A combinatorial proof of a formula for Betti numbers of a stacked polytope

Suyoung Choi*
Department of Mathematical Sciences
KAIST, Republic of Korea
choisy@kaist.ac.kr
(Current) Department of Mathematics
Osaka City University, Japan
choi@sci.osaka-cu.ac.jp

Jang Soo Kim†
Department of Mathematical Sciences
KAIST, Republic of Korea
jskim@kaist.ac.kr
(Current) LIAFA
University of Paris 7, France

Submitted: Aug 8, 2009; Accepted: Dec 13, 2009; Published: Jan 5, 2010
Mathematics Subject Classifications: 05A15, 05E40, 05E45, 52B05

Abstract

For a simplicial complex Δ, the graded Betti number $\beta_{i,j}(\mathbf{k}[\Delta])$ of the Stanley-Reisner ring $\mathbf{k}[\Delta]$ over a field $\mathbf{k}$ has a combinatorial interpretation due to Hochster. Terai and Hibi showed that if Δ is the boundary complex of a d-dimensional stacked polytope with n vertices for $d \geqslant 3$, then $\beta_{k-1,k}(\mathbf{k}[\Delta]) = (k-1)\binom{n-d}{k}$. We prove this combinatorially.

1 Introduction

A *simplicial complex* Δ on a finite set V is a collection of subsets of V satisfying

1. if $v \in V$, then $\{v\} \in \Delta$,
2. if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$.

Each element $F \in \Delta$ is called a *face* of Δ. The *dimension* of F is defined by $\dim(F) = |F| - 1$. The *dimension* of Δ is defined by $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$. For a subset $W \subset V$, let Δ_W denote the simplicial complex $\{F \cap W : F \in \Delta\}$ on W.

*The research of the first author was carried out with the support of the Japanese Society for the Promotion of Science (JSPS grant no. P09023) and the Brain Korea 21 Project, KAIST.

†The second author was supported by the SRC program of Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (No. R11-2007-035-01002-0).

Let Δ be a simplicial complex on V. Two elements $v, u \in V$ are said to be *connected* if there is a sequence of vertices $v = u_0, u_1, \ldots, u_r = u$ such that $\{u_i, u_{i+1}\} \in \Delta$ for all $i = 0, 1, \ldots, r-1$. A *connected component* C of Δ is a maximal nonempty subset of V such that every two elements of C are connected.

Let $V = \{x_1, x_2, \ldots, x_n\}$ and let R be the polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$ over a fixed field $\mathbf{k}$. Then R is a graded ring with the standard grading $R = \oplus_{i \geqslant 0} R_i$. Let $R(-j) = \oplus_{i \geqslant 0}(R(-j))_i$ be the graded module over R with $(R(-j))_i = R_{j+i}$. The *Stanley-Reisner ring* $\mathbf{k}[\Delta]$ of Δ over $\mathbf{k}$ is defined to be R/I_Δ, where I_Δ is the ideal of R generated by the monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. A *finite free resolution* of $\mathbf{k}[\Delta]$ is an exact sequence

$$0 \longrightarrow F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} \mathbf{k}[\Delta] \longrightarrow 0\,, \tag{1}$$

where $F_i = \oplus_{j \geqslant 0} R(-j)^{\beta_{i,j}}$ and each ϕ_i is degree-preserving. A finite free resolution (1) is *minimal* if each $\beta_{i,j}$ is smallest possible. There is a minimal finite free resolution of $\mathbf{k}[\Delta]$ and it is unique up to isomorphism. If (1) is minimal, then the (i,j)*-th graded Betti number* $\beta_{i,j}(\mathbf{k}[\Delta])$ of $\mathbf{k}[\Delta]$ is defined to be $\beta_{i,j}(\mathbf{k}[\Delta]) = \beta_{i,j}$. Hochster's theorem says

$$\beta_{i,j}(\mathbf{k}[\Delta]) = \sum_{\substack{W \subset V \\ |W| = j}} \dim_{\mathbf{k}} \widetilde{H}_{j-i-1}(\Delta_W; \mathbf{k}).$$

We refer the reader to [1, 5] for the details of Betti numbers and Hochster's theorem. Since $\dim_{\mathbf{k}} \widetilde{H}_0(\Delta_W; \mathbf{k})$ is the number of connected components of Δ_W minus 1, we can interpret $\beta_{i-1,i}(\mathbf{k}[\Delta])$ in a purely combinatorial way.

Definition 1.1. Let Δ be a simplicial complex on a finite nonempty set V. Let k be a nonnegative integer. The *k-th special graded Betti number* $b_k(\Delta)$ of Δ is defined to be

$$b_k(\Delta) = \sum_{\substack{W \subset V \\ |W| = k}} (\mathrm{cc}(\Delta_W) - 1)\,, \tag{2}$$

where $\mathrm{cc}(\Delta_W)$ denotes the number of connected components of Δ_W.

Note that since there is no connected component in $\Delta_\emptyset = \{\emptyset\}$, we have $b_0(\Delta) = -1$. If $k > |V|$, then $b_k(\Delta) = 0$ because there is nothing in the sum in (2). Thus we have

$$b_k(\Delta) = \begin{cases} \beta_{k-1,k}(\mathbf{k}[\Delta]), & \text{if } k \geqslant 1, \\ -1, & \text{if } k = 0. \end{cases}$$

We refer the reader to [7] for the basic notions of convex polytopes. Let P be a simplicial polytope with vertex set V. The *boundary complex* $\Delta(P)$ is the simplicial complex Δ on V such that $F \in \Delta$ for some $F \subset V$ if and only if $F \neq V$ and the convex hull of F is a face of P. Note that if the dimension of P is d, then $\dim(\Delta(P)) = d - 1$.

For a d-dimensional simplicial polytope P, we can attach a d-dimensional simplex to a facet of P. A *stacked polytope* is a simplicial polytope obtained in this way starting with a d-dimensional simplex.

Let P be a d-dimensional stacked polytope with n vertices. Hibi and Terai [6] showed that $\beta_{i,j}(\mathbf{k}[\Delta(P)]) = 0$ unless $i = j-1$ or $i = j-d+1$. Since $\beta_{i-1,i}(\mathbf{k}[\Delta(P)]) = \beta_{n-i-d+1,n-i}(\mathbf{k}[\Delta(P)])$, it is sufficient to determine $\beta_{i-1,i}(\mathbf{k}[\Delta(P)])$ to find all $\beta_{i,j}(\mathbf{k}[\Delta(P)])$. In the same paper, they found the following formula for $\beta_{k-1,k}(\mathbf{k}[\Delta(P)])$:

$$\beta_{k-1,k}(\mathbf{k}[\Delta(P)]) = (k-1)\binom{n-d}{k}. \tag{3}$$

Herzog and Li Marzi [4] gave another proof of (3).

The main purpose of this paper is to prove (3) combinatorially. In the meanwhile, we get as corollaries the results of Bruns and Hibi [2] : a formula of $b_k(\Delta)$ if Δ is a tree (or a cycle) considered as a 1-dimensional simplicial complex.

2 Definition of t-connected sum

In this section we define a t-connected sum of simplicial complexes, which gives another equivalent definition of the boundary complex of a stacked polytope. See [3] for the details of connected sums. And then, we extend the definition of t-connected sum to graphs, which has less restrictions on the construction. Every graph in this paper is simple.

2.1 A t-connected sum of simplicial complexes

Let V and V' be finite sets. A *relabeling* is a bijection $\sigma : V \to V'$. If Δ is a simplicial complex on V, then $\sigma(\Delta) = \{\sigma(F) : F \in \Delta\}$ is a simplicial complex on V'.

Definition 2.1. Let Δ_1 and Δ_2 be simplicial complexes on V_1 and V_2 respectively. Let $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$ be maximal faces with $|F_1| = |F_2|$. Let V_2' be a finite set and $\sigma : V_2 \to V_2'$ a relabeling such that $V_1 \cap V_2' = F_1$ and $\sigma(F_2) = F_1$. Then the *connected sum* $\Delta_1 \#_\sigma^{F_1,F_2} \Delta_2$ of Δ_1 and Δ_2 with respect to (F_1, F_2, σ) is the simplicial complex $(\Delta_1 \cup \sigma(\Delta_2)) \setminus \{F_1\}$ on $V_1 \cup V_2'$. If $\Delta = \Delta_1 \#_\sigma^{F_1,F_2} \Delta_2$ and $|F_1| = |F_2| = t$, then we say that Δ is a *t-connected sum* of Δ_1 and Δ_2.

Note that if Δ_1 and Δ_2 are $(d-1)$-dimensional pure simplicial complexes, i.e., the dimension of each maximal face is $d-1$, then we can only define a d-connected sum of them.

Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be simplicial complexes. A simplicial complex Δ is said to be a t-connected sum of $\Delta_1, \ldots, \Delta_n$ if there is a sequence of simplicial complexes $\Delta_1', \Delta_2', \ldots, \Delta_n'$ such that $\Delta_1' = \Delta_1$, Δ_i' is a t-connected sum of Δ_{i-1}' and Δ_i for $i = 2, 3, \ldots, n$, and $\Delta_n' = \Delta$.

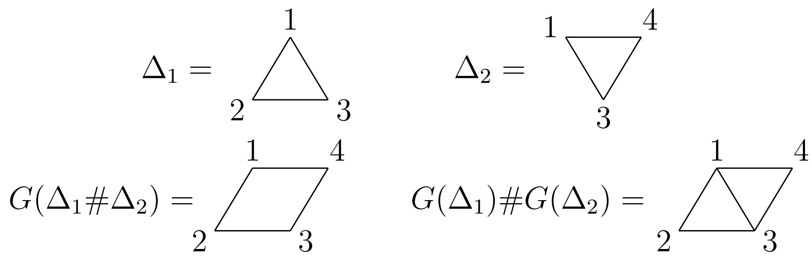

Figure 1: The 1-skeleton of a 2-connected sum of Δ_1 and Δ_2 is not a 2-connected sum of $G(\Delta_1)$ and $G(\Delta_2)$.

2.2 A t-connected sum of graphs

Let G be a graph with vertex set V and edge set E. Let $W \subset V$. Then the *induced subgraph* $G|_W$ of G with respect to W is the graph with vertex set W and edge set $\{\{x, y\} \in E : x, y \in W\}$. Let

$$b_k(G) = \sum_{\substack{W \subset V \\ |W|=k}} (\mathrm{cc}(G|_W) - 1)\,,$$

where $\mathrm{cc}(G|_W)$ denotes the number of connected components of $G|_W$.

Let Δ be a simplicial complex on V. The 1-*skeleton* $G(\Delta)$ of Δ is the graph with vertex set V and edge set $E = \{F \in \Delta : |F| = 2\}$. By definition, the connected components of Δ_W and $G(\Delta)|_W$ are identical for all $W \subset V$. Thus $b_k(\Delta) = b_k(G(\Delta))$.

Now we define a t-connected sum of two graphs.

Definition 2.2. Let G_1 and G_2 be graphs with vertex sets V_1 and V_2, and edge sets E_1 and E_2 respectively. Let $F_1 \subset V_1$ and $F_2 \subset V_2$ be sets of vertices such that $|F_1| = |F_2|$, and $G_1|_{F_1}$ and $G_2|_{F_2}$ are complete graphs. Let V_2' be a finite set and $\sigma : V_2 \to V_2'$ a relabeling such that $V_1 \cap V_2' = F_1$ and $\sigma(F_2) = F_1$. Then the *connected sum* $G_1 \#_{\sigma}^{F_1,F_2} G_2$ of G_1 and G_2 with respect to (F_1, F_2, σ) is the graph with vertex set $V_1 \cup V_2'$ and edge set $E_1 \cup \sigma(E_2)$, where $\sigma(E_2) = \{\{\sigma(x), \sigma(y)\} : \{x, y\} \in E_2\}$. If $G = G_1 \#_{\sigma}^{F_1,F_2} G_2$ and $|F_1| = |F_2| = t$, then we say that G is a *t-connected sum* of G_1 and G_2.

Note that in contrary to the definition of t-connected sum of simplicial complexes, it is not required that F_1 and F_2 are maximal, and we do not remove any element in $E_1 \cup \sigma(E_2)$. We define a t-connected sum of $G_1, G_2, \ldots, G_n$ as we did for simplicial complexes.

It is easy to see that, if $|F_1| = |F_2| \geqslant 3$, then $G(\Delta_1 \#_{\sigma}^{F_1,F_2} \Delta_2) = G(\Delta_1) \#_{\sigma}^{F_1,F_2} G(\Delta_2)$. Thus we get the following proposition.

Proposition 2.3. *For $t \geqslant 3$, if Δ is a t-connected sum of $\Delta_1, \Delta_2, \ldots, \Delta_n$, then $G(\Delta)$ is a t-connected sum of $G(\Delta_1), G(\Delta_2), \ldots, G(\Delta_n)$.*

Note that Proposition 2.3 is not true if $t = 2$ as the following example shows.

Example 2.4. Let $\Delta_1 = \{12, 23, 13\}$ and $\Delta_2 = \{13, 34, 14\}$ be simplicial complexes on $V_1 = \{1, 2, 3\}$ and $V_2 = \{1, 3, 4\}$. Here 12 means the set $\{1, 2\}$. Let $F_1 = F_2 = \{1, 3\}$ and let σ be the identity map from V_2 to itself. Then the edge set of $G(\Delta_1 \#_{\sigma}^{F_1,F_2} \Delta_2)$ is $\{12, 23, 34, 14\}$, but the edge set of $G(\Delta_1) \#_{\sigma}^{F_1,F_2} G(\Delta_2)$ is $\{12, 23, 34, 14, 13\}$. See Figure 1.

3 Main results

In this section we find a formula of $b_k(G)$ for a graph G which is a t-connected sum of two graphs. To do this let us introduce the following notation. For a graph G with vertex set V, let

$$c_k(G) = \sum_{\substack{W \subset V \\ |W|=k}} \mathrm{cc}(G|_W).$$

Note that $c_k(G) = b_k(G) + \binom{|V|}{k}$.

Lemma 3.1. *Let G_1 and G_2 be graphs with n_1 and n_2 vertices respectively. Let t be a positive integer and let G be a t-connected sum of G_1 and G_2. Then*

$$\begin{aligned} c_k(G) = &\sum_{i=0}^{k} \left(c_i(G_1)\binom{n_2-t}{k-i} + c_i(G_2)\binom{n_1-t}{k-i} \right) \\ &- \binom{n_1+n_2-t}{k} + \binom{n_1+n_2-2t}{k}. \end{aligned}$$

Proof. Let V_1 (resp. V_2) be the vertex set of G_1 (resp. G_2). We have $G = G_1 \#_\sigma^{F_1,F_2} G_2$ for some $F_1 \subset V_1$, $F_2 \subset V_2$, a vertex set V_2' and a relabeling $\sigma : V_1 \to V_2'$ such that $V_1 \cap V_2' = F_1$, $\sigma(F_2) = F_1$, and $G_1|_{F_1}$ and $G_2|_{F_2}$ are complete graphs on t vertices.

Let A be the set of pairs (C, W) such that $W \subset V_1 \cup V_2'$, $|W| = k$ and C is a connected component of $G|_W$. Let

$$A_1 = \{(C,W) \in A : C \cap V_1 \neq \emptyset\}, \quad A_2 = \{(C,W) \in A : C \cap V_2' \neq \emptyset\}.$$

Then $c_k(G) = |A| = |A_1| + |A_2| - |A_1 \cap A_2|$. It is sufficient to show that $|A_1| = \sum_{i=0}^k c_i(G_1)\binom{n_2-t}{k-i}$, $|A_2| = \sum_{i=0}^k c_i(G_2)\binom{n_1-t}{k-i}$ and $|A_1 \cap A_2| = \binom{n_1+n_2-t}{k} - \binom{n_1+n_2-2t}{k}$.

Let B_1 be the set of triples (C_1, W_1, X) such that $W_1 \subset V_1$, $X \subset V_2' \setminus V_1$, $|X| + |W_1| = k$ and C_1 is a connected component of $G_1|_{W_1}$. Let $\phi_1 : A_1 \to B_1$ be the map defined by $\phi_1(C,W) = (C \cap V_1, W \cap V_1, W \setminus V_1)$. Then ϕ_1 has the inverse map defined as follows. For a triple $(C_1, W_1, X) \in B_1$, $\phi_1^{-1}(C_1, W_1, X) = (C, W)$, where $W = W_1 \cup X$ and C is the connected component of $G|_W$ containing C_1. Thus ϕ_1 is a bijection and we get $|A_1| = |B_1| = \sum_{i=0}^k c_i(G_1)\binom{n_2-t}{k-i}$. Similarly we get $|A_2| = \sum_{i=0}^k c_i(G_2)\binom{n_1-t}{k-i}$.

Now let $B = \{W \subset V_1 \cup V_2' : W \cap F_1 \neq \emptyset\}$. Let $\psi : A_1 \cap A_2 \to B$ be the map defined by $\psi(C, W) = W$. We have the inverse map ψ^{-1} as follows. For $W \in B$, $\psi^{-1}(W) = (C, W)$, where C is the connected component of $G|_W$ containing $W \cap F_1$, which is guaranteed to exist since $G|_{F_1} = G_1|_{F_1}$ is a complete graph. Thus ψ is a bijection, and we get $|A_1 \cap A_2| = |B| = \binom{n_1+n_2-t}{k} - \binom{n_1+n_2-2t}{k}$. □

Theorem 3.2. *Let G_1 and G_2 be graphs with n_1 and n_2 vertices respectively. Let t be a positive integer and let G be a t-connected sum of G_1 and G_2. Then*

$$b_k(G) = \sum_{i=0}^{k} \left(b_i(G_1)\binom{n_2-t}{k-i} + b_i(G_2)\binom{n_1-t}{k-i} \right) + \binom{n_1+n_2-2t}{k}.$$

Proof. Since $c_k(G) = b_k(G) + \binom{n_1+n_2-t}{k}$, $c_i(G_1) = b_i(G_1) + \binom{n_1}{i}$ and $c_i(G_2) = b_i(G_2) + \binom{n_2}{i}$, by Lemma 3.1, it is sufficient to show that

$$2\binom{n_1+n_2-t}{k} = \sum_{i=0}^{k}\left(\binom{n_1}{i}\binom{n_2-t}{k-i} + \binom{n_2}{i}\binom{n_1-t}{k-i}\right),$$

which is immediate from the identity $\sum_{i=0}^{k}\binom{a}{i}\binom{b}{k-i} = \binom{a+b}{k}$. □

Recall that a t-connected sum G of two graphs depends on the choice of vertices of each graph and the identification of the chosen vertices. However, Theorem 3.2 says that $b_k(G)$ does not depend on them. Thus we get the following important property of a t-connected sum of graphs.

Corollary 3.3. *Let t be a positive integer and let G be a t-connected sum of graphs $G_1, G_2, \ldots, G_n$. If H is also a t-connected sum of $G_1, G_2, \ldots, G_n$, then $b_k(G) = b_k(H)$ for all k.*

Using Proposition 2.3, we get a formula for the special graded Betti number of a t-connected sum of two simplicial complexes for $t \geqslant 3$.

Corollary 3.4. *Let Δ_1 and Δ_2 be simplicial complexes on V_1 and V_2 respectively with $|V_1| = n_1$ and $|V_2| = n_2$. Let t be a positive integer and let Δ be a t-connected sum of Δ_1 and Δ_2. If $t \geqslant 3$, then*

$$b_k(\Delta) = \sum_{i=0}^{k}\left(b_i(\Delta_1)\binom{n_2-t}{k-i} + b_i(\Delta_2)\binom{n_1-t}{k-i}\right) + \binom{n_1+n_2-2t}{k}.$$

For an integer n, let K_n denote a complete graph with n vertices.

Let G be a graph with vertex set V. If H is a t-connected sum of G and K_{t+1}, then H is a graph obtained from G by adding a new vertex v connected to all vertices in W for some $W \subset V$ such that $G|_W$ is isomorphic to K_t. Thus H is determined by choosing such a subset $W \subset V$. Using this observation, we get the following lemma.

Theorem 3.5. *Let t be a positive integer. Let G be a t-connected sum of n K_{t+1}'s. Then*

$$b_k(G) = (k-1)\binom{n}{k}.$$

Proof. We construct a sequence of graphs $H_1, \ldots, H_n$ as follows. Let H_1 be the complete graph with vertex set $\{v_1, v_2, \ldots, v_{t+1}\}$. For $i \geqslant 2$, let H_i be the graph obtained from H_{i-1} by adding a new vertex v_{t+i} connected to all vertices in $\{v_1, v_2, \ldots, v_t\}$. Then H_n is a t-connected sum of n K_{t+1}'s, and we have $b_k(G) = b_k(H_n)$ by Corollary 3.3. In H_n, the vertex v_i is connected to all the other vertices for $i \leqslant t$, and v_j and $v_{j'}$ are not connected to each other for all $t+1 \leqslant j, j' \leqslant t+n$. Thus $b_k(H_n) = (k-1)\binom{n}{k}$. □

Observe that every tree with $n+1$ vertices is a 1-connected sum of n K_2's. Thus we get the following nontrivial property of trees which was observed by Bruns and Hibi [2].

Corollary 3.6. *[2, Example 2.1. (b)] Let T be a tree with $n+1$ vertices. Then $b_k(T)$ does not depend on the specific tree T. We have*

$$b_k(T) = (k-1)\binom{n}{k}.$$

Corollary 3.7. *[2, Example 2.1. (c)] Let G be an n-gon. If $k = n$, then $b_k(G) = 0$; otherwise,*

$$b_k(G) = \frac{n(k-1)}{n-k}\binom{n-2}{k}.$$

Proof. It is clear for $k = n$. Assume $k < n$. Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of G. Then

$$\begin{aligned}(n-k)\cdot b_k(G) &= \sum_{\substack{W\subset V\\ |W|=k}} (\mathrm{cc}(G|_W) - 1) \sum_{v\in V\setminus W} 1 \\ &= \sum_{v\in V} \sum_{\substack{W\subset V\setminus\{v\}\\ |W|=k}} (\mathrm{cc}(G|_W) - 1) \\ &= \sum_{v\in V} b_k(G|_{V\setminus\{v\}}).\end{aligned}$$

Since each $G|_{V\setminus\{v\}}$ is a tree with $n-1$ vertices, we are done by Corollary 3.6. □

Remark 3.8. Bruns and Hibi [2] obtained Corollary 3.6 and Corollary 3.7 by showing that if Δ is a tree (or an n-gon), considered as a 1-dimensional simplicial complex, then $\mathbf{k}[\Delta]$ has a pure resolution. Since $\mathbf{k}[\Delta]$ is Cohen-Macaulay and it has a pure resolution, the Betti numbers are determined by its type (c.f. [1]).

Now we can prove (3). Note that, for $d \geqslant 3$, if P is a d-dimensional simplicial polytope and Q is a simplicial polytope obtained from P by attaching a d-dimensional simplex S to a facet of P, then $\Delta(Q)$ is a d-connected sum of $\Delta(P)$ and $\Delta(S)$, and thus the 1-skeleton $G(\Delta(Q))$ is a d-connected sum of $G(\Delta(P))$ and K_{d+1}. Hence the 1-skeleton of the boundary complex of a d-dimensional stacked polytope is a d-connected sum of K_{d+1}'s.

Theorem 3.9. *Let P be a d-dimensional stacked polytope with n vertices. If $d \geqslant 3$, then*

$$b_k(\Delta(P)) = (k-1)\binom{n-d}{k}.$$

If $d = 2$, then

$$b_k(\Delta(P)) = \begin{cases} 0, & \text{if } k = n, \\ \frac{n(k-1)}{n-k}\binom{n-2}{k}, & \text{otherwise.}\end{cases}$$

Proof. Assume $d \geqslant 3$. Then the 1-skeleton $G(\Delta(P))$ is a d-connected sum of $n-d$ K_{d+1}'s. Thus by Theorem 3.5, we get $b_k(\Delta(P)) = b_k(G(\Delta(P))) = (k-1)\binom{n-d}{k}$.

Now assume $d = 2$. Then $G(\Delta(P))$ is an n-gon. Thus by Corollary 3.7 we are done. □

References

[1] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[2] Winfried Bruns and Takayuki Hibi. Cohen-Macaulay partially ordered sets with pure resolutions. *European J. Combin.*, 19(7):779–785, 1998.

[3] Victor M. Buchstaber and Taras E. Panov. *Torus actions and their applications in topology and combinatorics*, volume 24 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.

[4] Jürgen Herzog and Enzo Maria Li Marzi. Bounds for the Betti numbers of shellable simplicial complexes and polytopes. In *Commutative algebra and algebraic geometry (Ferrara)*, volume 206 of *Lecture Notes in Pure and Appl. Math.*, pages 157–167. Dekker, New York, 1999.

[5] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[6] Naoki Terai and Takayuki Hibi. Computation of Betti numbers of monomial ideals associated with stacked polytopes. *Manuscripta Math.*, 92(4):447–453, 1997.

[7] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.