-complete. Then $\delta(G) \geqslant 9$. If $\delta(G) \geqslant 10$, then we are done and so we may assume $\delta(G) = 9$, say $\deg(x) = 9$. In this case it follows from Proposition 3.12 that the complement $\overline{G_x}$ consists of cycles (at least one) and isolated vertices (possibly none). An argument similar to the argument given in the proof of Theorem 6.1 shows that G_x is contractible to K_6. Since x dominates every vertex of $V(G_x)$, then G itself is contractible to K_7. □

The problem of proving that every double-critical 8-chromatic graph is contractible to K_8 remains open.

8 Double-edge-critical graphs and mixed-double-critical graphs

A natural variation on the theme of double-critical graphs is to consider double-edge-critical graphs. A vertex-critical graph G is called *double-edge-critical* if the chromatic number of G decreases by at least two whenever two non-incident edges are removed from G, that is,

$$\chi(G - e_1 - e_2) \leqslant \chi(G) - 2 \text{ for any two non-incident edge } e_1, e_2 \in E(G) \tag{3}$$

It is easily seen that $\chi(G - e_1 - e_2)$ can never be strictly less that $\chi(G) - 2$ and so we may require $\chi(G - e_1 - e_2) = \chi(G) - 2$ in (3). The only critical k-chromatic graphs for $k \in \{1, 2\}$ are K_1 and K_2, therefore we assume $k \geqslant 3$ in the following.

Theorem 8.1. *A graph G is k-chromatic double-edge-critical if and only if it is the complete k-graph.*

Proof. It is straightforward to verify that any complete graph is double-edge-critical. Conversely, suppose G is a k-chromatic ($k \geqslant 3$) double-edge-critical graph. Then G is connected. If G is a complete graph, then we are done. Suppose G is not a complete graph. Then G contains an induced 3-path $P : wxy$. Since G is vertex-critical, $\delta(G) \geqslant k-1 \geqslant 2$, and so y is adjacent to some vertex z is $V(G)\backslash\{w, x, y\}$. Now the edges wx and yz are not incident, and so $\chi(G-wx-yz) = k-2$. Let φ denote a $(k-2)$-colouring of $G-wx-yz$. Then the vertices w and x (and y and z) are assigned the same colours, since otherwise G would be $(k-1)$-colourable. We may assume that φ assigns the colour $k-3$ to the vertices w and x, and the colour $k-2$ to the vertices y and z. Now define the $(k-1)$-colouring φ' such that $\varphi'(v) = \varphi(v)$ except $\varphi'(w) = k-1$ and $\varphi'(y) = k-1$. The colouring φ' is a proper $(k-1)$-colouring, since w and y are non-adjacent in G. This contradicts the fact that G is k-chromatic and therefore G must be a complete graph. □

A vertex-critical k-chromatic graph G is called *mixed-double-critical* if for any vertex $x \in G$ and any edge $e = uv \in E(G - x)$,

$$\chi(G - x - e) \leqslant \chi(G) - 2 \tag{4}$$

Theorem 8.2. *A graph G is k-chromatic mixed-double-critical if and only if it is the complete k-graph.*

The proof of Theorem 8.2 is straightforward and similar to the proof of Theorem 8.1.

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