On the sharpness of some results relating cuts and crossing numbers

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Abstract

It is already known that for very small edge cuts in graphs, the crossing number of the graph is at least the sum of the crossing number of (slightly augmented) components resulting from the cut. Under stronger connectivity condition in each cut component that was formalized as a graph operation called zip product, a similar result was obtained for edge cuts of any size, and a natural question was asked, whether this stronger condition is necessary. In this paper, we prove that the relaxed condition is not sufficient when the size of the cut is at least four, and we prove that the gap can grow quadratically with the cut size.

1 Introduction

Crossing number of graphs (see [13] for basic definitions) has been extensively studied for about sixty years and is still a notorious problem in graph theory. While determining or bounding the crossing number of graphs used to be the main issue at the beginning, the focus is now shifting to structural aspects of the crossing number problem. These include the study of several variants of crossing number [5, 8, 18, 19, 21, 22], crossing-critical graphs [9, 23, 24], or the properties of drawings with a bounded number of crossings per edge [20].

Very early at the development of the crossing number theory, Leighton realized that cuts in graphs play an important role in determining the crossing number of a graph. Combining them with the Lipton-Tarjan planar separator theorem [17], he used edge

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cuts in graphs to provide upper bounds on crossing numbers [1], whereas in the bisection method [15, 16], he used this structure to derive lower bounds for the crossing number.

Both the upper and the lower bound arising from graph cuts are general methods that apply to every graph, but neither provides the sharpness needed to yield exact bounds. This issue was resolved by the introduction of the zip product of graphs in [2, 3], which led to exact crossing number of several two-parameter graph families, most general being the crossing number of the Cartesian product of any sub-cubic tree with any star $K_{1,n}$. This family includes, as a subfamily, the product of any path and any star, resolving a long-standing conjecture by Jendroĺ and Ščerbova in [12]. Besides, it has also been helpful for other works concerning exact crossing numbers (as in [25, 26]), but also regarding crossing-critical graphs (see [4, 10]). It is natural to ask about its behavior with respect to other graph invariants.

The zip product approach, however, assumes a technical condition of having two coherent bundles in the zipped graphs (we formalize this condition later). In this paper, we examine the possible weakenings of this condition and prove that having only one bundle in each graph is not sufficient to establish superadditivity of crossing number with regard to the zip product. Moreover, we are able to achieve any gap between both quantities and show that the gap can grow quadratically with the number of edges involved in the zip product.

In the first section, we state the definitions of zip product and bundles and recall precedent results. In the second part, we describe the families of graphs that we use in the proof of our main result. In the last section, we point out a contradiction of our results with some arguments of Chimani, Gutwenger, and Mutzel [7]. These contradictions do not disprove their results, but only render their argument invalid.

2 The zip product

For $i = 1, 2$, let G_i be a simple graph (we will see how zip product can be extended for multiple edges) and $v_i \in V(G_i)$ such that both v_1 and v_2 have the same degree d. Let $N_i = N_{G_i}(v_i)$ be the set of neighboring vertices of v_i in G_i, and let $\sigma : N_1 \to N_2$ be a bijection. We call σ a *zip function* of the graphs G_1 and G_2 at vertices v_1 and v_2. The *zip product* of G_1 and G_2 according to σ is the graph $G_1 \odot_\sigma G_2$ obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ after adding edges $u\sigma(u)$ for any $u \in N_1$. With $G_1\, {}_{v_1}\!\odot_{v_2} G_2$, we denote the set of graphs that can be obtained as $G_1 \odot_\sigma G_2$ for some bijection σ between the neighborhoods of G_1 and G_2.

Let $v \in V(G)$ be a vertex of degree d in G. A *bundle* of v is a set B of d edge-disjoint paths from v to some vertex $u \in V(G)$, $u \neq v$. Vertex v is the *source* of the bundle and u is its *sink*. Other vertices on the paths of B are *internal vertices* of the bundle. Let $\breve{E}(B) = E(B) \cap E(G - v)$ denote the set of edges of B that are not incident with v. They are called *distant edges* of B. Two bundles B_1 and B_2 of v are *coherent* if their sets of distant edges are disjoint.

The following result was established in [3]:

Theorem 2.1 [3] *For $i = 1, 2$, let G_i be a graph, $v_i \in V(G_i)$ a vertex of degree d, and $N_i = N_{G_i}(v_i)$. Also assume that v_i has two coherent bundles $B_{i,1}$ and $B_{i,2}$ in G_i. Then, $\mathrm{cr}(G_1 \odot_\sigma G_2) \geqslant \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ for any bijection $\sigma : N_1 \to N_2$.*

Note that, as stated above, the zip product is defined to involve vertices that have no incident multiple edges. However, the reader shall have no difficulty establishing the same result for graphs with multiple edges: if the edges in each multiple edge are subdivided, the crossing number is preserved and the resulting graph is simple. The zip product is done using these simple graphs. We suppress the degree-two edges after the zip product, and obtain back the multiple edges. In this manner, the new vertices of the subdivision play only the role of placeholders for specifyin the matching between multigraphs G_1 and G_2, whereas the coherent bundles of v_i in G_i are allowed to share (multi)edes incident with v_i.

3 Two families of graphs

We define two families of graphs with specific crossing number, such that in each a chosen vertex has a single bundle.

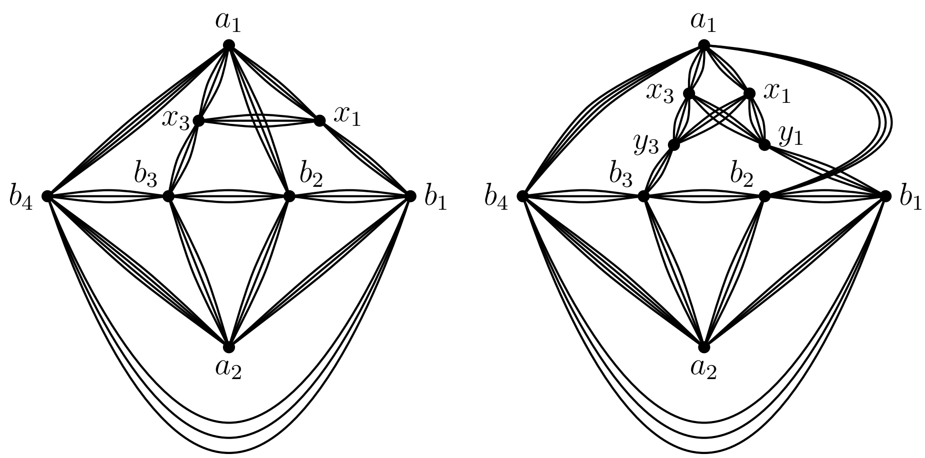
Definition 3.1 *Given any integer $p \geqslant 4$, let $p = 4k + r$. Let $K = K_{2,4}$ with bipartition a_i, $i = 1, 2$ and b_j, $j = 1, 2, 3, 4$. The graph $H_{k,r}$ is obtained from K with the following steps:*

1. *adding a cycle $C = b_1b_2b_3b_4$,*
2. *subdividing the edge a_1b_i with x_i, $i = 1, 3$,*
3. *adding the edge x_1x_3,*
4. *replacing every edge with k parallel edges,*
5. *adding r more edges to the multiedges a_ib_4, $i = 1, 2$.*

Figure 1(a) depicts $H_{3,1}$.

Lemma 3.2 *For $k \geqslant 1$ and $r \in \{0, 1, 2, 3\}$, $\mathrm{cr}(H_{k,r}) = k^2$.*

Proof. It is easy to find a subdivision of $K_{3,3}$ in $H_{1,r}$, thus $\mathrm{cr}(H_{1,r}) \geqslant 1$. Figure 1(a) gives a natural way of drawing $H_{1,0}$ with 1 crossing, establishing $\mathrm{cr}(H_{1,0}) = 1$. Furthermore, it is obvious that $\mathrm{cr}(H_{k,0}) = k^2 \,\mathrm{cr}(H_{1,0}) = k^2$. Since $H_{k,0}$ is a subgraph of $H_{k,r}$, we have $\mathrm{cr}(H_{k,r}) \geqslant \mathrm{cr}(H_{k,0})$. On the other hand, one can alter an optimal drawing of $H_{k,0}$ to a drawing of $H_{k,r}$ with the same number of crossings, as in Figure 1(a), concluding $\mathrm{cr}(H_{k,r}) = \mathrm{cr}(H_{k,0}) = k^2$. □



(a) Graph $H_{3,1}$ (b) Graph $G_{3,1}$

Figure 1: Two families of graphs: $G_{k,r}$ and $H_{k,r}$

Definition 3.3 *Given any integer $p \geqslant 4$, let $p = 4k + r$. Let $K = K_{2,4}$ with bipartition a_i, $i = 1, 2$ and b_j, $j = 1, 2, 3, 4$. The graph $G_{k,r}$ is obtained from K with the following steps:*

1. *adding a cycle $C = b_1b_2b_3b_4$,*
2. *subdividing the edge a_1b_i twice, with x_i, y_i, $i = 1, 3$,*
3. *adding the edges x_1y_3 and x_3y_1,*
4. *replacing every edge with k parallel edges,*
5. *adding r more edges to the multiedges a_ib_4, $i = 1, 2$.*

Figure 1(b) depicts $G_{3,1}$.

Lemma 3.4 *For $k \geqslant 1$ and $r \in \{0, 1, 2, 3\}$, $\mathrm{cr}(G_{k,r}) = 2k^2$.*

Proof. The subgraph $G_{1,0}$ contains a subdivision of a graph, obtained from $K_{3,4}$ by splitting two vertices of degree 4. This graph has crossing number two [6], and the drawing in Figure 1(b) gives a way of drawing $G_{1,0}$ establishing $\mathrm{cr}(G_{1,0}) = 2$. By construction, $\mathrm{cr}(G_{k,0}) = k^2 \,\mathrm{cr}(G_{1,0}) = 2k^2$. Furthermore, it is easy to modify the optimal drawing of $G_{k,0}$ to a drawing of $G_{k,r}$ with the same number of crossings as in Figure 1(b). As $G_{k,r}$ has $G_{k,0}$ as a subgraph, we have $\mathrm{cr}(G_{k,r}) = \mathrm{cr}(G_{k,0}) = 2k^2$. □

4 The Zip Product Gap

Proposition 4.1 *For $i = 1, 2$, let G_i be a graph such that $v_i \in V(G_i)$ of degree d has one bundle B_i in G_i, σ a bijection among neighborhoods of v_1 and v_2. If there exists an optimal drawing D of $G_1 \odot_\sigma G_2$, such that no edge of $G_1 - v_1$ crosses an edge of $G_2 - v_2$, then* $\operatorname{cr}(G_1 \odot_\sigma G_2) \geqslant \operatorname{cr}(G_1) + \operatorname{cr}(G_2)$.

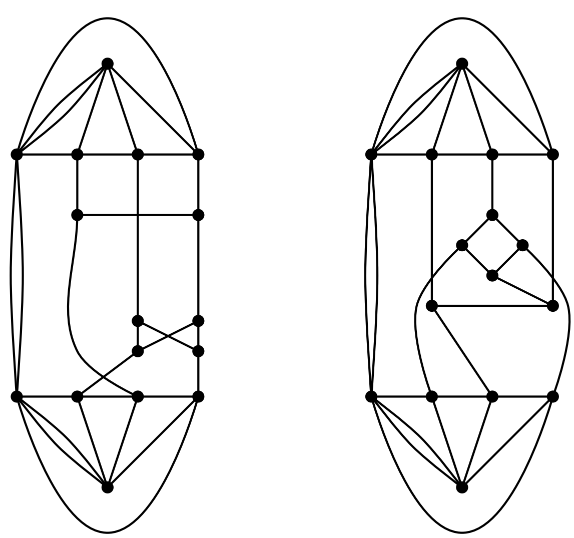
Proof. Without loss of generality, we may assume that each G_i is connected. Due to the bundles, $G_i - v_i$ is connected, too. Let D_i be obtained from $D[(G_i - v_i) \cup B_{3-i}]$ by contracting any of its faces whose boundary contains only segments of edges of B_{3-i}. Then D_i is a drawing of G_i, with the contracted region representing the vertex v_i.

Clearly, each crossing of D_i appears in D, and by assumption, no crossing of D appears in both D_1 and D_2. Thus $\operatorname{cr}(G_1 \odot_\sigma G_2) = \operatorname{cr}(D) \geqslant \operatorname{cr}(D_1) + \operatorname{cr}(D_2) \geqslant \operatorname{cr}(G_1) + \operatorname{cr}(G_2)$. □

The following theorem establishes that in general, one bundle at each vertex used in the zip product does not suffice for preservation of the crossing number:

Theorem 4.2 *For any $d \geqslant 4$, there exist graphs G_i, $i = 1, 2$, such that G_i has a vertex v_i with a bundle in G_i, $d_{G_i}(v_i) = d$, and there is a graph $G \in G_1\, {}_{v_1}\!\odot_{v_2} G_2$, such that* $\operatorname{cr}(G) < \operatorname{cr}(G_1) + \operatorname{cr}(G_2)$.

Proof. Let $d = 4k + r$ and set $G_1 = H_{k,r}$, $G_2 = G_{k,r}$, and let v_i, $i = 1, 2$, be the vertex a_1 from the definition of the respective graph. Then $\operatorname{cr}(G_1) = k^2$, $\operatorname{cr}(G_2) = 2k^2$.



(a) $G_{1,1} \odot H_{1,1}$ (b) Better drawing of $G_{1,1} \odot H_{1,1}$

Figure 2: $G_{1,1} \odot H_{1,1}$

The graph G in Figure 2(a) is clearly an element of $G_1\, {}_{v_1}\!\odot_{v_2} G_2$. Its better drawing in Figure 2(b) establishes $\mathrm{cr}(G) \leqslant 2k^2 < 3k^2 = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$. $\square$

Combining Theorem 4.2 with Proposition 4.1, we obtain the following:

Corollary 4.3 *For any $d \geqslant 4$, there exist graphs G_i, $i = 1, 2$, such that G_i has a vertex v_i with a bundle, $d_{G_i}(v_i) = d$, and a graph $G \in G_1\, {}_{v_1}\!\odot_{v_2} G_2$, such that some edge of $G_1 - v_1$ crosses some edge of $G_2 - v_2$ in any optimal drawing of G.*

Let graphs G_1 and G_2 be *d-compatible*, if each G_i contains a vertex v_i of degree d, such that v_i has a bundle in G_i. Define

$$g(d) = \max_{G_1, G_2} [\mathrm{cr}(G_1) + \mathrm{cr}(G_2) - \mathrm{cr}(G)],$$

where the maximum runs over all d-compatible pairs G_1, G_2 and all graphs $G \in G_1\, {}_{v_1}\!\odot_{v_2} G_2$. The proof of Theorem 4.2 establishes the following corollary:

Corollary 4.4 *Let $g(d)$ be defined as above. Then $g(d) = \Omega(d^2)$.*

Thus, we have a lower bound on the possible crossing number gap between the two original graphs and their zip product in terms of the size of the edge cut between the zipped graphs. An upper bound, on the other hand, is far from clear, as the edges of the two bundles can possibly cross each other arbitrarily often in an optimal drawing of the zip product, and optimal drawings of the original graphs can have different structure than the corresponding subdrawings of optimal drawings of the zipped graph. We summarize this discussion in the following problem:

Problem 4.5 *Find an upper bound on $g(d)$. Is $g(d) = O(d^2)$?*

Theorem 2.1 establishes that if the vertices involved in the zip product have two coherent bundles, then the crossing number is preserved. On the other hand, Theorem 4.2 establishes that whenever their degree is at least four, just one bundle at each vertex is in general not enough. We denote by $\mathrm{cr}(G_1\, {}_{v_1}\!\odot_{v_2} G_2)$ the maximum $\mathrm{cr}(G)$ taken over all G in $G_1\, {}_{v_1}\!\odot_{v_2} G_2$. It is easy to see that, if $d = 1$, then $\mathrm{cr}(G_1\, {}_{v_1}\!\odot_{v_2} G_2) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$. For $d = 2$, Leaños and Salazar established the same result in [14]. Therefore, two natural problems remain open:

Problem 4.6 *Let G_1, G_2 be graphs, such that G_i has a vertex v_i with a bundle in G_i and $d_{G_i}(v_i) = 3$. Is $\mathrm{cr}(G_1\, {}_{v_1}\!\odot_{v_2} G_2) \geqslant \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$?*

Problem 4.7 *Let G_1, G_2 be graphs, such that G_i has a vertex v_i with $d_{G_i}(v_i) = d$. Assume that v_1 has two coherent bundles in G_1, but v_2 has just one bundle in G_2. Is $\mathrm{cr}(G_1\, {}_{v_1}\!\odot_{v_2} G_2) \geqslant \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$?*

5 Some more open problems

In [7], the authors claim to have proved that if $C \subseteq E(G)$ is a minimum s,t-cut in a graph G and G_s and G_t are the components of $G - C$, then there exists an optimal drawing of G in which no edge of G_s crosses an edge of G_t. Since C is a minimum s,t-cut in G, G can be considered a zip product of graphs G_s^x and G_t^y, respectively obtained from G_s and G_t by adding a vertex x or y and connecting it to the endvertices of C in G_s or G_t. By Menger's theorem, x and y each has a bundle in the respective graph, yielding a contradiction to Corollary 4.3. Upon closer examination, the aforementioned result is supposed to follow from a proof, which contains invalid arguments. Thus Corollary 4.3 presents counterexamples to that claim.

Nevertheless, the paper [7] contains several original ideas, which could perhaps lead to a solution of Problems 4.6 or 4.7. Although our counterexample shows that the proof in [7] has a flaw, it does not disprove any of the main statements in that paper. These thus share the fate of the oldest result in the field of crossing numbers, the (still open) Zarankiewicz conjecture [11, 27]. We state them for the sake of completeness. In the following problem, a *planarization* of G is a graph, obtained from a drawing of G by replacing every crossing with a vertex. A *crossing minimal planarization* is a planarization, obtained from an optimal drawing.

Problem 5.1 ([7]) *Let G be a connected graph and let s and t be two distinct vertices in G. Then there exists a crossing minimal planarization P of G, such that the size of the minimum s,t-cut in P is the same as the size of the minimum s,t-cut in G. Moreover, any crossing minimum planarization of G can be transformed into a crossing minimum planarization of G with the above property in linear time.*

References

[1] S. N. Bhatt and F. T. Leighton, A framework for solving VLSI graph layout problems, *J. Comput. System Sci.* **28** (1984), 300–343.

[2] D. Bokal, On the crossing number of Cartesian products with paths, *J. Combin. Theory Ser. B* **97** (2007) 381–384.

[3] D. Bokal, On the crossing numbers of Cartesian products with trees, *J. Graph Theory* **56** (2007) 287–300.

[4] D. Bokal, Infinite families of crossing-critical graphs with prescribed average degree and crossing number, to appear in *J. Graph Theory.*

[5] D. Bokal, G. Fijavž and B. Mohar, The minor crossing number, *SIAM J. Disc. Math.* **20** (2006) 344–356.

[6] D. Bokal, B. Oporowski, R. B. Richter and G. Salazar, Characterization of 2-crossing-critical graphs I: Low connectivity or no V_8-minor, in preparation.

[7] M. Chimani, C. Gutwenger and P. Mutzel, On the minimum cut of planarizations, *Electron. Notes Discrete Math.* **28** (2007) 177–184.

[8] E. Czabarka, O. Sýkora, L. A. Székely and I. Vrťo, Biplanar crossing numbers I: a survey of results and problems, `http://www.math.sc.edu/~szekely/survey4.pdf`.

[9] P. Hliněný, Crossing-critical graphs have bounded path-width, *J. Combin. Theory Ser. B* **88** (2003), 347–367.

[10] P. Hliněný, New infinite families of crossing-critical graphs, *Electronic J. Combin.* **15** (2008), R102.

[11] R. K. Guy, The decline and fall of Zarankiewicz's theorem, in: F. Harary, ed., Proof Techniques in Graph Theory, Academic Press, New York, 1969, 63–69.

[12] S. Jendroĺ and M. Ščerbova, On the crossing number of $S_m \times P_n$ and $S_m \times C_n$, *Čas. Pest. Mat.* **107** (1982), 225–230.

[13] M. Klešč, The crossing number of $K_{2,3}\square C_3$, *Discrete Math.* **251** (2002), 109–117.

[14] J. Leaños, G. Salazar, On the additivity of crossing numbers of graphs, *J. Knot Theory Ramifications* **17** (2008), 1043–1050.

[15] F. T. Leighton, *Complexity Issues in VLSI*, MIT Press, Cambridge, 1983.

[16] F. T. Leighton, New lower bound techniques for VLSI, *Math. Systems Theory* **17** (1984), 47–70.

[17] R. Lipton and R. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979), 177–189.

[18] B. Mohar, The genus crossing number, submitted.

[19] J. Pach and G. Tóth, Degenerate crossing numbers, *In*: N. Amenta and O. Chedong, eds. *Proceedings of the twenty-second annual symposium on Computational geometry*, ACM, New York, 2006: 255–258.

[20] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17** (1997), 427–439.

[21] J. Pach and G. Tóth, Which crossing number is it anyway?, *J. Combin. Theory Ser. B* **80** (2000), 225–246.

[22] M. J. Pelsmajer, M. Schaefer and D. Štefankovič, Odd Crossing Number Is Not Crossing Number, *In*: P. Healy and N. S. Nikolov, eds., *Graph Drawing:* 13^{th} *International Symposium*, Lecture Notes in Comput. Sci. **3843**, Springer, Berlin, 2006: 386–396.

[23] R. B. Richter and C. Thomassen, Minimal graphs with crossing number at least k, *J. Combin. Theory Ser. B* **58** (1993), 217–224.

[24] G. Salazar, Infinite families of crossing-critical graphs with given average degree, *Discrete Math.* **271** (2003), 343–350.

[25] Tang Ling, Lu Shengxiang and Huang Yuanqiu, The Crossing Number of Cartesian Products of Complete Bipartite Graphs $K_{2,m}$ with Paths P_n, *Graphs Combin.* **23** (2007) 659–666.

[26] Zheng Wenping, Lin Xiaohui and Yang Yuansheng, The crossing number of $K_{2,m}\square P_n$, *Discrete Math.* **308** (2008) 6639–6644.

[27] K. Zarankiewicz, On a problem of P. Turán concerning graphs, *Fund. Math.* **41** (1954), 137–145.