

An optimal strongly identifying code in the infinite triangular grid

Iiro Honkala*

Department of Mathematics
University of Turku
20014 Turku, Finland

`honkala@utu.fi`

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Abstract

Assume that $G = (V, E)$ is an undirected graph, and $C \subseteq V$. For every $\mathbf{v} \in V$, we denote by $I(\mathbf{v})$ the set of all elements of C that are within distance one from \mathbf{v} . If the sets $I(\mathbf{v}) \setminus \{\mathbf{v}\}$ for $\mathbf{v} \in V$ are all nonempty, and, moreover, the sets $\{I(\mathbf{v}), I(\mathbf{v}) \setminus \{\mathbf{v}\}\}$ for $\mathbf{v} \in V$ are disjoint, then C is called a strongly identifying code. The smallest possible density of a strongly identifying code in the infinite triangular grid is shown to be $6/19$.

Keywords: Graph, identifying code, triangular grid, density.

1 Introduction

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E . A subset $C \subseteq V$ is called a **code** in G , and its elements are called **codewords**.

The distance $d(\mathbf{u}, \mathbf{v})$ between two vertices \mathbf{u} and \mathbf{v} is the number of edges on any shortest path between them.

For all $\mathbf{v} \in V$ we denote

$$I(\mathbf{v}) = \{\mathbf{c} \in C : d(\mathbf{c}, \mathbf{v}) \leq 1\}.$$

If we denote by $B_r(\mathbf{v})$ the ball of radius r with centre \mathbf{v} , then $I(\mathbf{v}) = C \cap B_1(\mathbf{v})$.

If all the sets $I(\mathbf{v})$ are nonempty and pairwise different, then C is called an **identifying code**. This concept was introduced in [8] in connection with studying multiprocessor

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architectures. Such an architecture can be viewed as a graph, where each vertex represents a processor, and each edge represents a dedicated link between two processors. Assume that at most one of the processors is malfunctioning. Each of the chosen codewords \mathbf{c} tests the sets $B_1(\mathbf{c})$ and reports YES if it detects a problem and NO otherwise. The fact that C is identifying implies that based on the reports, we can uniquely identify the one malfunctioning processor or tell that everything is fine.

Strongly identifying codes were introduced in [7] (in a more general form); cf. also [14].

Definition 1. *A code C in the graph $G = (V, E)$ is called **strongly identifying** if all the sets $I(\mathbf{v}) \setminus \{\mathbf{v}\}$ for $\mathbf{v} \in V$ are nonempty, and, moreover, the sets $\{I(\mathbf{v}), I(\mathbf{v}) \setminus \{\mathbf{v}\}\}$ for $\mathbf{v} \in V$ are disjoint.*

Here the idea is that a malfunctioning processor may or may not be able to send a correct report, and we need a slightly stronger code. Clearly, a strongly identifying code is also identifying.

Strongly identifying codes have also been studied in [9] and [10].

The concept of a **locating-dominating** set introduced by Slater [13] (see also [3]) is closely related to that of identifying codes.

Identifying codes and locating-dominating codes have been extensively studied: see the Internet bibliography [11] maintained by Antoine Lobstein. For results on the triangular grid, see, e.g., [1], [2], [4], [6] and [5].

In the square grid \mathbb{Z}^2 it is easy to see that the smallest possible density of a strongly identifying code equals $2/5$. Indeed, the code $\{(x, y) \in \mathbb{Z}^2 : x \equiv 1 \text{ or } 3 \pmod{5}\}$ is strongly identifying and has density $2/5$. The lower bound $2/5$ on the density is an immediate corollary of [12, Theorem 14].

In the hexagonal mesh the smallest possible density of a strongly identifying code equals $1/2$. Indeed, if we delete all the vertical edges in the hexagonal mesh, and take as codewords all the vertices on every second of the resulting (infinitely many) doubly infinite paths, we clearly get a strongly identifying code with density $1/2$. The lower bound on the density is an immediate corollary of [12, Theorem 14].

From now on we consider the infinite triangular grid. The vertex set of the infinite triangular grid T is $V = \{v(i, j) : i, j \in \mathbb{Z}\}$, where

$$v(i, j) = i(1, 0) + j\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

and two vertices are adjacent if their Euclidean distance is 1. Denote by T_n the set of vertices $v(i, j)$ with $|i| \leq n$ and $|j| \leq n$. The density of a code C in T is defined to be

$$D(C) = \limsup_{n \rightarrow \infty} \frac{|C \cap T_n|}{|T_n|}.$$

The smallest density of an identifying code in the infinite triangular grid is $1/4$; see [8].

It is easy to check that the code given in Figure 1 is strongly identifying and has density of $6/19$. We always denote codewords by black dots. The code is obtained as a doubly periodic tiling and the tile is shown in the figure.

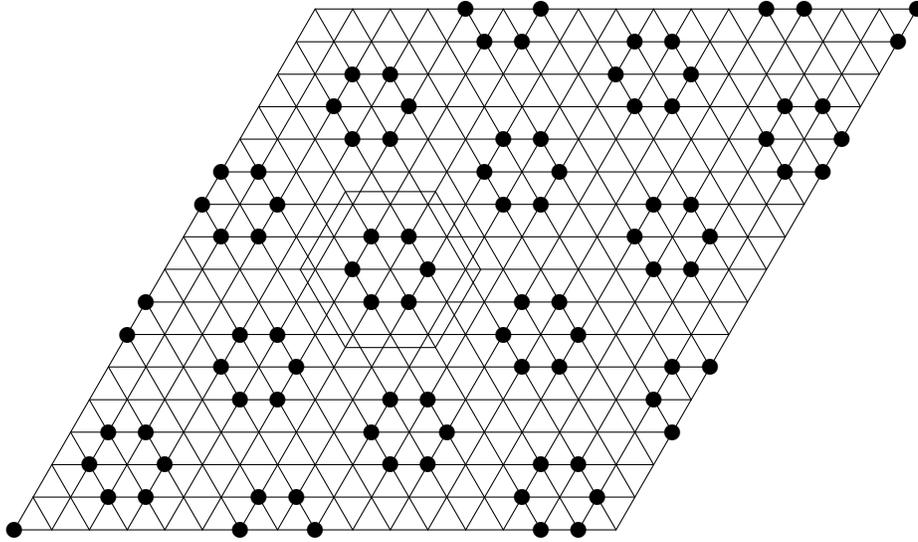


Figure 1: A strongly identifying code with density $6/19$.

The purpose of this note is to prove that this code is optimal, i.e., the density of every strongly identifying code in the infinite triangular grid is at least $6/19$.

2 The proof

From now on we assume that C is a strongly identifying code in T .

Denote

$$C_i = \{\mathbf{c} \in C : |I(\mathbf{c})| = i\},$$

and

$$C_{\geq j} = \bigcup_{i \geq j} C_i.$$

We also denote

$$N_i = \{\mathbf{v} \notin C : |I(\mathbf{v})| = i\},$$

and

$$N_{\geq j} = \bigcup_{i \geq j} N_i.$$

Trivially, $C_0 = C_1 = \emptyset$.

Following Slater [14] we define the **share** of a codeword $\mathbf{c} \in C$ — which we denote by $s(\mathbf{c})$ — by the formula

$$s(\mathbf{c}) = \sum_{\mathbf{v} \in B_1(\mathbf{c})} \frac{1}{|I(\mathbf{v})|}.$$

We now introduce a voting scheme using which we perform an averaging over the shares of the codewords.

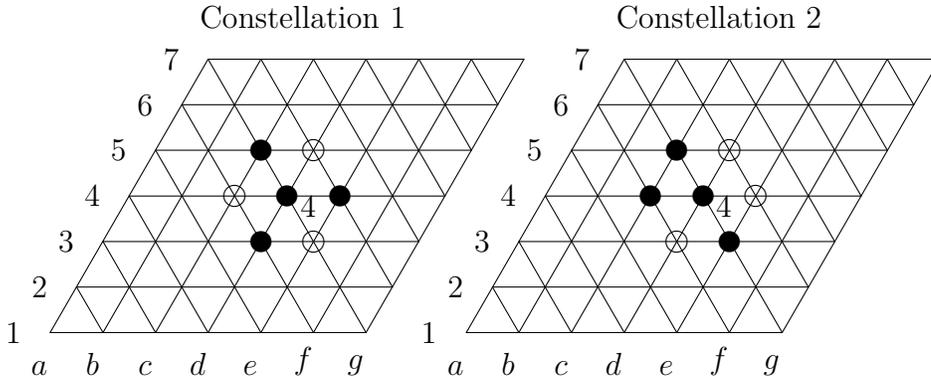


Figure 2: Constellations 1 and 2.

Rule 1: Every codeword in $C_{\geq 4}$ gives $1/3$ of a vote to every codeword neighbour in C_2 .

In fact, no codeword can have more than one codeword neighbour that belongs to C_2 : if $\mathbf{a} \in C_2$ and $\mathbf{b} \in C_2$ were two different codeword neighbours of $\mathbf{c} \in C$, then

$$I(\mathbf{a}) \setminus \{\mathbf{a}\} = \{\mathbf{c}\} = I(\mathbf{b}) \setminus \{\mathbf{b}\},$$

and the code would not be strongly identifying.

Rule 2: Every codeword $\mathbf{c} \in C_{\geq 4}$ gives $1/12$ of a vote to every codeword neighbour in C_3 and all their codeword neighbours in C_3 (i.e., if S is the subset of codewords of C_3 referred to above, then each element of S gets $1/12$ of a vote from \mathbf{c}).

Clearly, if $\mathbf{c} \in C_{\geq 4}$ and $\mathbf{a} \in C_3$ is its codeword neighbour, then \mathbf{a} can have at most one codeword neighbour in C_3 .

These two rules describe the voting behaviour of the vertices in $C_{\geq 4}$.

Lemma 1. *If $\mathbf{c} \in C_{\geq 4}$, then $s(\mathbf{c}) \leq 19/6$ and \mathbf{c} gives at most $19/6 - s(\mathbf{c})$ votes.*

Proof. Consider first the vertices $\mathbf{c} \in C_4$. There are essentially three different cases.

Assume first that none of the codeword neighbours of \mathbf{c} are adjacent. Without loss of generality, \mathbf{c} is the vertex d4 of Constellation 1 in Figure 2. As we already remarked after introducing Rule 1, at most one of the codewords c5, d3 and e4 is in C_2 , and the others are in $C_{\geq 3}$. The vertices c4, d5 and e3 are all in $N_{\geq 3}$, and therefore

$$s(\mathbf{c}) \leq \frac{1}{4} + 5 \cdot \frac{1}{3} + \frac{1}{2} = \frac{29}{12},$$

and $19/6 - s(\mathbf{c}) \geq 9/12$. The number of votes given by \mathbf{c} is clearly at most $1/3 + 4 \cdot 1/12 = 8/12 \leq 9/12$ as claimed.

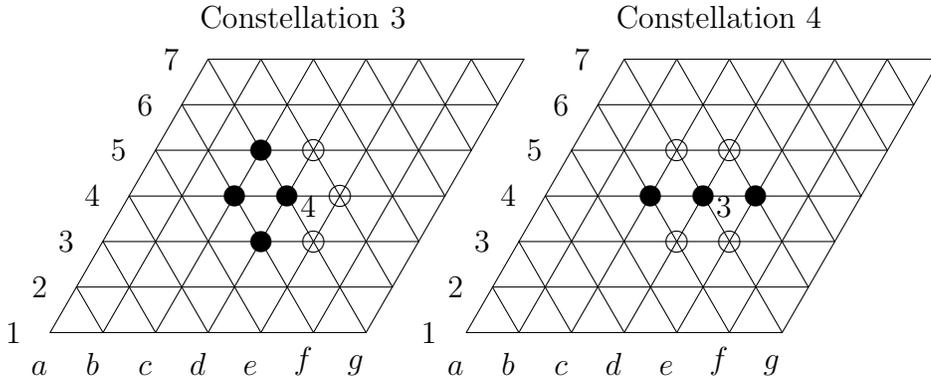


Figure 3: Constellations 3 and 4.

Assume then that \mathbf{c} is like d4 in Constellation 2 in Figure 2. Then $c4, c5 \in C_{\geq 3}$, and at least one of the vertices $c4$ and $c5$ is in $C_{\geq 4}$ (otherwise $I(c4) = I(c5) = \{c4, c5, d4\}$). If $e3 \in C_2$, then $e4 \in N_{\geq 3}$ (otherwise $I(e3) = I(e4) = \{e3, d4\}$); and for the same reason if $e4 \in N_2$, then $e3 \in C_{\geq 3}$. Anyway,

$$s(\mathbf{c}) \leq 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = \frac{5}{2},$$

and $19/6 - s(\mathbf{c}) \geq 2/3$. The number of votes given by \mathbf{c} is clearly at most $2/3$.

Assume then that \mathbf{c} is like the vertex d4 in Constellation 3 in Figure 3. It is possible that $e4 \in N_1$. However, at least one of the codewords d3 and c5 belongs to $C_{\geq 4}$: if both of them were in C_3 , then $I(d3) \setminus \{d3\} = I(c5) \setminus \{c5\}$, contradicting the fact that C is a strongly identifying code. This implies that

$$s(\mathbf{c}) \leq 3 \cdot \frac{1}{4} + \frac{1}{3} + 2 \cdot \frac{1}{2} + 1 = \frac{37}{12},$$

and $19/6 - s(\mathbf{c}) \geq 1/12$, but it also implies that \mathbf{c} gives at most $1/12$ of a vote: if neither $c5$ nor $d3$ is in C_3 , then \mathbf{c} gives no votes at all; if $c5$, say, is in C_3 , then it gets $1/12$ of a vote from \mathbf{c} , but neither of the codeword neighbours of $c5$ is in C_3 .

If $\mathbf{c} \in C_{\geq 5}$, then trivial calculations show that in all cases $s(\mathbf{c}) \leq 5/2$ and that \mathbf{c} gives at most $2/3$ of a vote. \square

The final three voting rules tell how the vertices of C_3 vote.

Rule 3: If a codeword $\mathbf{c} \in C_3$ has a codeword neighbour $\mathbf{a} \in C_2$, then \mathbf{c} gives $1/3$ of a vote to \mathbf{a} , if \mathbf{c} and its two codeword neighbours are collinear, and $1/4$ of a vote, otherwise.

Rule 4: Assume that $\mathbf{c} \in C_3$ and that \mathbf{c} has a codeword neighbour that belongs to C_2 or that \mathbf{c} and its two codewords neighbours are collinear. Then \mathbf{c} gives $1/12$ of a vote to every codeword neighbour in C_3 and all their codeword neighbours in $C_3 \setminus \{\mathbf{c}\}$.

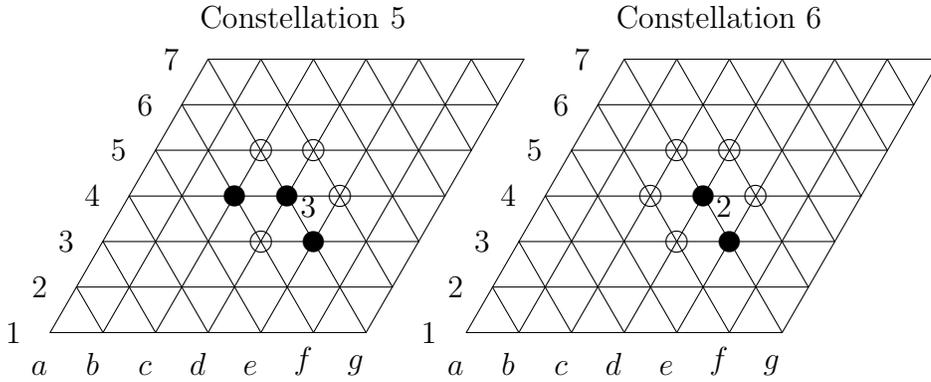


Figure 4: Constellations 5 and 6.

Rule 5: Assume that \mathbf{c} and its two codeword neighbours all belong to C_3 but are not collinear. If the share of \mathbf{c} is at most $37/12$ and the share of exactly one of its codeword neighbours is bigger than $19/6$, then that codeword gets $1/12$ of a vote from \mathbf{c} .

Lemma 2. Assume that $\mathbf{c} \in C_3$ and none of its neighbours belongs to C_2 , and that \mathbf{c} gives votes. Then $s(\mathbf{c}) \leq 19/6$ and \mathbf{c} gives at most $19/6 - s(\mathbf{c})$ votes.

Proof. The case when \mathbf{c} gives votes according to Rule 5 is trivial; so assume that \mathbf{c} does not give any votes according to Rule 5. Then \mathbf{c} gives votes according to Rule 4, and hence \mathbf{c} and its two codeword neighbours $\mathbf{a} \in C_{\geq 3}$ and $\mathbf{b} \in C_{\geq 3}$ are collinear. Without loss of generality, \mathbf{c} is the vertex d4 in Constellation 4 in Figure 3. Then at least one of the vertices d3 and c5 belongs to $N_{\geq 3}$; likewise at least one of the vertices e3 and d5 belongs to $N_{\geq 3}$. Consequently,

$$s(\mathbf{c}) \leq 5 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = \frac{8}{3},$$

and $19/6 - s(\mathbf{c}) \geq 1/2$. According to Rule 4, \mathbf{c} gives $1/12$ of a vote to at most four vertices, and the claim is clear. \square

Lemma 3. Assume that $\mathbf{c} \in C_3$ has a neighbour $\mathbf{a} \in C_2$. Then $s(\mathbf{c}) \leq 19/6$ and \mathbf{c} gives at most $19/6 - s(\mathbf{c})$ votes.

Proof. Assume first that \mathbf{c} and its two codeword neighbours are collinear. Without loss of generality, \mathbf{c} is again the vertex d4 in Constellation 4 and c4 belongs to C_2 (and then obviously e4 is in $C_{\geq 3}$). Then both d3 and c5 belong to $N_{\geq 3}$; and at least one of the vertices e3 and d5 belongs to $N_{\geq 3}$. But again we see that $s(\mathbf{c}) \leq 8/3$, and $19/6 - s(\mathbf{c}) \geq 1/2$, and the total number of votes given by \mathbf{c} is at most $1/3 + 2 \cdot 1/12 = 1/2$.

Assume second that \mathbf{c} and its two codeword neighbours are not collinear. Without loss of generality \mathbf{c} is the vertex d4 in Constellation 5 in Figure 4, and c4 belongs to C_2 . Again, e3 is in $C_{\geq 3}$. Because C is strongly identifying, we know that $I(c4) \setminus \{c4\} \neq I(d5)$,

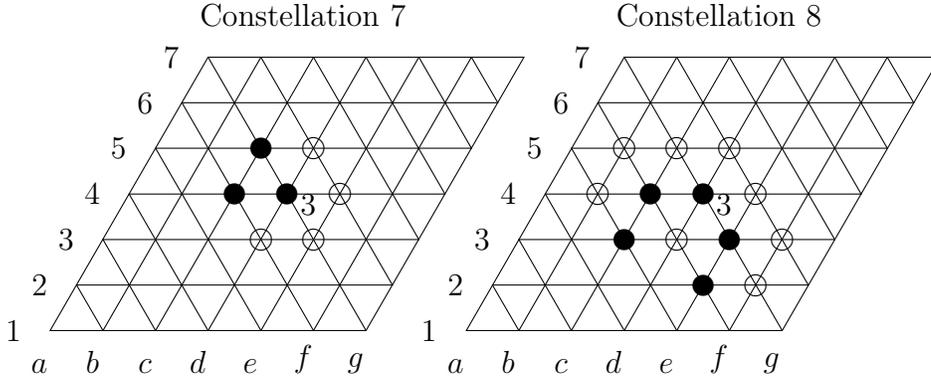


Figure 5: Constellations 7 and 8.

and therefore $d5 \in N_{\geq 2}$. Because $I(c5) \neq I(c4)$, we know that $c5 \in N_{\geq 3}$; and because $I(d3) \neq I(d4)$, we know that $d3 \in N_{\geq 4}$. All in all,

$$s(\mathbf{c}) \leq \frac{1}{4} + 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} = \frac{11}{4},$$

and $19/6 - s(\mathbf{c}) \geq 5/12$. According to Rules 3 and 4 the vertex \mathbf{c} gives $1/4$ of a vote to $c4$ and $1/12$ of a vote to at most two vertices, and hence at most $5/12$ of a vote altogether. \square

Lemma 4. *If $\mathbf{c} \in C_2$, then \mathbf{c} gets at least $s(\mathbf{c}) - 19/6$ votes.*

Proof. Without loss of generality, assume that $s(\mathbf{c}) > 19/6$ and that \mathbf{c} is the vertex $d4$ in Constellation 6 in Figure 4. Because C is identifying, we know that $I(d3) \neq I(d4)$ and hence $d3 \in N_{\geq 3}$; and similarly, $e3 \in C_{\geq 3}$ and $e4 \in N_{\geq 3}$. At most one of the vertices $c4$, $c5$ and $d5$ belongs to N_1 and at least two of them belong to $N_{\geq 2}$. Hence

$$s(\mathbf{c}) \leq 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} + 1 = \frac{7}{2}.$$

Therefore, if $e3$ gives $1/3$ of a vote to $d4$, then the claim certainly holds. By Rules 1 and 3 this is true, unless $e3$ belongs to C_3 and its remaining codeword neighbour is either $e2$ or $f3$. These are symmetrical cases, so assume that $e2$ is in C and $f2$ and $f3$ are non-codewords. Then $I(d3) \neq I(e3)$ implies that $d3$ is in $N_{\geq 4}$. Hence $s(\mathbf{c}) \leq 41/12$, and $s(\mathbf{c}) - 19/6 \leq 1/4$, and the claim is again valid, because $e3$ now gives $1/4$ of a vote to $d4$ by Rule 3. \square

Lemma 5. *If $\mathbf{c} \in C_3$ and $s(\mathbf{c}) > 19/6$, then \mathbf{c} gets at least $s(\mathbf{c}) - 19/6$ votes.*

Proof. Because $\mathbf{c} \in C_3$ and $s(\mathbf{c}) > 19/6$, we know that \mathbf{c} and its two codeword neighbours cannot be collinear, and without loss of generality \mathbf{c} is the vertex $d4$ in Constellation 5 or in Constellation 7.

Consider first Constellation 7. Here, c_4 and c_5 both belong to $C_{\geq 4}$ and at most one of the vertices e_3 and e_4 can belong to N_1 . Hence $s(\mathbf{c}) \leq 10/3$, and it suffices to show that d_4 gets at least $1/6$ of a vote. But, indeed, both c_4 and c_5 give $1/12$ of a vote to d_4 by Rule 2.

Assume therefore that \mathbf{c} is the vertex d_4 in Constellation 5. The fact that $s(\mathbf{c}) > 19/6$ implies that d_5 belongs to N_1 . Because $I(c_4) \setminus \{c_4\} \neq I(d_5)$, we know that $c_4 \in C_{\geq 3}$ and similarly $e_3 \in C_{\geq 3}$. Because $I(d_3) \neq I(d_4)$, we know that $d_3 \in N_{\geq 4}$. Therefore

$$s(\mathbf{c}) \leq \frac{1}{4} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} + 1 = \frac{13}{4}.$$

The fact that $s(\mathbf{c}) > 19/6$ now implies that c_5 and e_4 are both in N_2 . Therefore b_5 and f_3 are both non-codewords. The claim is clearly true if d_4 gets at least $1/12$ of a vote. If c_4 or e_3 is in $C_{\geq 4}$, then this is true by Rule 2, so we can assume that they both belong to C_3 . If b_4 is in C , then c_4 gives $1/12$ of a vote to d_4 by Rule 4, so we can assume that b_4 is not in C ; similarly, we can assume that f_2 is not in C . But then c_3 and e_2 are both in C and we have Constellation 8 in Figure 5.

If c_3 is in C_2 , then c_4 gives $1/12$ of a vote to d_4 by Rule 4; so assume that c_3 is in $C_{\geq 3}$. If c_3 is in $C_{\geq 4}$, then c_3 gives $1/12$ of a vote to d_4 by Rule 2; so assume that c_3 is in C_3 .

If b_3 is in C , then $s(c_3) < 19/6$ and $s(c_4) \leq 37/12$ and therefore c_4 gives $1/12$ of a vote to d_4 by Rule 5; so assume that b_3 is not in C .

Because $s(\mathbf{c}) > 19/6$, we know that d_2 is not in C . But then c_2 must be in C and c_3 gives $1/12$ of a vote to d_4 by Rule 4. \square

Theorem 1. *The density of a strongly identifying code in the infinite triangular grid is at least $6/19$.*

Proof. Assume that C is a strongly identifying code in the infinite triangular grid. Let $n \geq 5$ be fixed, and consider the set T_n .

Consider now the voting process described above. For all codewords \mathbf{c} , define $m(\mathbf{c})$ as the total number of votes given minus the total number of votes received.

We first check that for all $\mathbf{c} \in C$ we have

$$s(\mathbf{c}) + m(\mathbf{c}) \leq \frac{19}{6}. \tag{1}$$

There are several cases to consider:

- If $\mathbf{c} \in C_{\geq 4}$, then \mathbf{c} gives at most $19/6 - s(\mathbf{c})$ votes by Lemma 1, and the sum of $s(\mathbf{c})$ and the number of votes given by \mathbf{c} is at most $19/6$.
- If $\mathbf{c} \in C_2$, then \mathbf{c} does not give any votes and by Lemma 4, the number of votes received by \mathbf{c} is at least $s(\mathbf{c}) - 19/6$, and again (1) holds.
- If $\mathbf{c} \in C_3$ **gives votes**, then by Lemmas 2 and 3, \mathbf{c} gives at most $19/6 - s(\mathbf{c})$ votes, and (1) holds.

- Finally, assume that $\mathbf{c} \in C_3$ **does not give any votes**. If $s(\mathbf{c}) \leq 19/6$, then (1) trivially holds. If $s(\mathbf{c}) > 19/6$, then by Lemma 5, \mathbf{c} gets at least $s(\mathbf{c}) - 19/6$ votes and again (1) holds.

Consider the sum

$$\sum_{\mathbf{c} \in C \cap T_n} (s(\mathbf{c}) + m(\mathbf{c})).$$

Except for the votes received from codewords not in T_n and votes given to codewords not in T_n , the number of votes given by the codewords in T_n is the same as the number of votes received by the codewords in T_n . From the voting rules we immediately see that if a codeword gives votes to another, their distance is at most 2. Consequently,

$$\sum_{\mathbf{c} \in C \cap T_n} m(\mathbf{c}) \geq -4|T_{n+2} \setminus T_n| = -4(16n + 24), \quad (2)$$

where $4|T_{n+2} \setminus T_n|$ is an upper bound (cf. Lemmas 1, 2 and 3) on the total number of votes received by the codewords in T_n from the codewords not in T_n .

On the other hand, if we consider the sum $\sum_{\mathbf{c} \in C \cap T_n} s(\mathbf{c})$, then every vertex $\mathbf{v} \in T_{n-1}$ with $|I(\mathbf{v})| = i$ contributes the summand $1/i$ to $s(\mathbf{c})$ for all the i codewords \mathbf{c} within distance one from \mathbf{v} (and these codewords \mathbf{c} all belong to T_n). Hence

$$\begin{aligned} \sum_{\mathbf{c} \in C \cap T_n} s(\mathbf{c}) &\geq |T_{n-1}| \\ &= |T_n| - 8n. \end{aligned} \quad (3)$$

From (1) we see that $s(\mathbf{c}) + m(\mathbf{c}) \leq 19/6$ for all $\mathbf{c} \in C \cap T_n$ and therefore

$$\sum_{\mathbf{c} \in C \cap T_n} (s(\mathbf{c}) + m(\mathbf{c})) \leq \frac{19}{6}|C \cap T_n|. \quad (4)$$

Using (2), (3) and (4) we now see that

$$|T_n| - 8n - 4(16n + 24) \leq \sum_{\mathbf{c} \in C \cap T_n} (s(\mathbf{c}) + m(\mathbf{c})) \leq \frac{19}{6}|C \cap T_n|,$$

i.e.,

$$|T_n| - (72n + 96) \leq \frac{19}{6}|C \cap T_n|,$$

i.e.,

$$\frac{|C \cap T_n|}{|T_n|} \geq \frac{6}{19} - \frac{6(72n + 96)}{19(2n + 1)^2},$$

from which we see that the claim is true. \square

So we have proved:

Theorem 2. *The smallest possible density of a strongly identifying code in the infinite triangular grid is $6/19$.* \square

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