Quantized dual graded graphs

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Abstract

We study quantized dual graded graphs, which are graphs equipped with linear operators satisfying the relation $DU - qUD = rI$. We construct examples based upon: the Fibonacci differential poset, permutations, standard Young tableau, and plane binary trees.

1 Introduction

Fomin's *dual graded graphs* [Fom] and Stanley's *differential posets* [Sta] are constructions developed to understand and generalize the enumerative consequences of the Robinson-Schensted algorithm. The key relation in these constructions is $DU - UD = rI$, where U, D are up and down operators acting on the graphs or posets[1]. In this article we develop some of the basic theory of *quantized dual graded graph*, which are equipped with up-down operators U, D satisfying the q-Weyl relation $DU - qUD = rI$. Here q can be considered a parameter which the graph depends upon, and which can be specialized. One of the motivations for the current work were the signed differential posets developed in [Lam], which correspond to the relation $DU + UD = rI$. Thus quantized dual graded graphs specialize to usual dual graded graphs at $q = 1$, and to signed differential posets (or their dual graded graph equivalent) at $q = -1$.

The motivating enumerative identity in the subject developed by Fomin and Stanley is

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

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[1]Fomin also considered more general relations of the form $DU = f(UD)$

where the sum is over partitions of n, and f^λ is the number of standard Young tableau of shape λ. The corresponding analogue (Theorem 4) for a quantized dual graded graph (Γ, Γ') reads

$$\sum_v f_\Gamma^v(q)\, f_{\Gamma'}^v(q) = r^n [n]_q! \tag{1}$$

where the sum is over vertices of height n, the polynomials $f_\Gamma^v(q)$ and $f_{\Gamma'}^v(q)$ are weighted enumerations of paths in Γ and Γ', and $[n]_q!$ is the q-analogue of $n!$.

We explicitly construct examples of quantized dual graded graphs and interpret (1). These examples are based on various combinatorial objects: the Fibonacci differential poset (also called the Young-Fibonacci lattice), permutations, standard Young tableau, and plane binary trees. Unfortunately, we have been unable to quantize Young's lattice. More examples will be given in joint work [BLL] with Bergeron and Li, where in some cases a representation theoretic explanation for the identities $DU - qUD = I$ and (1) will be given.

We remark that in [Ter] Terwilliger introduced the notion of a *uniform poset*, a different generalization of differential posets, which includes quantized dual graded graphs as a special case.

2 Quantized dual graded graphs

Let $\Gamma = (V, E, m, h)$ be a *graded graph* with edge weights $m(v, w) \in \mathbb{N}[q]$. That is, Γ is a directed graph with vertex set V, directed edge set E, edge weights $\{m(v, w) \mid (v, w) \in E\}$, and equipped with a height function $h : V \to \mathbb{N}$ such that if $(v, w) \in E$ then $h(w) = h(v) + 1$. Furthermore, each edge has a weight $m(v, w) \in \mathbb{N}[q]$ which is a non-zero polynomial in q with nonnegative coefficients. We shall assume that Γ is locally finite in the following sense: for each v, there are finitely many edges entering and leaving (this differs slightly from the usual definition of locally finiteness for posets). Because each edge has a weight, we shall assume that there are no multiple edges.

Let $\widehat{\mathbb{C}(q)[V]}$ be the $\mathbb{C}(q)$-vector space of formal linear combinations of the vertex set V. A linear operator on $\widehat{\mathbb{C}(q)[V]}$ is *continuous* if it is compatible with arbitrary linear combinations. Define continuous linear operators $U, D : \widehat{\mathbb{C}(q)[V]} \to \widehat{\mathbb{C}(q)[V]}$ by

$$U(v) = \sum_{w:(v,w)\in E} m(v, w)\, w$$
$$D(w) = \sum_{v:(v,w)\in E} m(v, w)\, v.$$

and extending by linearity and continuity. We define a pairing $\langle ., . \rangle : \widehat{\mathbb{C}(q)[V]} \times \mathbb{C}(q)[V] \to \mathbb{C}(q)$ by $\langle v, w \rangle = \delta_{v,w}$ for $v, w \in E$. Then U and D are adjoint with respect to this pairing.

Let $(\Gamma = (V, E, m, h), \Gamma' = (V, E', m', h))$ be a pair of graded graphs with the same vertex set. Then (Γ, Γ') is a pair of *quantized dual graded graphs* (qDGG for short) if we have the identity

$$D_{\Gamma'} U_\Gamma - q U_\Gamma D_{\Gamma'} = rI \tag{2}$$

for some positive integer $r \in \{1, 2, 3, \ldots\}$, called the *differential coefficient*. In the sequel, we will often write U and D for U_Γ and $D_{\Gamma'}$. When $q = 1$, we obtain the dual graded graphs of [Fom], which are equipped with the relation $DU - UD = rI$. We should note that Fomin also considered the more general relation $DU = f(UD)$ for arbitrary functions f; however, he did not focus on (2) where q is a formal parameter. Strictly speaking, our quantized dual graded graphs are not graded graphs in Fomin's sense, as Fomin only allows (nonnnegative) integer edge weights.

If $(\Gamma(q), \Gamma'(q))$ are a pair of quantized dual graded graphs then we say that $(\Gamma(q), \Gamma'(q))$ is a quantization of $(\Gamma(1), \Gamma'(1))$. The basic example of a dual graded graph is Young's lattice of partitions, ordered by containment; see [Fom, Sta]. Thus one should consider the following as the basic problem for quantized dual graded graphs.

Problem 1. *Find a quantization of Young's lattice.*

Paul Terwilliger has shown (private communication) that Young's lattice cannot be quantized with weights in $\mathbb{N}[q]$, as required by our definitions here. However, quantization allowing more general weights, such as weights in $\mathbb{N}[q^{1/2}, q^{-1/2}]$, may still exist. Since all the examples in the present paper have weights in $\mathbb{N}[q]$, we shall not discuss the situation with more general edge weights.

In [LS], we constructed in joint work with Mark Shimozono dual graded graphs from the strong (Bruhat) and weak orders of the Weyl group of a Kac-Moody algebra. The dual graded graphs constructed this way include Young's lattice, and closely related graphs such as the shifted Young's lattice.

Problem 2. *Find a quantization of Kac-Moody dual graded graphs.*

It would be extremely interesting to solve the above problem (again allowing more general weights) using the representation theory of quantized Kac-Moody algebras, generalizing the construction of [LS] which relied on the root system combinatorics of a Kac-Moody algebra.

Remark 1. Equation (2) specializes to $DU + UD = I$ when $q = -1$ and $r = 1$. Graphs satisfying this relation were studied in [Lam]. More specifically, in [Lam] we studied only such graphs, called signed differential posets, which arose from labeled posets. The examples constructed in the present paper can also be specialized at $q = -1$, giving what would be called "signed dual graded graphs". The main example in [Lam] was the construction of a signed differential poset structure on Young's lattice. Since we have been unable to quantize Young's lattice, we have stopped short of explicitly writing the examples in the current article using the notation in [Lam].

3 q-derivatives and enumeration on quantized dual graded graphs

Let $f(t) = \sum_{n \geqslant 0} a_n t^n \in \mathbb{C}[[t]]$ be a formal power series in one variable. Define the *q-derivative* as follows:

$$f^q(t) = \sum_{n \geqslant 0} [n]_q a_n t^{n-1}.$$

Here $[n]_q = 1 + q + \ldots + q^{n-1}$ denotes the q-analogue of n. We also set $[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q$. Let U, D be formal, non-commuting variables satisfying the relation $DU - qUD = r$. We assume that U and D commute with the variable q. The following Lemma explains the relationship between the relation $DU - qUD = r$ and q-derivatives.

Lemma 3. *Suppose $f(U) \in \mathbb{C}[[U]]$ is a formal power series in the variable U. Then $Df(U) = r\, f^q(U) + f(qU)D$.*

Proof. By linearity and continuity it suffices to prove the statement for $f(U) = U^n$. For $n = 0$, the formula is trivially true. The inductive step follows from the calculation

$$\begin{aligned} DU^n &= (r[n-1]_q U^{n-2} + q^{n-1}U^{n-1}D)U \\ &= r[n-1]_q U^{n-1} + q^{n-1}U^{n-1}(r + qUD) \\ &= r[n]_q U^{n-1} + q^n U^n D, \end{aligned}$$

using $[n]_q = [n-1]_q + q^{n-1}$. $\square$

We now suppose (Γ, Γ') is a qDGG with a unique minimum (source) $\emptyset$, which we assume has height $h(\emptyset) = 0$. Let us denote the weight generating function of paths in Γ from $\emptyset$ to a vertex $v \in V$ by $f_\Gamma^v = (U^n\emptyset, v)$, where $n = h(v)$. It is not difficult to see that we have

$$(D^n U^n \emptyset, \emptyset) = \sum_{v\colon h(v)=n} f_\Gamma^v\, f_{\Gamma'}^v.$$

The following is an analogue of [Fom, Corollary 1.5.4]; see also [Sta, Corollary 3.9].

Theorem 4. *Let (Γ, Γ') be a* qDGG *with a unique minimum $\emptyset$. Then*

$$\sum_{v\colon h(v)=n} f_\Gamma^v\, f_{\Gamma'}^v = r^n [n]_q!.$$

Proof. By Lemma 3 we have

$$D^n U^n\, \emptyset = D^{n-1}(r[n]_q U^{n-1} + q^n U^n D)\, \emptyset = r[n]_q D^{n-1} U^{n-1} \emptyset$$

from which the result follows by induction. $\square$

More generally, let $f(\emptyset \to v \to w)$ denote the weight generating function of paths beginning at $\emptyset$, going up to v in Γ, then going down to w in Γ'. For fixed w with $h(w) = m \leqslant n$, we then have

$$\sum_{v:\, h(v)=n} f(\emptyset \to v \to w) = (D^{n-m}U^n\,\emptyset, w) = r^{n-m}([n]_q[n-1]_q \cdots [m+1]_q)\, f_\Gamma^w.$$

Other path generating function problems can be solved by studying the "normal ordering problem" for the relation $DU - qUD = r$, that is, the problem of rewriting a word in the letters U and D as a linear combination of terms U^iD^j. We shall not pursue this direction here, but see for example [Var].

4 q-reflection-extension

Let $(\Gamma_n = (V, E, m, h), \Gamma'_n = (V, E', m', h))$ be a pair of graded graphs with height function taking values in $[0, n]$, and such that (2) holds for some fixed r, when applied to all vertices v such that $h(v) < n$. We call such a pair a *partial* qDGG of height n. We will construct a partial qDGG $(\Gamma_{n+1}, \Gamma'_{n+1})$ of height $n+1$, and such that they agree with (Γ_n, Γ'_n) up to height n.

Let us write $V_i = \{v \mid h(v) = i\}$. The height $n+1$ vertices of (both) Γ_{n+1} and Γ'_{n+1} will be given by the set $V_{n+1} = \{v^1, v^2, \ldots, v^r \mid v \in V_n\} \cup \{w' \mid w \in V_{n-1}\}$. There will be two kinds of edges. For Γ_{n+1}, we construct

1. r edges (v, v^1), $(v, v^2), \ldots, (v, v^r)$ for each $v \in V_n$ which have weight $m(v, v^j) := 1$.
2. An edge (v, w') for each edge (w, v) of Γ', where $v \in V_n$ and $w \in V_{n-1}$. This edge has weight $m(v, w') := q\, m'(w, v)$.

And for Γ'_{n+1}, we construct

1. r edges (v, v^1), $(v, v^2), \ldots, (v, v^r)$ for each $v \in V_n$ which have weight $m'(v, v') := 1$.
2. An edge (v, w') for each edge (w, v) of Γ', where $v \in V_n$ and $w \in V_{n-1}$. This edge has weight $m'(v, w') := m(w, v)$.

We omit the proof of the following, which is the same as the corresponding result for differential posets [Sta] or signed differential posets [Lam].

Proposition 5. *Suppose* (Γ_n, Γ'_n) *is a partial* qDGG *of height* n *and differential coefficient* r*. Then* $(\Gamma_{n+1}, \Gamma'_{n+1})$ *is a partial* qDGG *of height* $n+1$ *and differential coefficient* r*. Furthermore,* $(\Gamma, \Gamma') = \lim_{n\to\infty}(\Gamma_n, \Gamma'_n)$ *is a well-defined* qDGG *with differential coefficient* r*.*

5 The quantized Fibonacci poset

Let (Γ, Γ') be a qDGG. If the edge sets of Γ and of Γ' are identical and in addition every edge weight $m(v,w)$ (and $m'(v,w)$) of Γ (and Γ') is a single power q^i then we call (Γ, Γ') a *quantized differential poset*. For then, $\Gamma(1)$ would be a differential poset in the sense of Stanley [Sta].

Remark 2. We could insist that $\Gamma = \Gamma'$ as graded graphs, but then in the construction of a quantization of the Fibonacci differential posets we would need to use half powers of q.

Define $\Gamma_0 = \Gamma'_0$ to be the graded graph with a single vertex $\emptyset$ with height 0. For each $r \in \{1, 2, \ldots\}$, we define the *quantized r-Fibonacci poset* to be the corresponding qDGG $(\mathrm{Fib}_{(r)}, \mathrm{Fib}'_{(r)})$ obtained from (Γ_0, Γ'_0) via Proposition 5. The qDGG $(\mathrm{Fib}_{(r)}, \mathrm{Fib}'_{(r)})$ is a quantization of the Fibonacci differential poset of Stanley [Sta], or the Young-Fibonacci graph of Fomin [Fom]. We now describe $(\mathrm{Fib}_{(r)}, \mathrm{Fib}'_{(r)})$ explicitly, suppressing the parameter r in most of the notation.

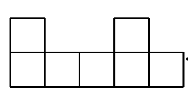
The vertex set V of $(\mathrm{Fib}_{(r)}, \mathrm{Fib}'_{(r)})$ consists of words w in the letters $1_1, 1_2, \ldots, 1_r, 2$ with height function given by summing the letters in the word (all the 1's have the same value). In the notation of the q-reflection algorithm, the vertices $v^1, \ldots, v^r$ are obtained from v by prepending $1_1, 1_2, \ldots, 1_r$ respectively; the vertices w' are obtained from w by prepending the letter 2. The edges (v,w) are of one of the two forms:

1. v is obtained from w by removing the first 1 (one of the letters $1_1, 1_2, \ldots, 1_r$);

2. v is obtained from w by changing a 2 to one of the 1's, such that all letters to the left of this 2 is also a 2.

In either case, let $s(v,w)$ denote the number of letters preceding the letter which is changed or removed to go from w to v. The edges (v,w) of form (1) have edge weight $m(v,w) = m'(v,w) = q^{s(v,w)}$ in both $\mathrm{Fib}_{(r)}$ and $\mathrm{Fib}'_{(r)}$. The edges of form (2) have edge weight $m(v,w) = q^{s(v,w)+1}$ in $\mathrm{Fib}_{(r)}$, and edge weight $m'(v,w) = q^{s(v,w)}$ in $\mathrm{Fib}'_{(r)}$.

For the rest of this section, we will restrict ourselves to $r = 1$, and write 1 instead of 1_1. We now describe the weight of a path from $\emptyset$ to a word w in $\mathrm{Fib} = \mathrm{Fib}_{(1)}$ or $\mathrm{Fib}' = \mathrm{Fib}'_{(1)}$. Given a word $w \in \mathrm{Fib}$ one has a *snakeshape* ([Fom]) consisting of a series of columns of height one or two. For example, for $w = 21121$ we have the shape


.

Given such a *snakeshape* λ, following Fomin [Fom] we say that a Young-Fibonacci-tableau of shape λ is a bijective filling of λ with the numbers $\{1, 2 \ldots, n\}$ so that:

1. In any height two column the lower number is smaller.

2. To the right of a height two column containing the numbers a and b none of the numbers in $[a, b]$ occur.

3. To the right of a height one column containing the number a, no numbers greater than a occur.

For each number $i \in \{1, 2 \ldots, n\}$, let $p_i(T)$ denote the number of columns C in T to the left of the column containing i, and such that C contains a number less than i. Then set

$$\mathrm{wt}(T) = \prod_{i \in \text{lower row}} q^{p_i(T)} \prod_{i \in \text{upper row}} q^{p_i(T)+1} \quad \text{and} \quad \mathrm{wt}'(T) = \prod_i q^{p_i(T)}.$$

Fomin [Fom] described a bijection between Young-Fibonacci-tableau T of shape $\lambda = \lambda(w)$ and paths from $\emptyset$ to w in Fib (or Fib′). For example, the tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 7 & 6 & 4 & 1 \\ \hline \end{array}$$

corresponds to the path $\emptyset \to 1 \to 11 \to 21 \to 211 \to 221 \to 2121 \to 21121$, and we have $\mathrm{wt}(T) = q^0q^0q^1q^1q^1 \cdot q^1q^2 = q^6$ and $\mathrm{wt}'(T) = q^0q^0q^0q^1q^1q^1q^1 = q^4$.

Lemma 6. *Under this bijection the weight of the path is equal to* $\mathrm{wt}(T)$ *in* Fib*, and equal to* $\mathrm{wt}'(T)$ *in* Fib′.

Proof. This is straightforward, using the description of the bijection on [Fom, p.394]. □

Thus we have $f^w_{\mathrm{Fib}} = \sum_T \mathrm{wt}(T)$ and $f^w_{\mathrm{Fib}'} = \sum_T \mathrm{wt}'(T)$ where the sum is over Young-Fibonacci tableau with shape $\lambda(w)$. It is not clear whether there is a simple way to write the identity that results from Theorem 4.

6 The qDGG on permutations

Let $V = \sqcup_{n \geqslant 0} S_n$ be the disjoint union of all permutations equipped with the height function $h(w) = n$ if $w \in S_n$. Define a graded graph Perm with vertex set V and edge set E consisting of edges (v, w) whenever $v \in S_{n-1}$ is obtained from $w \in S_n$ by deleting the letter n; define $m(v, w) := q^{n-s}$, where $1 \leqslant s \leqslant n$ is the position of the letter n in w. Define Perm′ with the same vertex set and edges (v, w) whenever $v \in S_{n-1}$ is obtained from $w \in S_n$ by deleting the last letter, followed by reducing all letters greater than the deleted letter by one; define $m'(v, w) := 1$ always.

For example, in Perm there is an edge from 4123 to 41523 with weight q^3. In Perm′ there is an edge from 3142 to 41523 with weight 1. The following result is a straightforward verification of the definitions.

Proposition 7. *The pair* (Perm, Perm′) *is a* qDGG *with differential coefficient* $r = 1$.

Let $\mathrm{inv}(w)$ denote the number of inversions of a permutation w. For the pair of quantized dual graded graphs (Perm, Perm′), we have

$$f^w_{\mathrm{Perm}} = q^{\mathrm{inv}(w)} \quad \text{and} \quad f^w_{\mathrm{Perm}'} = 1.$$

Thus Theorem 4 expresses the identity (see [EC1, Corollary 1.3.10])

$$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!. \tag{3}$$

7 The qDGG on tableaux

Let Y_n denote the set of standard Young tableau P of size n with any shape (see [EC2]). We assume the reader is familiar with tableaux, and with Schensted insertion.

Let $V = \cup_{i\geqslant 0} Y_i$ with the obvious height function. Define Tab to be the graded graph with vertex set V, and edges $(P, P') \in Y_n \times Y_{n+1}$ whenever there is some $k \in \{1, 2, \ldots, n+1\}$ so that P' is obtained from P by first increasing the numbers greater than or equal to k inside P by 1, and then Schensted inserting k; declare $m(P, P') = q^{n+1-k}$. Define Tab$'$ to be the graded graph with vertex set V and edges $(P, P') \in Y_n \times Y_{n+1}$ whenever P' is obtained from P by removing n; declare $m(P, P') = 1$. The following result is straightforward.

Proposition 8. *The pair* $(\mathrm{Tab}, \mathrm{Tab}')$ *is a* qDGG *with differential coefficient* $r = 1$*.*

Fix a standard Young tableau $P \in Y_n$. There is a bijection from the set of paths p from the empty tableau $\emptyset$ to P in Tab, to the set of standard Young tableau of shape equal to the shape of P. The bijection is obtained by taking the sequence of shapes encountered along p, or equivalently, by taking the recording tableau of the sequence of Schensted insertions given by p. The following Lemma is immediate.

Lemma 9. *Suppose p is a path from $\emptyset$ to P, corresponding to a standard Young tableau Q. Then the weight of p in* Tab *is equal to $q^{\mathrm{inv}(w(P,Q))}$, where $w(P, Q) \leftrightarrow (P, Q)$ under the Robinson-Schensted bijection.*

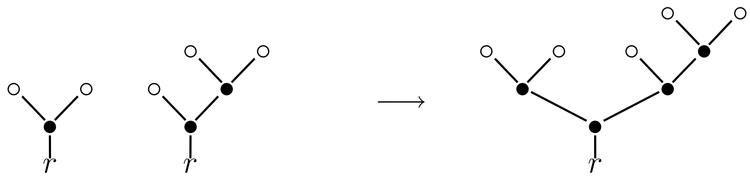
It follows that Theorem 4 applied to Proposition 8 gives (3), with the terms labeled by permutations w on the left hand side grouped according to the insertion tableau of w.

8 The qDGG on plane binary trees

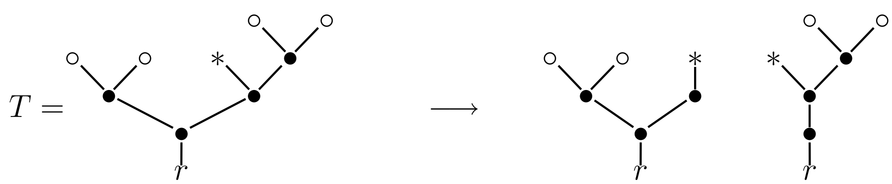
We assume the reader has some familiarity with the Hopf algebra of plane binary trees; see [AS, Section 1.1, Section 1.5] for further details. A *plane binary tree* (also called a rooted plane binary tree) is a tree T embedded into the plane which has three kinds of vertices: (a) a unique *root node* r which has exactly 1 child, (b) a number of *internal nodes* with two children, and (c) a number of *leaves* with no children. The leaves are numbered $\{0, 1, \ldots, n\}$ from left to right, where n is the number of internal nodes. Let $\mathcal{T}_n$ denote the set of plane binary trees with n internal nodes. By definition, $\mathcal{T}_0$ consists of the tree $\emptyset$, which has a root r, no internal nodes, and a single leaf 0. The cardinality $|\mathcal{T}_n|$ is equal to the Catalan number $\frac{(2n)!}{n!(n+1)!}$; see for example [AS, p.3].

We now describe a number of combinatorial operations on plane binary trees. Given two plane binary trees $T_1 \in \mathcal{T}_p$ and $T_2 \in \mathcal{T}_q$ we can *graft* a new plane binary tree $T_1 \vee T_2 \in \mathcal{T}_{p+q+1}$ by placing T_1 to the left of T_2 in the plane, identifying the two root nodes r_1 and

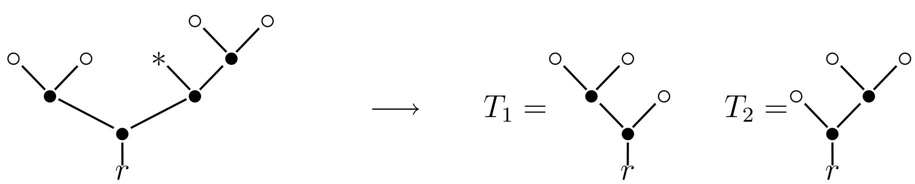
r_2 to form a new internal node, and attaching a new root to this internal node:



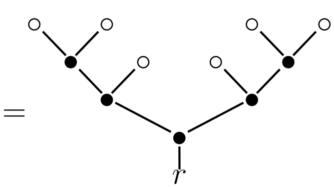
Given a tree $T \in \mathcal{T}_p$ and a position $i \in \{0, 1, \ldots, p\}$ indexing a leaf $v \in T$ we can *slice* T at v to obtain two trees $T_1 \in \mathcal{T}_i$ and $T_2 \in \mathcal{T}_{p-i}$ as follows: draw the unique path P from v to the root r. Then the edges of T weakly to the left of P form a rooted planar (but possibly non-binary) tree T_L, which can be transformed into a plane binary tree T_1 by removing internal nodes with less than two children, and merging some edges. Similarly the edges of T weakly to the right of P give a tree T_R that can be made into a plane binary tree T_2. Note that the number of internal nodes of T is equal to the sum of the number of internal nodes of T_1 and T_2. The following shows T_L and T_R for a tree T, when T is sliced at the $*$-ed leaf:



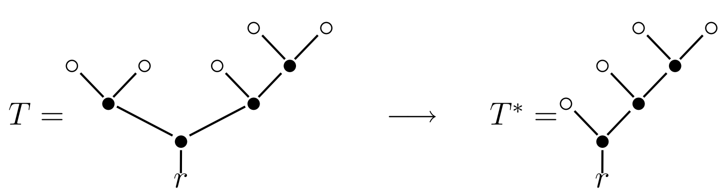
Thus the trees T_1 and T_2 obtained by slicing T at the $*$-ed leaf is given as follows:



We write $\mathrm{SG}(T, i) = T_1 \vee T_2$ to denote the composition of slicing and grafting, so in the above example $\mathrm{SG}(T, 2)$ would be the tree

$\mathrm{SG}(T, 2) =$ 

Given a non-empty tree $T \in \mathcal{T}_p$, we can obtain another tree $T^* \in \mathcal{T}_{p-1}$ from T by removing the leftmost (or 0) leaf v and erasing the node w which is joined to v:



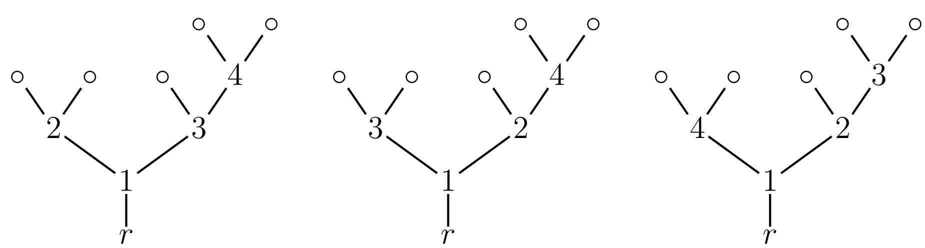
Let $V = \cup_{i \geqslant 0} \mathcal{T}_i$, with the obvious height function $h : V \to \mathbb{N}$. Define a graded graph Tree with vertex set V, and edges (T, T') whenever $T' = \mathrm{SG}(T, i)$ for some i; declare that $m(T', T) := q^i$. Define a graded graph Tree$'$ with vertex set V, and edges (T^*, T) for every $T \neq \emptyset$; declare that $m'(T^*, T) := 1$.

Proposition 10. *The pair* (Tree, Tree$'$) *is a* qDGG *with differential coefficient* $r = 1$*.*

Proof. Let $T \in \mathcal{T}_p$. Let $T' = \mathrm{SG}(T, i)$, where $i \in \{1, 2, \ldots, p\}$. Then it is not difficult to see that $(T')^* = \mathrm{SG}(T^*, i-1)$. This cancels out all the terms in $(D_{\mathrm{Tree}'} U_{\mathrm{Tree}} - q U_{\mathrm{Tree}} D_{\mathrm{Tree}'})T$ except for the one corresponding to $\mathrm{SG}(T, 0)^* = T$ which has coefficient $q^0 = 1$. $\square$

To describe the identity of Theorem 4 explicitly, let us define a linear extension of $T \in \mathcal{T}_p$ to be a bijective labeling $e : T \to \{1, 2, \ldots, p\}$ of the internal nodes of T with $\{1, 2, \ldots, p\}$, so that children are labeled with numbers bigger than those of their ancestors. Let $E(T)$ denote the set of linear extensions of T. Also, let us say that an internal node v is *to the left* (resp. *to the right*) of an internal node w if v belongs to the left (resp. right) branch and w belongs to the right (resp. left) branch of their closest (youngest) common ancestor.

If e is a linear extension of $T \in \mathcal{T}_p$, then we may define a permutation $w_e \in S_p$ by reading the labels of the internal nodes from left to right. It is well known (see for example [LR, Proposition 2.3]) that as T varies over $\mathcal{T}_p$ and e varies over $E(T)$ we obtain every $w \in S_p$ exactly once in this way. For example, the following are the three linear extensions of the same tree T:



corresponding to the permutations 2134, 3124, and 4123. Thus $f^T_{\mathrm{Tree}} = q + q^2 + q^3$.

Lemma 11. *Let* $T \in \mathcal{T}_p$*. Then*

$$f^T_{\mathrm{Tree}} = \sum_{e \in E(T)} q^{\mathrm{inv}(w_e)} \qquad \text{and} \qquad f^T_{\mathrm{Tree}'} = 1.$$

Proof. The claim for Tree$'$ is clear. For Tree, we will describe a bijection between $E(T)$ and paths from $\emptyset$ to T.

Let e' be a linear extension of T' and suppose that $T = T_1 \vee T_2$ is obtained from grafting a slice of T'. We may treat T_1 and T_2 as subtrees of T', and in particular restrict e' to T_1 and T_2. Thus we may define a labeling e (depending on e', T', T_1, and T_2) of T by declaring it to be equal to $e' + 1$ on $T_1 \cup T_2$, and equal to 1 on the new internal node present in T but absent in T'. It is straightforward to see that $e \in E(T)$. Conversely, given $e \in E(T)$, it is easy to recover e' and T' by comparing the labels along the leftmost branch of T_2 with the labels along the rightmost branch of T_1.

Recursively applying this procedure we obtain the desired bijection between $E(T)$ and paths from $\emptyset$ to T. Finally, the number of new inversions created in each step of this procedure is equal to the number of internal nodes of T_1, which in turn is the exponent of q in $m(T', T)$. This completes the proof. $\square$

Thus Theorem 4 for $(\mathrm{Tree}, \mathrm{Tree}')$ amounts to grouping together the terms of the left hand side of (3) into Catalan number many terms.

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