# Properties of the Steiner Triple Systems of Order 19

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**Abstract**

Properties of the 11 084 874 829 Steiner triple systems of order 19 are examined. In particular, there is exactly one 5-sparse, but no 6-sparse, STS(19); there is exactly one uniform STS(19); there are exactly two STS(19) with no almost parallel classes; all STS(19) have chromatic number 3; all have chromatic index 10, except for 4 075 designs with chromatic index 11 and two with chromatic index 12; all are 3-resolvable; and there are exactly two 3-existentially closed STS(19).

**Keywords:** automorphism, chromatic index, chromatic number, configuration, cycle structure, existential closure, independent set, partial parallel class, rank, Steiner triple system of order 19.

# 1 Introduction

A *Steiner triple system* (STS) is a pair $(X, \mathcal{B})$, where $X$ is a finite set of *points* and $\mathcal{B}$ is a collection of 3-subsets of points, called *blocks* or *triples*, with the property that every 2-subset of points occurs in exactly one block. The size of the point set, $v := |X|$, is the *order* of the design, and an STS of order $v$ is commonly denoted by STS($v$). Steiner triple systems form perhaps the most fundamental family of combinatorial designs; it is well known that they exist exactly for orders $v \equiv 1, 3 \pmod 6$ [31].

Two STS($v$) are *isomorphic* if there is a bijection between their point sets that maps blocks onto blocks. Denoting the number of isomorphism classes of STS($v$) by $N(v)$, we have $N(3) = 1$, $N(7) = 1$, $N(9) = 1$, $N(13) = 2$ and $N(15) = 80$. Indeed, due to their relatively small number, the STSs up to order 15 have been studied in detail and are rather well understood. An extensive study of their properties was carried out by Mathon, Phelps and Rosa in the early 1980s [35].

For the next admissible parameter, we have $N(19) = 11\,084\,874\,829$, obtained in [26]. Of course, this huge number prohibits a discussion of each individual design. Because the designs are publicly available in compressed form [28], however, examination of some of their properties can be easily automated. Computing resources set a strict limit on what is feasible: one CPU year permits 2.8 milliseconds on average for each design.

Many properties of interest can nonetheless be treated. In Section 2, results, mainly of a computational nature, are presented. They show, amongst other things, that there is exactly one 5-sparse, but no 6-sparse, STS(19); that there is one uniform STS(19); that there are two STS(19) with no almost parallel classes; that all STS(19) have chromatic number 3; that all have chromatic index 10, except for 4 075 designs with chromatic index 11 and two with chromatic index 12; that all STS(19) are 3-resolvable; and that there are two 3-existentially closed STS(19). Some tables from the original classification [26] are repeated for completeness. In Section 3, some properties that remain open are mentioned, and the computational resources needed in the current work are briefly discussed.

Table 1: Automorphism group order

| \|Aut\| | # | \|Aut\| | # | \|Aut\| | # | \|Aut\| | # |
|---|---|---|---|---|---|---|---|
| 1 | 11 084 710 071 | 8 | 101 | 19 | 1 | 96 | 1 |
| 2 | 149 522 | 9 | 19 | 24 | 11 | 108 | 1 |
| 3 | 12 728 | 12 | 37 | 32 | 3 | 144 | 1 |
| 4 | 2 121 | 16 | 13 | 54 | 2 | 171 | 1 |
| 6 | 182 | 18 | 11 | 57 | 2 | 432 | 1 |

# 2 Properties

## 2.1 Automorphisms

The automorphisms and automorphism groups of the STS(19) were studied in [6, 26]; we reproduce the results here (with a correction in our Table 2).

Representing an automorphism as a permutation of the points, the nonidentity automorphisms can be divided into two types based on their order. The automorphisms of prime order have six cycle types

$$19^1, \quad 1^1 2^9, \quad 1^1 3^6, \quad 1^3 2^8, \quad 1^7 2^6, \quad 1^7 3^4,$$

and the automorphisms of composite order have nine cycle types

$$1^1 9^2, \quad 1^1 6^3, \quad 1^1 3^2 6^2, \quad 1^1 2^1 4^4, \quad 1^1 2^1 8^2, \quad 1^3 8^2, \quad 1^3 4^4, \quad 1^3 2^2 6^2, \quad 1^3 2^2 4^3.$$

Table 1 gives the order of the automorphism group for each isomorphism class. Tables 2 and 3 partition the possible orders of the automorphism groups into classes based on the types of prime and composite automorphisms that occur in the group. Compared with [26], Table 2 has been corrected by transposing the classes 18c and 18d, and the classes 12a and 12b (this correction is incorporated in the table reproduced in [4]).

A list of the 104 STS(19) having an automorphism group of order at least 9 is given in compact notation in the supplement to [6]. Cyclic STS(19) were first enumerated in [1] and 2-rotational ones (automorphism cycle type $1^1 9^2$) in [38]; these systems are listed in [35]. The 184 reverse STS(19) (automorphism cycle type $1^1 2^9$), together with their automorphism groups, were determined in [10].

In this paper, certain STS(19) are identified as follows: A1–A4 are the cyclic systems as listed in [35]; B1–B10 are the 2-rotational STS(19) as listed in [35]; and S1–S7 are the sporadic STS(19) listed in the Appendix. In addition, an STS(19) can be identified by the order of its automorphism group when this is unique (the listings in [6] are useful for retrieving such designs). Design A4, with an automorphism group of order 171, is both cyclic and 2-rotational and is therefore also listed as B8 in [35]; it is the *Netto triple system* [39]. A reader interested in copies of STS(19) that are not included among the sporadic examples here will apparently need to carry out some computational work, perhaps utilizing the catalogue from [28]—the authors of the current work are glad to provide consultancy for such an endeavour.

Table 2: Automorphisms (prime order)

| Order | Class | $19^1$ | $1^12^9$ | $1^13^6$ | $1^32^8$ | $1^72^6$ | $1^73^4$ | # |
|---|---|---|---|---|---|---|---|---|
| 432 | | | | * | * | * | * | 1 |
| 171 | | * | | * | | | | 1 |
| 144 | | | | * | * | * | | 1 |
| 108 | | | | * | * | * | * | 1 |
| 96 | | | | * | * | * | | 1 |
| 57 | | * | | * | | | | 2 |
| 54 | | | | * | | * | * | 2 |
| 32 | | | | | * | * | | 3 |
| 24 | | | | * | * | * | | 11 |
| 19 | | * | | | | | | 1 |
| 18 | a | | * | * | | | | 1 |
| | b | | | * | * | | * | 2 |
| | c | | | * | | * | * | 6 |
| | d | | | * | | * | | 2 |
| 16 | | | | | * | * | | 13 |
| 12 | a | | | * | * | * | | 8 |
| | b | | | * | * | | | 7 |
| | c | | | * | | * | | 12 |
| | d | | | | * | * | * | 10 |
| 9 | | | | * | | | | 19 |
| 8 | a | | | | * | * | | 84 |
| | b | | | | * | | | 17 |
| 6 | a | | * | * | | | | 14 |
| | b | | | * | * | | | 14 |
| | c | | | * | | * | | 116 |
| | d | | | | * | | * | 10 |
| | e | | | | | * | * | 28 |
| 4 | a | | | | * | * | | 839 |
| | b | | | | * | | | 662 |
| | c | | | | | * | | 620 |
| 3 | a | | | * | | | | 12 664 |
| | b | | | | | | * | 64 |
| 2 | a | | * | | | | | 169 |
| | b | | | | * | | | 78 961 |
| | c | | | | | * | | 70 392 |
| # | | 4 | 184 | 12 885 | 80 645 | 72 150 | 124 | 164 758 |

Table 3: Automorphisms (composite order)

| Class | $1^1 9^2$ | $1^1 6^3$ | $1^1 3^2 6^2$ | $1^1 2^1 4^4$ | $1^1 2^1 8^2$ | $1^3 8^2$ | $1^3 4^4$ | $1^3 2^2 6^2$ | $1^3 2^2 4^3$ | # |
|---|---|---|---|---|---|---|---|---|---|---|
| 432 | | | * | | | * | * | * | | 1 |
| 171 | * | | | | | | | | | 1 |
| 144 | | | * | * | * | | * | | | 1 |
| 108 | | | * | | | | | * | | 1 |
| 96 | | | * | | | | * | | | 1 |
| 57 | | | | | | | | | | 2 |
| 54 | | | * | | | | | | | 2 |
| 32 | | | | * | | | * | | | 3 |
| 24 | | | * | | | | | | | 11 |
| 19 | | | | | | | | | | 1 |
| 18a | | * | | | | | | | | 1 |
| 18b | | | | | | | | * | | 2 |
| 18c | | | * | | | | | | | 6 |
| 18d | | | * | | | | | | | 2 |
| 16 | | | | * | * | | * | | | 5 |
| 16 | | | | * | | | | | | 6 |
| 16 | | | | | | * | * | | | 1 |
| 16 | | | | | | | * | | | 1 |
| 12a | | | * | | | | | | | 8 |
| 12b | | | | | | | | | | 7 |
| 12c | | | | | | | | | | 12 |
| 12d | | | | | | | | * | | 10 |
| 9 | * | | | | | | | | | 9 |
| 9 | | | | | | | | | | 10 |
| 8a | | | | | | | * | | | 2 |
| | | | | | | | | | | 82 |
| 8b | | | | * | | | * | | | 5 |
| | | | | | * | | * | | | 10 |
| | | | | | | * | * | | | 2 |
| 6a | | * | | | | | | | | 14 |
| 6b | | | | | | | | | | 14 |
| 6c | | | * | | | | | | | 104 |
| | | | | | | | | | | 12 |
| 6d | | | | | | | | * | | 10 |
| 6e | | | | | | | | | | 28 |
| 4a | | | | | | | | | | 839 |
| 4b | | | | * | | | | | | 498 |
| | | | | | | | * | | | 153 |
| | | | | | | | | | | 11 |
| 4c | | | | | | | | | * | 48 |
| | | | | | | | | | | 572 |
| # | 10 | 15 | 137 | 518 | 16 | 4 | 185 | 24 | 48 | |

Table 4: Number of subsystems

| STS(7) | STS(9) | # | STS(7) | STS(9) | # |
|---|---|---|---|---|---|
| 0 | 0 | 10 997 902 498 | 3 | 1 | 45 |
| 0 | 1 | 270 784 | 4 | 0 | 2 449 |
| 1 | 0 | 86 101 058 | 4 | 1 | 25 |
| 1 | 1 | 12 956 | 6 | 0 | 75 |
| 2 | 0 | 572 471 | 6 | 1 | 5 |
| 2 | 1 | 641 | 12 | 0 | 2 |
| 3 | 0 | 11 819 | 12 | 1 | 1 |

## 2.2 Subsystems and Ranks

A *subsystem* in an STS is a subset of blocks that forms an STS on a subset of the points. A subsystem in an STS($v$) has order at most $(v-1)/2$; hence a subsystem in an STS(19) has order 3, 7 or 9. Moreover, the intersection of two subsystems is a subsystem. It follows that each STS(19) has at most one subsystem of order 9, with equality for 284 457 isomorphism classes [42]. The number of subsystems of each order in each isomorphism class was determined in [29] and these results are collected in Table 4. The STS(19) with 12 subsystems of order 7 and 1 subsystem of order 9 is the system having an automorphism group of order 432, and the other two STS(19) with 12 subsystems of order 7 are the systems having automorphism groups of orders 108 and 144.

The *rank* of an STS is the linear rank of its point–block incidence matrix over GF(2). In this setting, a nonempty set of points is (linearly) dependent if every block intersects the set in an even number of points. Counting the point–block incidences in a dependent set in two different ways, one finds that a dependent set necessarily consists of $(v+1)/2$ points so that its complement is the point set of a subsystem of order $(v-1)/2$. An in-depth study of the rank of STSs has been carried out in [11].

In particular, for $v = 19$ there is at most one dependent set, with equality if and only if there exists a subsystem of order 9. It follows that the rank of an STS(19) is 18 if there exists a subsystem of order 9 (284 457 isomorphism classes) and 19 otherwise (11 084 590 372 isomorphism classes).

The rank over GF(2) gives the dimension of the binary code generated by the (rows or columns of) the incidence matrix. The code generated by the rows of a point–block incidence matrix is the *point code* of the STS. There exist nonisomorphic STS(19) that have equivalent point codes [27].

## 2.3 Small Configurations

A *configuration* $\mathcal{C}$ in an STS $(X, \mathcal{B})$ is a subset of blocks $\mathcal{C} \subseteq \mathcal{B}$. Small configurations in STSs have been studied extensively; see [8, Chapter 13], [17] and [19]. The number of any configuration of size at most 3 is a function of the order of the STS. We address small configurations with some particular properties.

A configuration $\mathcal{C}$ with $|\mathcal{C}| = \ell$ and $|\cup_{C\in\mathcal{C}} C| = k$ is a $(k,\ell)$-*configuration*. A configuration is *even* if each of its points occurs in an even number of blocks. If no point of a configuration occurs in exactly one block, then the configuration is *full*.

The only even (and only full) configuration of size 4 is the *Pasch configuration*, the $(6,4)$-configuration depicted in Figure 1. The numbers of Pasch configurations in the STS(19) were tabulated in [26]; for completeness, we repeat the result in Table 5.

Table 5: Number of Pasches

| Pasch | # | Pasch | # | Pasch | # | Pasch | # |
|---|---|---|---|---|---|---|---|
| 0 | 2 591 | 17 | 954 710 609 | 34 | 2 190 166 | 51 | 366 |
| 1 | 35 758 | 18 | 845 596 671 | 35 | 1 301 951 | 52 | 482 |
| 2 | 263 646 | 19 | 716 603 299 | 36 | 775 233 | 53 | 78 |
| 3 | 1 315 161 | 20 | 583 321 976 | 37 | 452 306 | 54 | 278 |
| 4 | 4 958 687 | 21 | 457 755 898 | 38 | 267 642 | 55 | 69 |
| 5 | 15 095 372 | 22 | 347 324 307 | 39 | 152 122 | 56 | 137 |
| 6 | 38 481 050 | 23 | 255 589 428 | 40 | 92 056 | 57 | 24 |
| 7 | 84 328 984 | 24 | 182 938 899 | 41 | 51 019 | 58 | 104 |
| 8 | 162 045 054 | 25 | 127 614 183 | 42 | 31 587 | 59 | 6 |
| 9 | 276 886 518 | 26 | 87 003 115 | 43 | 16 974 | 60 | 41 |
| 10 | 426 050 673 | 27 | 58 052 942 | 44 | 11 827 | 62 | 47 |
| 11 | 596 271 997 | 28 | 38 010 203 | 45 | 6 008 | 64 | 3 |
| 12 | 765 958 741 | 29 | 24 457 073 | 46 | 4 629 | 66 | 18 |
| 13 | 910 510 124 | 30 | 15 492 114 | 47 | 2 151 | 70 | 5 |
| 14 | 1 008 615 673 | 31 | 9 663 499 | 48 | 2 099 | 78 | 2 |
| 15 | 1 047 850 033 | 32 | 5 956 712 | 49 | 724 | 84 | 3 |
| 16 | 1 027 129 335 | 33 | 3 623 356 | 50 | 991 | | |

Three STS(19) with 84 Pasch configurations were found in [23]. Indeed, 84 is the maximum possible number of Pasch configurations and the list of such STS(19) in [23] is complete. The three systems are those having automorphism groups of order 108, 144 and 432, also encountered in Section 2.2.

Replacing the blocks of a Pasch configuration, say $\mathcal{P} = \{\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}\}$, by the blocks of $\mathcal{P}' = \{\{x,y,z\},\{x,b,c\},\{a,y,c\},\{a,b,z\}\}$ transforms an STS into another STS. This operation is a *Pasch switch*. All but one of the 80 isomorphism classes of STS(15) contain at least one Pasch configuration. Any one of these can be transformed to any other by some sequence of Pasch switches [16, 22]. A natural question is whether the same is true for the STS(19), that is, if each STS(19) containing at least one Pasch configuration can be transformed to any other such design via Pasch switches. The answer is in the negative.

In [21] the concept of *twin Steiner triple systems* was introduced. These are two STSs each of which contains precisely one Pasch configuration that when switched produces the other system. If in addition the twin systems are isomorphic we have *identical twins*. In

[20] nine pairs of twin STS(19) are given. By examining all STS(19) containing a single Pasch configuration, we have established that there are in total 126 pairs of twins, but no identical twins.

We also consider STSs that contain precisely two Pasch configurations, say $\mathcal{P}$ and $\mathcal{Q}$, such that when $\mathcal{P}$ (respectively $\mathcal{Q}$) is switched what is obtained is an STS containing just one Pasch configuration $\mathcal{P}'$ (respectively $\mathcal{Q}'$). There are precisely 9 such systems. In every case the two single Pasch systems obtained by the Pasch switches are nonisomorphic. One such system is S1 (in the Appendix).

For size 6, there are two even configurations, known as the *grid* and the *prism* (or *double triangle*); these $(9, 6)$-configurations are depicted in Figure 1.

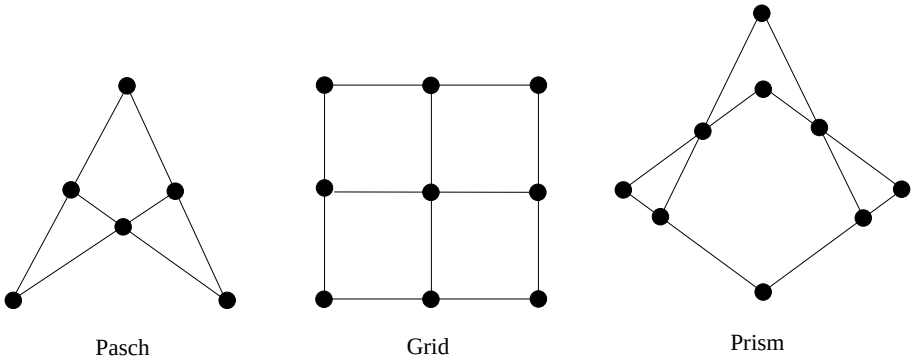


Figure 1: The even configurations of size at most 6

Every STS contains an even configuration of size at most 8, see [15]. However, no STS(19) missing either a grid or a prism was known. Indeed, a complete enumeration of grids and prisms establishes that there is no such STS(19). The distribution of the numbers of grids is shown in Table 9 and that for prisms in Table 10. The smallest number of grids in an STS(19) is 21 (design S4) and the largest is 384 (the STS(19) with automorphism group order 432). The smallest number of prisms is 171 (design A4) and the largest is 1 152 (the designs with automorphism group orders 108, 144 and 432). In particular, then, every STS(19) contains both even $(9, 6)$-configurations.

An STS is *k-sparse* if it does not contain any $(n + 2, n)$-configuration for any $4 \leqslant n \leqslant k$. In studying $k$-sparse systems it suffices to focus on full configurations, because an $(n + 2, n)$-configuration that is not full contains an $(n + 1, n - 1)$-configuration. Because $k$-sparse STS(19) with $k \geqslant 4$ are anti-Pasch, one could simply check the 2 591 anti-Pasch STS(19). A more extensive tabulation of small $(n + 2, n)$-configurations was carried out in this work.

There is one full $(7, 5)$-configuration (the *mitre*) and two full $(8, 6)$-configurations, known as the *hexagon* (or 6-*cycle*) and the *crown*. These are drawn in Figure 2, and their numbers are presented in Tables 11, 12 and 13.

The existence of a 5-sparse STS(19) was known [7]. By Table 11 there are exactly four nonisomorphic anti-mitre STS(19). Moreover, by Tables 12 and 13 there is a unique STS(19) with no hexagon and exactly four with no crown. Considering the intersections

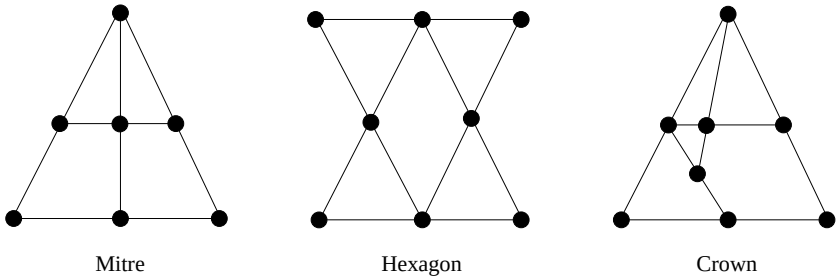


Figure 2: The full $(7,5)$- and $(8,6)$-configurations

of the classes of STS(19) with these properties, and the anti-Pasch ones, only two STS(19) are in more than one of the classes: one has no Pasch and no mitre, and one has no Pasch and no crown.

**Theorem 1.** *The numbers of* 4*-sparse,* 5*-sparse and* 6*-sparse* STS(19) *are* 2 591*,* 1 *and* 0*, respectively.*

The unique 5-sparse—that is, anti-Pasch and anti-mitre—STS(19) is A4. The unique STS(19) having no Pasch and no crown is A2, and the unique STS(19) with no hexagon is S5. The other three anti-mitre systems are B4, S6 and A3, and the other three anti-crown systems are those with automorphism group orders 108, 144 and 432. The largest number of mitres, hexagons and crowns in an STS(19) is 144 (for the three STS(19) with automorphism group orders 108, 144 and 432), 171 (for A4) and 314 (for S7), respectively.

## 2.4 Cycle Structure and Uniform Systems

Any two distinct points $x, y \in X$ of an STS determine a *cycle graph* in the following way. The points $x, y$ occur in a unique block $\{x, y, z\}$. The cycle graph has one vertex for each point in $X \setminus \{x, y, z\}$ and an edge between two vertices if and only if the corresponding points occur together with $x$ or $y$ in a block.

A cycle graph of an STS is 2-regular and consists of a set of cycles of even length. Hence they can be specified as integer partitions of $v-3$ using even integers greater than or equal to 4. For $v = 19$, the possible partitions are $l_1 = 4+4+4+4$, $l_2 = 4+4+8$, $l_3 = 4+6+6$, $l_4 = 4+12$, $l_5 = 6+10$, $l_6 = 8+8$ and $l_7 = 16$. The cycle vector of an STS is a tuple showing the distribution of the cycle graphs; for STS(19) we have $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ with $\sum_{i=1}^{7} a_i = \binom{19}{2} = 171$, where $a_i$ denotes the number of occurrences of the partition $l_i$.

The cycle vector $(0, 0, 0, 0, 0, 0, 171)$ is of particular interest; an STS all of whose cycle graphs consist of a single cycle is *perfect*. It is known [25] that there is no perfect STS(19). A more general family consists of the STSs with $a_i = \binom{v}{2}$ for some $i$; such STSs are *uniform*. Uniform STS(19) are known to exist [39].

An extensive investigation of the cycle vectors of STS(19) was carried out. The results are summarized in Table 6, where the designs are grouped according to the support of the cycle vector, that is, $\{i : a_i \neq 0\}$. Only 28 out of 128 possible combinations of cycle graphs are actually realised.

Table 6: Combinations of cycle graphs

| Type | # | Type | # | Type | # |
|---|---|---|---|---|---|
| 5 | 1 | 3567 | 125 | 24567 | 75 786 636 |
| 57 | 5 | 4567 | 5 009 893 | 34567 | 174 351 058 |
| 134 | 3 | 12347 | 39 | 123457 | 51 146 |
| 347 | 1 | 12457 | 56 | 123467 | 15 |
| 357 | 1 | 12467 | 1 | 124567 | 8 658 874 |
| 457 | 17 | 13457 | 89 | 134567 | 11 039 468 |
| 567 | 2 585 | 13467 | 2 | 234567 | 8 685 731 027 |
| 1347 | 5 | 14567 | 135 588 | 1234567 | 2 124 060 807 |
| 2457 | 255 | 23457 | 46 863 | | |
| 3457 | 259 | 23567 | 10 | | |

The main observation from Table 6 is the following.

**Theorem 2.** *There is exactly one uniform* STS(19).

The following conclusions can also be drawn from Table 6. The anti-Pasch systems are one with cycle graph 5; five with cycle graphs 5 and 7; and 2 585 with cycle graphs 5, 6 and 7. The unique 6-cycle-free system has cycle graphs 1, 2, 4, 6 and 7. The numbers of $k$-cycle-free systems for $k = 4$, 6, 8, 10, 12 and 16 are 2 591, 1, 381, 66, 2 727 and 4, respectively. The unique uniform STS(19) is the 5-sparse system A4 of Theorem 1.

## 2.5 Independent Sets

An *independent set* $I \subseteq X$ in a Steiner triple system $(X, \mathcal{B})$ is a set of points with the property that no block of $\mathcal{B}$ is contained in $I$. A *maximum independent set* is an independent set of maximum size. There exists an STS(19) that contains a maximum independent set of size $m$ if and only if $m \in \{7, 8, 9, 10\}$, and $m = 10$ arises precisely when the design contains a subsystem of order 9; see [8, Chapter 17]. The following theorem collects the results of a complete determination.

**Theorem 3.** *The numbers of* STS(19) *with maximum independent set size* 7*,* 8*,* 9 *and* 10 *are* 2*,* 10 133 102 887*,* 951 487 483 *and* 284 457*, respectively.*

The two systems that have maximum independent set of size 7 are the (cyclic) systems A2 and A4.

## 2.6 Chromatic Number

A *colouring* of a Steiner triple system $(X, \mathcal{B})$ is a partition of $X$ into independent sets. A partition of $X$ into $k$ independent sets is a *$k$-colouring*. The *chromatic number* of an STS is the smallest integer $k$ such that the STS has a $k$-colouring, and corresponding colourings are *optimal*. Designs with a unique optimal colouring have been termed *uniquely colourable* [41]. A colouring is *equitable* if the cardinalities of the colour classes differ by at most one. An STS is *$k$-balanced* if every $k$-colouring is equitable.

No STS($v$) with $v > 3$ is 2-chromatic [40]. Moreover, every STS(19) is 4-colourable [13, Theorem 6.1]; see also [24, Theorem 5]. Consequently, the chromatic number of any STS(19) is either 3 or 4. No STS(19) with chromatic number 4 was known; indeed as we see next, none exists. An exhaustive search establishes the following.

**Theorem 4.** *Every* STS(19) *is* 3*-chromatic. More specifically,*

($i$) *every* STS(19) *has a* 3*-colouring with colour class sizes* $(7, 7, 5)$ *and*

($ii$) *every* STS(19) *except for designs* A2 *and* A4 *has a* 3*-colouring with colour class sizes* $(8, 6, 5)$.

Next we show that Theorem 4 completes the determination of the combinations of 3-colouring patterns that can occur in an STS(19). For a given 3-colouring of an STS(19), let the colour classes be $(C_1, C_2, C_3)$. Let $c_i = |C_i|$ for $1 \leqslant i \leqslant 3$. Without loss of generality suppose that $c_1 \geqslant c_2 \geqslant c_3$, and denote the pattern of colour class sizes by the corresponding integer triple $(c_1, c_2, c_3)$. Informally, we refer to the colour classes $C_1, C_2, C_3$ as red, yellow and blue. It is shown in [12, Section 2.4] and [13] that any 3-colouring of an STS(19) must have one of the six patterns

$$(7, 6, 6),\ (7, 7, 5),\ (8, 6, 5),\ (8, 7, 4),\ (9, 5, 5),\ (9, 6, 4),$$

and that certain reductions are possible.

**Lemma 1.** *An* STS(19) *that has a* 3*-colouring with colour class sizes*

($i$) $(7, 7, 5)$ *also has one with sizes* $(7, 6, 6)$,

($ii$) $(8, 6, 5)$ *either has one with sizes* $(7, 7, 5)$ *or one with sizes* $(7, 6, 6)$,

($iii$) $(8, 7, 4)$ *also has one with sizes* $(7, 7, 5)$,

($iv$) $(9, 5, 5)$ *either has one with sizes* $(9, 6, 4)$ *or one with sizes* $(8, 6, 5)$,

($v$) $(9, 6, 4)$ *also has one with sizes* $(8, 6, 5)$,

($vi$) $(8, 7, 4)$ *also has one with sizes* $(8, 6, 5)$,

($vii$) $(9, 5, 5)$ *also has one with sizes* $(8, 6, 5)$,

($viii$) $(9, 6, 4)$ *also has one with sizes* $(9, 5, 5)$,

($ix$) $(9, 6, 4)$ *also has one with sizes* $(8, 7, 4)$.

*Proof.* For $(i)$–$(v)$, see [12, Section 2.4] or [13, Section 4]. It remains only to prove $(vi)$–$(ix)$.

Let $x_{ijk}$, $1 \leqslant i \leqslant j \leqslant k$, denote the number of blocks containing points belonging to colour classes $C_i$, $C_j$ and $C_k$, with appropriate multiplicities. Thus, for example, $x_{122}$ is the number of blocks that contain a red point and two yellow points. Write $x$ for $x_{223}$. As in the proof of [12, Theorem 2.4.1] we can construct the following table by a straightforward computation.

| $(c_1, c_2, c_3)$ | $x_{122}$ | $x_{133}$ | $x_{112}$ | $x_{113}$ | $x_{223}$ | $x_{233}$ | $x_{123}$ |
|---|---|---|---|---|---|---|---|
| (7, 6, 6) | $15-x$ | $x$ | $3+x$ | $18-x$ | $x$ | $15-x$ | 6 |
| (7, 7, 5) | $21-x$ | $x-5$ | $1+x$ | $20-x$ | $x$ | $15-x$ | 5 |
| (8, 6, 5) | $15-x$ | $x-3$ | $7+x$ | $21-x$ | $x$ | $13-x$ | 4 |
| (8, 7, 4) | $21-x$ | $x-7$ | $6+x$ | $22-x$ | $x$ | $13-x$ | 2 |
| (9, 5, 5) | $10-x$ | $x-2$ | $12+x$ | $24-x$ | $x$ | $12-x$ | 1 |
| (9, 6, 4) | $15-x$ | $x-6$ | $12+x$ | $24-x$ | $x$ | $12-x$ | 0 |

Suppose we have an $(8, 7, 4)$ 3-colouring of an STS(19). Then $x \geqslant 7$ since $x_{133} = x - 7 \geqslant 0$. Moreover, $x_{233} = 13 - x \leqslant 6$. Therefore we can find a yellow point to change to blue without creating a blue-blue-blue block. This proves $(vi)$.

Suppose we have a $(9, 5, 5)$ 3-colouring. Since $x_{122} + x_{133} = 8 < 9$ we can find a red point to be changed to either yellow or blue. This proves $(vii)$.

Suppose we have a $(9, 6, 4)$ 3-colouring. If $x_{233} < 6$, we can change a yellow point to blue. So we may assume that $x_{233} = 6$. Then $x_{133} = x_{123} = 0$. Hence each blue point occurs exactly three times in the yellow-blue-blue blocks and paired with three yellow points. So each blue point must occur paired with three yellow points in yellow-yellow-blue blocks. This is impossible; hence $(viii)$ is proved.

Again, suppose we have a $(9, 6, 4)$ 3-colouring. If $x_{122} < 9$, we can change a red point to yellow. Otherwise $x_{122} \geqslant 9$. This forces $x = x_{223} = x_{233} = 6$ and $x_{133} = x_{123} = 0$, which is impossible by the same argument as in the proof of $(viii)$. This proves $(ix)$. □

The main result of this section is a straightforward consequence of Theorem 4 and Lemma 1.

**Theorem 5.** *Any* STS(19) *is* 3*-colourable with one of the following six combinations of* 3*-colouring patterns:*

$$\begin{aligned}
\mathcal{C}_1 &= \{(7,6,6),(7,7,5)\},\\
\mathcal{C}_2 &= \{(7,6,6),(7,7,5),(8,6,5)\},\\
\mathcal{C}_3 &= \{(7,6,6),(7,7,5),(8,6,5),(8,7,4)\},\\
\mathcal{C}_4 &= \{(7,6,6),(7,7,5),(8,6,5),(9,5,5)\},\\
\mathcal{C}_5 &= \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5)\},\\
\mathcal{C}_6 &= \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5),(9,6,4)\}.
\end{aligned}$$

The first combination in Theorem 5, $\{(7,6,6),(7,7,5)\}$, occurs in only two STS(19), both of which are cyclic; in fact these are the two exceptions of Theorem 4($ii$), systems A2 and A4. The other two cyclic STS(19), A1 and A3, have the colouring pattern combination $\{(7,6,6),(7,7,5),(8,6,5)\}$. It is easy to find examples exhibiting each of the remaining combinations.

We are now able to answer the open problem of whether there exists a 3-balanced STS(19) [13, Problem 1]. By [13, Theorem 4.1] and Theorems 4 and 5 we immediately get the following.

**Corollary 1.** *Every* STS(19) *is* 3*-chromatic and has an equitable* 3*-colouring. There exists no* 3*-balanced* STS(19)*.*

In a separate computation we obtained the frequency of occurrence of each combination of 3-colouring patterns. We also obtained information concerning the size of maximum independent sets. Our results are presented in Table 7 in the form of a two-way frequency table of maximum independent set size against combinations of 3-colouring patterns $\mathcal{C}_i$ as defined in Theorem 5. The cell in row $\mathcal{C}_i$, column $j$ gives the number of STS(19) that have 3-colouring pattern combination $\mathcal{C}_i$ and maximum independent set size $j$. Observe that the total count for size 10 is in agreement with [42], and it is worth pointing out that the zero entries in rows $\mathcal{C}_2$ to $\mathcal{C}_6$ can be deduced by elementary arguments without the need for any extensive computation. In particular, it is not difficult to show that an independent set of size 10 excludes the possibility of a (9,5,5) 3-colouring.

Table 7: Colourings and maximum independent sets

| Colouring | 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|
| $\mathcal{C}_1$ | 2 | 0 | 0 | 0 | 2 |
| $\mathcal{C}_2$ | 0 | 53 680 512 | 2 650 830 | 1 241 | 56 332 583 |
| $\mathcal{C}_3$ | 0 | 10 079 422 375 | 421 936 849 | 283 216 | 10 501 642 440 |
| $\mathcal{C}_4$ | 0 | 0 | 2 912 144 | 0 | 2 912 144 |
| $\mathcal{C}_5$ | 0 | 0 | 464 995 662 | 0 | 464 995 662 |
| $\mathcal{C}_6$ | 0 | 0 | 58 991 998 | 0 | 58 991 998 |
| Total | 2 | 10 133 102 887 | 951 487 483 | 284 457 | 11 084 874 829 |

## 2.7 Almost Parallel Classes

A set of nonintersecting blocks that do not contain all points of the design is a *partial parallel class*, and a partial parallel class with $\lfloor v/3 \rfloor$ blocks is an *almost parallel class*. Consequently, six nonintersecting blocks of an STS(19) form an almost parallel class. For each STS(19) we determined the total number of almost parallel classes in the following way.

For each STS(19), the point to be missed by the almost parallel class is specified, after which the problem of finding the almost parallel classes can be formulated as instances

of the exact cover problem. In the exact cover problem, a set $U$ and a collection $\mathcal{S}$ of subsets of $U$ are given, and one wants to determine (one or all) partitions of $U$ using sets from $\mathcal{S}$. To solve instances of the exact cover problem, the libexact software [30], which implements ideas from work by Knuth [32], was utilized. The results are presented in Table 8.

There is a conjecture that for all $v \equiv 1, 3 \pmod 6$, $v \geqslant 15$, there exists an STS($v$) whose largest partial parallel class has fewer than $\lfloor v/3 \rfloor$ blocks [4, Conjecture 2.86], [8, Conjectures 19.4 and 19.5], [41, Section 3.1]. The results in the current work are in accordance with this conjecture.

In fact, Lo Faro already showed that every STS(19) has a partial parallel class with five blocks [33] and, constructively, that there indeed exists an STS(19) with no almost parallel class [34]. The current work shows that there are exactly two STS(19) with no almost parallel classes. These are A4 and the unique design with automorphism group of order 432. The largest number of almost parallel classes, 182, arises in S3.

A set of blocks of a design with the property that each point occurs in exactly $\alpha$ of these blocks is an *$\alpha$-parallel class*. A partition of all blocks into $\alpha$-parallel classes is an *$\alpha$-resolution*, and a design that admits an $\alpha$-resolution is *$\alpha$-resolvable*. A Steiner triple system whose order $v$ is not divisible by 3 cannot have a (1-)parallel class, but may have a 3-parallel class. The existence of Steiner triple systems of order at least 7 without a 3-parallel class is an open problem [8, p. 419].

A complete search demonstrates that every STS(19) not only has a 3-parallel class, but a 3-resolution. It is, however, not always the case that every 3-parallel class can be extended to a 3-resolution. That is, some STS(19) contain a 6-parallel class that is *nonseparable*, in that it does not further partition into two 3-parallel classes. Using [3], the largest $\alpha$ for which an STS($v$) contains a nonseparable $\alpha$-parallel class is 3, 1, 3, 5 and 6 for $v = 7$, 9, 13, 15 and 19, respectively.

## 2.8 Chromatic Index

While the chromatic number concerns colouring points, the chromatic index concerns colouring blocks. More precisely, the *chromatic index* of an STS is the smallest number of colours that can be used to colour the blocks so that no two intersecting blocks receive the same colour.

An STS($v$) is resolvable if and only if its chromatic index is $(v-1)/2$. Since 19 is not divisible by 3, there is no resolvable STS(19), and the smallest possible chromatic index for such a design is $\lceil 57/6 \rceil = 10$.

By elementary counting, an STS(19) with chromatic index 10 must have at least 7 disjoint almost parallel classes. Moreover, the chromatic index of an STS(19) with no almost parallel classes is at least $\lceil 57/5 \rceil = 12$. We now describe the computational approach used to show that 10, 11 and 12 are the only possible chromatic indices for an STS(19).

Exact algorithms and greedy algorithms for finding the chromatic index and upper bounds on the chromatic index of STSs were presented in the early 1980s [2, 5]. Now

Table 8: Number of almost parallel classes

| APC | # | APC | # | APC | # | APC | # |
|---|---|---|---|---|---|---|---|
| 0 | 2 | 79 | 764 738 | 110 | 526 902 725 | 141 | 43 290 |
| 36 | 1 | 80 | 1 224 282 | 111 | 495 595 995 | 142 | 25 609 |
| 40 | 1 | 81 | 1 924 007 | 112 | 458 547 878 | 143 | 14 838 |
| 48 | 5 | 82 | 2 974 055 | 113 | 417 254 801 | 144 | 8 604 |
| 50 | 1 | 83 | 4 513 033 | 114 | 373 408 256 | 145 | 4 827 |
| 51 | 1 | 84 | 6 737 331 | 115 | 328 678 489 | 146 | 2 907 |
| 52 | 2 | 85 | 9 882 490 | 116 | 284 606 260 | 147 | 1 581 |
| 54 | 5 | 86 | 14 239 039 | 117 | 242 381 171 | 148 | 1 028 |
| 56 | 14 | 87 | 20 170 633 | 118 | 203 039 046 | 149 | 522 |
| 57 | 6 | 88 | 28 071 379 | 119 | 167 316 900 | 150 | 386 |
| 58 | 16 | 89 | 38 411 235 | 120 | 135 654 277 | 151 | 210 |
| 59 | 6 | 90 | 51 637 134 | 121 | 108 190 905 | 152 | 173 |
| 60 | 31 | 91 | 68 231 490 | 122 | 84 895 844 | 153 | 75 |
| 61 | 27 | 92 | 88 611 342 | 123 | 65 517 542 | 154 | 85 |
| 62 | 58 | 93 | 113 110 188 | 124 | 49 778 191 | 155 | 32 |
| 63 | 65 | 94 | 141 933 285 | 125 | 37 203 375 | 156 | 53 |
| 64 | 158 | 95 | 175 017 943 | 126 | 27 381 347 | 157 | 6 |
| 65 | 225 | 96 | 212 214 494 | 127 | 19 807 367 | 158 | 22 |
| 66 | 476 | 97 | 252 843 760 | 128 | 14 108 068 | 159 | 6 |
| 67 | 774 | 98 | 296 203 531 | 129 | 9 891 578 | 160 | 24 |
| 68 | 1 606 | 99 | 341 097 019 | 130 | 6 829 506 | 162 | 5 |
| 69 | 2 801 | 100 | 386 153 551 | 131 | 4 633 657 | 164 | 12 |
| 70 | 5 363 | 101 | 429 813 668 | 132 | 3 105 171 | 166 | 3 |
| 71 | 9 930 | 102 | 470 269 272 | 133 | 2 044 697 | 167 | 1 |
| 72 | 18 098 | 103 | 505 968 628 | 134 | 1 327 796 | 168 | 1 |
| 73 | 32 270 | 104 | 535 235 668 | 135 | 847 519 | 172 | 4 |
| 74 | 56 959 | 105 | 556 712 827 | 136 | 536 040 | 174 | 4 |
| 75 | 98 415 | 106 | 569 489 811 | 137 | 332 998 | 180 | 1 |
| 76 | 168 833 | 107 | 572 707 805 | 138 | 203 608 | 182 | 1 |
| 77 | 284 405 | 108 | 566 389 062 | 139 | 123 411 | | |
| 78 | 470 557 | 109 | 550 847 618 | 140 | 74 672 | | |

modern algorithms for finding colourings and chromatic numbers of graphs can be used to determine the chromatic number of the line graph of the design, which equals the chromatic index of the design.

To find a 10-colouring, the algorithm starts by finding sets of 7 disjoint almost parallel classes. To do this, for each STS(19), all almost parallel classes are first found (as in Section 2.7). Using these, sets of 7 disjoint ones are obtained by an algorithm for finding cliques in graphs (form one vertex for each almost parallel class and place edges between disjoint classes). The Cliquer software [37] can be utilized to find the cliques. The final step is an exhaustive search for three partial parallel classes to partition the remaining $57 - 7 \cdot 6 = 15$ blocks.

A more general exhaustive search algorithm was applied to instances with chromatic index greater than 10. The final result is as follows.

**Theorem 6.** *The numbers of* STS(19) *that have chromatic index* 10*,* 11 *and* 12 *are* 11 084 870 752*,* 4 075 *and* 2*, respectively.*

Consequently, exactly the two STS(19) with no almost parallel classes (see Section 2.7) have chromatic index 12. Our results are consistent with the observation that no STS($v$) with $v > 7$ and chromatic index exceeding the minimum chromatic index by more than 2 is known to exist [8, pp. 366–367], [41, p. 411].

## 2.9 Existential Closure

The *block intersection graph* of an STS has one vertex for each block and an edge between two vertices exactly when the corresponding blocks intersect. A graph $G = (V, E)$ is *$n$-existentially closed* if for every $n$-element subset $S \subseteq V$ of vertices and for every subset $T \subseteq S$, there exists a vertex $x \notin S$ that is adjacent to every vertex in $T$ and nonadjacent to every vertex in $S \setminus T$.

In [14] $n$-existentially closed block intersection graphs of STSs are studied. The block intersection graph of an STS($v$) is 2-existentially closed if and only if $v \geqslant 13$, it cannot be 4-existentially closed [36, Theorem 1] for any $v$, and the only possible orders for which it can be 3-existentially closed are 19 and 21. In fact, two STS(19) possess 3-existentially closed block intersection graphs [14].

The following result from [14, Theorem 4.1] helps in designing an algorithm for determining whether the block intersection graph of an STS is 3-existentially closed.

**Theorem 7.** *The block intersection graph of an* STS($v$) *is* 3*-existentially closed if and only if*

(*i*) *the* STS($v$) *contains no subsystem* STS(7)*,*

(*ii*) *the* STS($v$) *contains no subsystem* STS(9)*,*

(*iii*) *for every set of three nonintersecting blocks, if* $v < 19$ *there exists a block that intersects none of the three, and if* $v \geqslant 19$ *there exists a block that intersects all three.*

No STS(19) other than those discovered in [14] is 3-existentially closed.

**Theorem 8.** *The number of* 3*-existentially closed* STS(19) *is* 2.

The two 3-existentially closed STS(19) are A3 and S2.

# 3 Conclusions

The main aim of the current work has been to compute all kinds of properties of STS(19) and collect them in a single place. However, it is impossible to accomplish this task in an exhaustive manner, so we omit discussion of properties that (1) we do not consider to have large general interest, (2) we are not able to present in a compact manner, or (3) we simply are not able to compute at the present time.

For example, we consider various kinds of colouring problems, such as those studied in [9, 18], to be of the first type. Any properties that have been used as invariants for STSs cannot, by definition, be tabulated in a compact way and are of the second type; examples of this type include various forms of so-called trains.

The third type of problems contain some very interesting open problems, including those of determining intersection numbers of STSs, maximal sets of disjoint STSs, and whether all STSs are derived. Further information on these problems can be found in [4, 8]. For example, just determining whether a single STS is derived remains a major challenge.

The problems were addressed using three different computational environments (in Canada, Finland and Great Britain), so we do not try to give exact details about the computations. The computational resources needed partition the problems roughly into three groups: those taking days or at most a couple of weeks ("easy"), those taking up to a couple of years ("intermediate") and those taking up to ten years ("hard"). These CPU times are roughly the times needed for one core of a "contemporary microprocessor".

The intermediate calculations were those of determining subconfigurations (10 CPU weeks), determining the almost parallel classes (1.5 CPU years), constructing the frequency table of maximum independent set size against 3-colouring pattern combination (12 CPU weeks), showing existence of 3-parallel classes (7 CPU months) and searching for 3-existentially closed designs (9 CPU months). The only one belonging to the category of hard calculations was the determination of the chromatic indices, which consumed just under 8 CPU years. All remaining calculations were "easy".

# Appendix

We use the same method for compressing STSs as in the supplement to [6]. That is, for the points we use the symbols `a-s` and represent an STS by a string of 57 symbols $x_1x_2\cdots x_{57}$. The symbol $x_i$ is the largest element in the $i$th block. The other two symbols in the $i$th block are the smallest pair of symbols not occurring in earlier blocks under the *colexicographic* ordering of pairs: a pair $y, z$ with $y < z$ is smaller than a pair $y', z'$ with

$y' < z'$ iff $z < z'$, or $z = z'$ and $y < y'$. The order of the automorphism group is given after each design.

S1: `edgfhghijkllmnljompqporqsnsloqprmrsnnopsrqqprosqsrpsqrrss` (1)
S2: `cefggfhijijklmnokppqmrsolrsqnqpsnrmornsoqpsqporpqrsrsqsrs` (8)
S3: `cefghngjljrikoqplrnqmskmsnonsmrlpmoprqpqosopqsrrpsqqsrsrs` (3)
S4: `cefghigpojlijqmplrqokomsnnqpslrommnsrqprnsoprqsrspqqsrsrs` (1)
S5: `cefghfgjoiksmrlpnksqkmpsnlrnoqmmnqposrprqoorpqsrspqqrssrs` (6)
S6: `cefghigomjsinksllsjqkmropnlqrpomnrpqpqornsopqrsrpqsqsrsrs` (9)
S7: `cefihkgsojosmiqmnrlpjqklospnqlpormprnsprqonsoprqsrqqrssrs` (1)

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Table 9: Number of grids

| Grid | # | Grid | # | Grid | # | Grid | # |
|---|---|---|---|---|---|---|---|
| 21 | 1 | 58 | 421 406 261 | 95 | 5 466 378 | 132 | 19 595 |
| 22 | 1 | 59 | 455 538 873 | 96 | 4 452 414 | 133 | 17 568 |
| 23 | 1 | 60 | 483 962 320 | 97 | 3 625 512 | 134 | 17 390 |
| 24 | 6 | 61 | 505 587 977 | 98 | 2 964 501 | 135 | 15 125 |
| 25 | 27 | 62 | 519 737 441 | 99 | 2 419 681 | 136 | 14 765 |
| 26 | 44 | 63 | 525 975 481 | 100 | 1 984 363 | 137 | 12 845 |
| 27 | 156 | 64 | 524 399 635 | 101 | 1 625 523 | 138 | 12 707 |
| 28 | 403 | 65 | 515 397 821 | 102 | 1 340 634 | 139 | 10 911 |
| 29 | 1 012 | 66 | 499 528 245 | 103 | 1 103 378 | 140 | 10 689 |
| 30 | 2 577 | 67 | 477 877 986 | 104 | 915 322 | 141 | 9 228 |
| 31 | 6 067 | 68 | 451 447 963 | 105 | 756 727 | 142 | 9 097 |
| 32 | 13 721 | 69 | 421 183 378 | 106 | 629 794 | 143 | 7 629 |
| 33 | 29 607 | 70 | 388 549 216 | 107 | 522 121 | 144 | 7 495 |
| 34 | 62 549 | 71 | 354 553 810 | 108 | 439 478 | 145 | 6 593 |
| 35 | 125 648 | 72 | 320 163 173 | 109 | 365 162 | 146 | 6 407 |
| 36 | 246 636 | 73 | 286 220 933 | 110 | 310 349 | 147 | 5 325 |
| 37 | 461 547 | 74 | 253 571 165 | 111 | 256 766 | 148 | 5 266 |
| 38 | 840 481 | 75 | 222 621 207 | 112 | 219 625 | 149 | 4 318 |
| 39 | 1 484 562 | 76 | 193 840 439 | 113 | 183 979 | 150 | 4 386 |
| 40 | 2 534 581 | 77 | 167 454 239 | 114 | 157 625 | 151 | 3 507 |
| 41 | 4 196 398 | 78 | 143 611 784 | 115 | 133 530 | 152 | 3 515 |
| 42 | 6 739 474 | 79 | 122 366 578 | 116 | 115 251 | 153 | 2 820 |
| 43 | 10 522 877 | 80 | 103 592 757 | 117 | 97 139 | 154 | 2 838 |
| 44 | 15 960 510 | 81 | 87 177 751 | 118 | 85 923 | 155 | 2 265 |
| 45 | 23 562 586 | 82 | 72 978 536 | 119 | 72 545 | 156 | 2 455 |
| 46 | 33 871 296 | 83 | 60 813 771 | 120 | 65 014 | 157 | 1 830 |
| 47 | 47 412 716 | 84 | 50 428 258 | 121 | 55 582 | 158 | 1 905 |
| 48 | 64 736 436 | 85 | 41 665 785 | 122 | 50 393 | 159 | 1 433 |
| 49 | 86 205 567 | 86 | 34 306 651 | 123 | 43 478 | 160 | 1 552 |
| 50 | 112 103 389 | 87 | 28 141 430 | 124 | 40 275 | 161 | 1 124 |
| 51 | 142 489 811 | 88 | 23 037 710 | 125 | 34 759 | 162 | 1 284 |
| 52 | 177 059 163 | 89 | 18 809 436 | 126 | 32 578 | 163 | 913 |
| 53 | 215 192 146 | 90 | 15 344 880 | 127 | 28 746 | 164 | 1 010 |
| 54 | 256 144 342 | 91 | 12 489 931 | 128 | 27 080 | 165 | 766 |
| 55 | 298 709 622 | 92 | 10 159 180 | 129 | 23 884 | 166 | 843 |
| 56 | 341 446 147 | 93 | 8 261 382 | 130 | 23 163 | 167 | 557 |
| 57 | 382 864 465 | 94 | 6 721 096 | 131 | 20 281 | 168 | 664 |

Table 9: Number of grids (cont.)

| Grid | # | Grid | # | Grid | # | Grid | # |
|---|---|---|---|---|---|---|---|
| 169 | 490 | 194 | 80 | 219 | 2 | 249 | 3 |
| 170 | 527 | 195 | 19 | 220 | 23 | 250 | 2 |
| 171 | 324 | 196 | 90 | 221 | 2 | 252 | 10 |
| 172 | 429 | 197 | 21 | 222 | 14 | 254 | 1 |
| 173 | 267 | 198 | 70 | 223 | 5 | 255 | 2 |
| 174 | 383 | 199 | 8 | 224 | 33 | 256 | 7 |
| 175 | 206 | 200 | 97 | 225 | 5 | 258 | 1 |
| 176 | 328 | 201 | 16 | 226 | 8 | 260 | 7 |
| 177 | 153 | 202 | 39 | 227 | 5 | 262 | 1 |
| 178 | 232 | 203 | 6 | 228 | 31 | 264 | 8 |
| 179 | 126 | 204 | 79 | 229 | 2 | 267 | 2 |
| 180 | 223 | 205 | 5 | 230 | 4 | 272 | 7 |
| 181 | 128 | 206 | 25 | 231 | 3 | 276 | 4 |
| 182 | 207 | 207 | 13 | 232 | 21 | 280 | 4 |
| 183 | 109 | 208 | 59 | 234 | 10 | 284 | 3 |
| 184 | 155 | 209 | 4 | 235 | 1 | 288 | 5 |
| 185 | 75 | 210 | 51 | 236 | 26 | 294 | 1 |
| 186 | 149 | 211 | 2 | 238 | 5 | 300 | 1 |
| 187 | 57 | 212 | 46 | 239 | 1 | 303 | 1 |
| 188 | 159 | 213 | 10 | 240 | 26 | 308 | 1 |
| 189 | 45 | 214 | 14 | 242 | 1 | 312 | 3 |
| 190 | 91 | 215 | 2 | 243 | 1 | 320 | 2 |
| 191 | 44 | 216 | 38 | 244 | 7 | 336 | 2 |
| 192 | 123 | 217 | 3 | 245 | 1 | 384 | 1 |
| 193 | 36 | 218 | 15 | 248 | 11 | | |

Table 10: Number of prisms

| Prism | # | Prism | # | Prism | # | Prism | # |
|---|---|---|---|---|---|---|---|
| 171 | 1 | 250 | 75 976 | 287 | 42 388 161 | 324 | 198 341 505 |
| 189 | 1 | 251 | 98 127 | 288 | 46 639 711 | 325 | 196 983 412 |
| 200 | 1 | 252 | 125 286 | 289 | 51 169 522 | 326 | 195 225 803 |
| 207 | 1 | 253 | 158 108 | 290 | 55 931 715 | 327 | 193 085 136 |
| 211 | 1 | 254 | 200 729 | 291 | 60 918 787 | 328 | 190 605 951 |
| 216 | 2 | 255 | 253 967 | 292 | 66 151 873 | 329 | 187 795 686 |
| 217 | 1 | 256 | 318 185 | 293 | 71 586 084 | 330 | 184 649 280 |
| 219 | 6 | 257 | 397 908 | 294 | 77 237 835 | 331 | 181 212 592 |
| 221 | 1 | 258 | 492 617 | 295 | 83 032 700 | 332 | 177 549 753 |
| 222 | 6 | 259 | 610 716 | 296 | 88 988 957 | 333 | 173 586 201 |
| 223 | 17 | 260 | 753 345 | 297 | 95 089 060 | 334 | 169 440 136 |
| 224 | 22 | 261 | 921 675 | 298 | 101 293 200 | 335 | 165 109 202 |
| 225 | 27 | 262 | 1 126 793 | 299 | 107 579 627 | 336 | 160 640 418 |
| 226 | 25 | 263 | 1 368 838 | 300 | 113 892 453 | 337 | 155 982 892 |
| 227 | 41 | 264 | 1 655 279 | 301 | 120 225 453 | 338 | 151 293 063 |
| 228 | 73 | 265 | 1 993 377 | 302 | 126 496 164 | 339 | 146 440 917 |
| 229 | 130 | 266 | 2 390 574 | 303 | 132 753 692 | 340 | 141 569 668 |
| 230 | 166 | 267 | 2 851 791 | 304 | 138 902 842 | 341 | 136 664 720 |
| 231 | 245 | 268 | 3 389 099 | 305 | 144 926 038 | 342 | 131 727 398 |
| 232 | 321 | 269 | 4 010 807 | 306 | 150 790 370 | 343 | 126 770 273 |
| 233 | 448 | 270 | 4 727 106 | 307 | 156 429 753 | 344 | 121 858 346 |
| 234 | 667 | 271 | 5 547 565 | 308 | 161 884 623 | 345 | 116 981 409 |
| 235 | 932 | 272 | 6 485 240 | 309 | 167 038 214 | 346 | 112 190 976 |
| 236 | 1 291 | 273 | 7 552 715 | 310 | 171 888 128 | 347 | 107 410 238 |
| 237 | 1 750 | 274 | 8 757 871 | 311 | 176 448 741 | 348 | 102 737 476 |
| 238 | 2 462 | 275 | 10 118 769 | 312 | 180 620 616 | 349 | 98 136 704 |
| 239 | 3 344 | 276 | 11 640 128 | 313 | 184 476 735 | 350 | 93 657 722 |
| 240 | 4 558 | 277 | 13 335 175 | 314 | 187 911 346 | 351 | 89 292 744 |
| 241 | 6 221 | 278 | 15 233 835 | 315 | 190 927 860 | 352 | 85 046 857 |
| 242 | 8 341 | 279 | 17 317 913 | 316 | 193 530 670 | 353 | 80 920 249 |
| 243 | 11 120 | 280 | 19 617 190 | 317 | 195 702 979 | 354 | 76 911 822 |
| 244 | 14 888 | 281 | 22 137 761 | 318 | 197 395 867 | 355 | 73 054 525 |
| 245 | 20 119 | 282 | 24 884 491 | 319 | 198 675 356 | 356 | 69 332 115 |
| 246 | 26 400 | 283 | 27 887 561 | 320 | 199 497 261 | 357 | 65 735 409 |
| 247 | 34 577 | 284 | 31 140 015 | 321 | 199 874 535 | 358 | 62 291 346 |
| 248 | 44 753 | 285 | 34 623 522 | 322 | 199 760 946 | 359 | 58 986 226 |
| 249 | 58 845 | 286 | 38 376 738 | 323 | 199 286 571 | 360 | 55 805 608 |

Table 10: Number of prisms (cont.)

| Prism | # | Prism | # | Prism | # | Prism | # |
|---|---|---|---|---|---|---|---|
| 361 | 52 776 788 | 398 | 5 210 998 | 435 | 424 445 | 472 | 95 566 |
| 362 | 49 877 144 | 399 | 4 870 806 | 436 | 400 992 | 473 | 92 467 |
| 363 | 47 109 094 | 400 | 4 555 184 | 437 | 375 930 | 474 | 89 604 |
| 364 | 44 477 939 | 401 | 4 255 687 | 438 | 356 584 | 475 | 86 116 |
| 365 | 41 956 665 | 402 | 3 975 185 | 439 | 335 932 | 476 | 83 388 |
| 366 | 39 596 950 | 403 | 3 710 635 | 440 | 318 533 | 477 | 80 516 |
| 367 | 37 316 718 | 404 | 3 468 155 | 441 | 300 617 | 478 | 78 206 |
| 368 | 35 158 337 | 405 | 3 235 022 | 442 | 286 646 | 479 | 74 644 |
| 369 | 33 131 446 | 406 | 3 021 856 | 443 | 271 545 | 480 | 72 289 |
| 370 | 31 199 621 | 407 | 2 817 205 | 444 | 258 555 | 481 | 68 924 |
| 371 | 29 360 909 | 408 | 2 632 611 | 445 | 245 429 | 482 | 67 293 |
| 372 | 27 626 089 | 409 | 2 454 635 | 446 | 235 409 | 483 | 63 891 |
| 373 | 25 997 783 | 410 | 2 292 545 | 447 | 224 067 | 484 | 62 065 |
| 374 | 24 455 068 | 411 | 2 137 919 | 448 | 214 575 | 485 | 58 964 |
| 375 | 22 993 528 | 412 | 1 995 564 | 449 | 205 399 | 486 | 56 790 |
| 376 | 21 604 049 | 413 | 1 861 521 | 450 | 197 610 | 487 | 54 505 |
| 377 | 20 310 057 | 414 | 1 737 449 | 451 | 188 729 | 488 | 52 492 |
| 378 | 19 075 074 | 415 | 1 616 932 | 452 | 182 542 | 489 | 49 354 |
| 379 | 17 916 453 | 416 | 1 509 591 | 453 | 176 060 | 490 | 47 536 |
| 380 | 16 819 109 | 417 | 1 404 929 | 454 | 168 815 | 491 | 45 253 |
| 381 | 15 795 662 | 418 | 1 314 772 | 455 | 162 976 | 492 | 43 832 |
| 382 | 14 826 839 | 419 | 1 225 935 | 456 | 158 019 | 493 | 40 816 |
| 383 | 13 907 432 | 420 | 1 144 721 | 457 | 152 147 | 494 | 39 536 |
| 384 | 13 050 725 | 421 | 1 067 065 | 458 | 148 600 | 495 | 37 181 |
| 385 | 12 241 906 | 422 | 995 655 | 459 | 142 312 | 496 | 35 949 |
| 386 | 11 482 906 | 423 | 927 859 | 460 | 138 498 | 497 | 33 708 |
| 387 | 10 762 834 | 424 | 868 000 | 461 | 134 174 | 498 | 32 268 |
| 388 | 10 084 561 | 425 | 811 642 | 462 | 130 272 | 499 | 30 063 |
| 389 | 9 453 238 | 426 | 758 276 | 463 | 125 969 | 500 | 28 901 |
| 390 | 8 853 538 | 427 | 709 328 | 464 | 122 632 | 501 | 27 030 |
| 391 | 8 294 860 | 428 | 663 317 | 465 | 117 860 | 502 | 25 906 |
| 392 | 7 771 024 | 429 | 619 097 | 466 | 115 901 | 503 | 24 000 |
| 393 | 7 269 785 | 430 | 582 159 | 467 | 111 021 | 504 | 23 162 |
| 394 | 6 806 485 | 431 | 544 981 | 468 | 108 594 | 505 | 21 754 |
| 395 | 6 363 581 | 432 | 513 193 | 469 | 104 985 | 506 | 20 937 |
| 396 | 5 960 984 | 433 | 479 631 | 470 | 101 572 | 507 | 19 322 |
| 397 | 5 569 324 | 434 | 452 765 | 471 | 98 344 | 508 | 18 497 |

Table 10: Number of prisms (cont.)

| Prism | # | Prism | # | Prism | # | Prism | # |
|---|---|---|---|---|---|---|---|
| 509 | 17 095 | 546 | 1 731 | 583 | 334 | 620 | 2 116 |
| 510 | 16 519 | 547 | 1 499 | 584 | 421 | 621 | 2 254 |
| 511 | 15 154 | 548 | 1 394 | 585 | 413 | 622 | 2 301 |
| 512 | 14 143 | 549 | 1 222 | 586 | 434 | 623 | 2 357 |
| 513 | 13 411 | 550 | 1 291 | 587 | 392 | 624 | 2 510 |
| 514 | 12 808 | 551 | 1 103 | 588 | 480 | 625 | 2 523 |
| 515 | 11 849 | 552 | 1 094 | 589 | 420 | 626 | 2 527 |
| 516 | 11 530 | 553 | 926 | 590 | 465 | 627 | 2 581 |
| 517 | 10 468 | 554 | 1 000 | 591 | 474 | 628 | 2 719 |
| 518 | 10 064 | 555 | 826 | 592 | 572 | 629 | 2 826 |
| 519 | 9 280 | 556 | 885 | 593 | 521 | 630 | 2 966 |
| 520 | 8 869 | 557 | 719 | 594 | 593 | 631 | 3 099 |
| 521 | 8 064 | 558 | 757 | 595 | 599 | 632 | 3 144 |
| 522 | 7 774 | 559 | 648 | 596 | 662 | 633 | 3 059 |
| 523 | 7 153 | 560 | 728 | 597 | 647 | 634 | 3 157 |
| 524 | 6 714 | 561 | 532 | 598 | 710 | 635 | 3 236 |
| 525 | 6 300 | 562 | 629 | 599 | 729 | 636 | 3 362 |
| 526 | 6 014 | 563 | 517 | 600 | 830 | 637 | 3 384 |
| 527 | 5 362 | 564 | 511 | 601 | 872 | 638 | 3 465 |
| 528 | 5 209 | 565 | 436 | 602 | 972 | 639 | 3 487 |
| 529 | 4 847 | 566 | 505 | 603 | 959 | 640 | 3 393 |
| 530 | 4 551 | 567 | 416 | 604 | 1 011 | 641 | 3 423 |
| 531 | 4 184 | 568 | 497 | 605 | 1 050 | 642 | 3 599 |
| 532 | 4 108 | 569 | 374 | 606 | 1 149 | 643 | 3 580 |
| 533 | 3 736 | 570 | 452 | 607 | 1 188 | 644 | 3 753 |
| 534 | 3 743 | 571 | 358 | 608 | 1 375 | 645 | 3 622 |
| 535 | 3 116 | 572 | 387 | 609 | 1 308 | 646 | 3 827 |
| 536 | 3 141 | 573 | 349 | 610 | 1 358 | 647 | 3 643 |
| 537 | 2 792 | 574 | 345 | 611 | 1 471 | 648 | 3 812 |
| 538 | 2 744 | 575 | 330 | 612 | 1 495 | 649 | 3 744 |
| 539 | 2 548 | 576 | 381 | 613 | 1 553 | 650 | 3 902 |
| 540 | 2 452 | 577 | 336 | 614 | 1 701 | 651 | 3 579 |
| 541 | 2 100 | 578 | 351 | 615 | 1 703 | 652 | 3 790 |
| 542 | 2 155 | 579 | 326 | 616 | 1 875 | 653 | 3 752 |
| 543 | 1 864 | 580 | 382 | 617 | 1 868 | 654 | 3 713 |
| 544 | 1 844 | 581 | 315 | 618 | 1 980 | 655 | 3 683 |
| 545 | 1 613 | 582 | 399 | 619 | 2 027 | 656 | 3 662 |

Table 10: Number of prisms (cont.)

| Prism | # | Prism | # | Prism | # | Prism | # |
|---|---|---|---|---|---|---|---|
| 657 | 3 649 | 694 | 1 495 | 731 | 194 | 768 | 27 |
| 658 | 3 597 | 695 | 1 380 | 732 | 211 | 769 | 22 |
| 659 | 3 637 | 696 | 1 400 | 733 | 175 | 770 | 21 |
| 660 | 3 667 | 697 | 1 250 | 734 | 177 | 771 | 14 |
| 661 | 3 567 | 698 | 1 324 | 735 | 164 | 772 | 21 |
| 662 | 3 416 | 699 | 1 141 | 736 | 152 | 773 | 12 |
| 663 | 3 464 | 700 | 1 136 | 737 | 154 | 774 | 24 |
| 664 | 3 326 | 701 | 1 010 | 738 | 147 | 775 | 10 |
| 665 | 3 370 | 702 | 1 024 | 739 | 116 | 776 | 16 |
| 666 | 3 370 | 703 | 931 | 740 | 116 | 777 | 5 |
| 667 | 3 294 | 704 | 935 | 741 | 88 | 778 | 13 |
| 668 | 3 155 | 705 | 833 | 742 | 123 | 779 | 3 |
| 669 | 3 170 | 706 | 844 | 743 | 89 | 780 | 5 |
| 670 | 3 123 | 707 | 729 | 744 | 97 | 781 | 8 |
| 671 | 3 023 | 708 | 759 | 745 | 75 | 782 | 10 |
| 672 | 3 036 | 709 | 669 | 746 | 103 | 783 | 6 |
| 673 | 2 903 | 710 | 666 | 747 | 68 | 784 | 9 |
| 674 | 2 895 | 711 | 636 | 748 | 90 | 785 | 5 |
| 675 | 2 735 | 712 | 624 | 749 | 79 | 786 | 10 |
| 676 | 2 797 | 713 | 597 | 750 | 91 | 787 | 5 |
| 677 | 2 606 | 714 | 564 | 751 | 56 | 788 | 9 |
| 678 | 2 600 | 715 | 511 | 752 | 60 | 789 | 2 |
| 679 | 2 416 | 716 | 531 | 753 | 44 | 790 | 8 |
| 680 | 2 493 | 717 | 433 | 754 | 65 | 791 | 8 |
| 681 | 2 302 | 718 | 455 | 755 | 43 | 792 | 12 |
| 682 | 2 238 | 719 | 439 | 756 | 45 | 793 | 1 |
| 683 | 2 215 | 720 | 394 | 757 | 46 | 795 | 2 |
| 684 | 2 072 | 721 | 359 | 758 | 39 | 796 | 2 |
| 685 | 2 115 | 722 | 366 | 759 | 42 | 797 | 2 |
| 686 | 2 023 | 723 | 334 | 760 | 35 | 798 | 3 |
| 687 | 1 880 | 724 | 326 | 761 | 28 | 799 | 1 |
| 688 | 1 868 | 725 | 262 | 762 | 30 | 800 | 2 |
| 689 | 1 724 | 726 | 306 | 763 | 23 | 801 | 1 |
| 690 | 1 645 | 727 | 229 | 764 | 40 | 805 | 1 |
| 691 | 1 620 | 728 | 253 | 765 | 15 | 806 | 5 |
| 692 | 1 595 | 729 | 253 | 766 | 16 | 807 | 1 |
| 693 | 1 497 | 730 | 218 | 767 | 19 | 808 | 4 |

Table 10: Number of prisms (cont.)

| Prism | # | Prism | # | Prism | # | Prism | # |
|---|---|---|---|---|---|---|---|
| 809 | 1 | 838 | 1 | 856 | 1 | 912 | 2 |
| 814 | 1 | 840 | 2 | 864 | 2 | 918 | 6 |
| 816 | 3 | 844 | 1 | 868 | 1 | 1152 | 3 |
| 818 | 1 | 846 | 2 | 870 | 4 | | |
| 822 | 14 | 850 | 1 | 878 | 1 | | |
| 832 | 1 | 852 | 1 | 888 | 2 | | |

Table 11: Number of mitres

| Mitre | # | Mitre | # | Mitre | # | Mitre | # |
|---|---|---|---|---|---|---|---|
| 0 | 4 | 29 | 666 856 068 | 56 | 699 975 | 83 | 39 |
| 3 | 11 | 30 | 726 726 670 | 57 | 427 224 | 84 | 83 |
| 4 | 27 | 31 | 765 630 873 | 58 | 261 965 | 85 | 16 |
| 5 | 94 | 32 | 780 912 655 | 59 | 162 576 | 86 | 47 |
| 6 | 463 | 33 | 771 673 239 | 60 | 105 125 | 87 | 20 |
| 7 | 1 587 | 34 | 739 625 001 | 61 | 68 560 | 88 | 34 |
| 8 | 5 196 | 35 | 688 305 207 | 62 | 47 177 | 89 | 7 |
| 9 | 16 130 | 36 | 622 481 814 | 63 | 32 413 | 90 | 54 |
| 10 | 45 051 | 37 | 547 576 707 | 64 | 23 643 | 91 | 1 |
| 11 | 119 156 | 38 | 468 917 351 | 65 | 16 778 | 92 | 19 |
| 12 | 292 925 | 39 | 391 303 591 | 66 | 12 393 | 93 | 9 |
| 13 | 685 985 | 40 | 318 424 938 | 67 | 8 661 | 94 | 7 |
| 14 | 1 502 196 | 41 | 252 876 637 | 68 | 6 489 | 96 | 27 |
| 15 | 3 122 990 | 42 | 196 124 480 | 69 | 4 295 | 98 | 2 |
| 16 | 6 160 011 | 43 | 148 685 094 | 70 | 3 264 | 99 | 2 |
| 17 | 11 527 121 | 44 | 110 224 646 | 71 | 2 181 | 100 | 6 |
| 18 | 20 542 885 | 45 | 79 959 174 | 72 | 1 700 | 102 | 7 |
| 19 | 34 903 297 | 46 | 56 803 086 | 73 | 990 | 104 | 2 |
| 20 | 56 577 514 | 47 | 39 545 210 | 74 | 909 | 105 | 2 |
| 21 | 87 700 390 | 48 | 26 981 662 | 75 | 469 | 108 | 5 |
| 22 | 130 128 895 | 49 | 18 067 853 | 76 | 465 | 112 | 3 |
| 23 | 185 013 010 | 50 | 11 873 632 | 77 | 270 | 114 | 1 |
| 24 | 252 364 501 | 51 | 7 665 089 | 78 | 263 | 116 | 2 |
| 25 | 330 721 805 | 52 | 4 870 654 | 79 | 122 | 120 | 2 |
| 26 | 416 700 734 | 53 | 3 046 823 | 80 | 191 | 144 | 3 |
| 27 | 505 540 524 | 54 | 1 883 004 | 81 | 72 | | |
| 28 | 591 121 831 | 55 | 1 150 672 | 82 | 96 | | |

Table 12: Number of hexagons

| Hexa | # | Hexa | # | Hexa | # | Hexa | # |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 34 | 724 247 745 | 66 | 436 234 | 98 | 110 |
| 2 | 1 | 35 | 714 131 642 | 67 | 326 333 | 99 | 62 |
| 4 | 8 | 36 | 685 867 252 | 68 | 239 208 | 100 | 77 |
| 5 | 2 | 37 | 642 422 184 | 69 | 179 527 | 101 | 33 |
| 6 | 18 | 38 | 587 540 455 | 70 | 134 495 | 102 | 74 |
| 7 | 42 | 39 | 525 307 321 | 71 | 100 405 | 103 | 17 |
| 8 | 275 | 40 | 459 726 499 | 72 | 75 980 | 104 | 35 |
| 9 | 1 060 | 41 | 394 271 746 | 73 | 57 056 | 105 | 28 |
| 10 | 3 888 | 42 | 331 862 444 | 74 | 43 803 | 106 | 28 |
| 11 | 13 543 | 43 | 274 475 233 | 75 | 31 922 | 107 | 10 |
| 12 | 42 046 | 44 | 223 366 811 | 76 | 26 629 | 108 | 136 |
| 13 | 119 420 | 45 | 179 088 397 | 77 | 17 366 | 109 | 10 |
| 14 | 315 586 | 46 | 141 683 536 | 78 | 13 996 | 110 | 17 |
| 15 | 769 997 | 47 | 110 703 052 | 79 | 9 867 | 111 | 10 |
| 16 | 1 750 488 | 48 | 85 587 484 | 80 | 8 815 | 112 | 14 |
| 17 | 3 711 050 | 49 | 65 546 910 | 81 | 5 888 | 113 | 1 |
| 18 | 7 390 282 | 50 | 49 813 749 | 82 | 5 139 | 114 | 17 |
| 19 | 13 851 974 | 51 | 37 586 617 | 83 | 3 120 | 115 | 1 |
| 20 | 24 536 316 | 52 | 28 199 864 | 84 | 2 880 | 116 | 18 |
| 21 | 41 147 211 | 53 | 21 046 347 | 85 | 1 883 | 117 | 4 |
| 22 | 65 593 940 | 54 | 15 677 184 | 86 | 2 264 | 118 | 1 |
| 23 | 99 604 643 | 55 | 11 622 883 | 87 | 1 127 | 120 | 10 |
| 24 | 144 448 598 | 56 | 8 623 668 | 88 | 1 016 | 121 | 1 |
| 25 | 200 532 422 | 57 | 6 370 044 | 89 | 615 | 122 | 4 |
| 26 | 266 967 992 | 58 | 4 713 086 | 90 | 1 645 | 124 | 8 |
| 27 | 341 559 277 | 59 | 3 483 045 | 91 | 436 | 126 | 16 |
| 28 | 420 712 045 | 60 | 2 580 662 | 92 | 408 | 128 | 1 |
| 29 | 499 765 074 | 61 | 1 909 874 | 93 | 249 | 132 | 3 |
| 30 | 573 401 076 | 62 | 1 419 396 | 94 | 234 | 144 | 12 |
| 31 | 636 579 383 | 63 | 1 050 752 | 95 | 150 | 171 | 1 |
| 32 | 684 620 989 | 64 | 786 486 | 96 | 248 | | |
| 33 | 714 416 762 | 65 | 577 280 | 97 | 75 | | |

Table 13: Number of crowns

| Crown | # | Crown | # | Crown | # | Crown | # |
|---|---|---|---|---|---|---|---|
| 0 | 4 | 75 | 84 | 112 | 39 276 | 149 | 11 026 967 |
| 24 | 8 | 76 | 225 | 113 | 46 162 | 150 | 12 418 598 |
| 28 | 1 | 77 | 98 | 114 | 55 376 | 151 | 13 951 975 |
| 32 | 7 | 78 | 230 | 115 | 65 172 | 152 | 15 649 589 |
| 34 | 1 | 79 | 146 | 116 | 78 169 | 153 | 17 504 989 |
| 36 | 17 | 80 | 303 | 117 | 92 109 | 154 | 19 550 198 |
| 40 | 4 | 81 | 184 | 118 | 110 533 | 155 | 21 784 052 |
| 42 | 2 | 82 | 352 | 119 | 130 711 | 156 | 24 217 202 |
| 44 | 3 | 83 | 271 | 120 | 155 188 | 157 | 26 857 968 |
| 45 | 1 | 84 | 507 | 121 | 183 383 | 158 | 29 735 229 |
| 46 | 3 | 85 | 409 | 122 | 218 318 | 159 | 32 838 784 |
| 48 | 17 | 86 | 625 | 123 | 256 913 | 160 | 36 187 030 |
| 49 | 2 | 87 | 538 | 124 | 304 546 | 161 | 39 768 756 |
| 50 | 4 | 88 | 788 | 125 | 357 058 | 162 | 43 644 429 |
| 51 | 3 | 89 | 745 | 126 | 420 855 | 163 | 47 762 633 |
| 52 | 19 | 90 | 1 103 | 127 | 493 066 | 164 | 52 146 277 |
| 54 | 23 | 91 | 997 | 128 | 580 012 | 165 | 56 809 902 |
| 55 | 5 | 92 | 1 448 | 129 | 678 149 | 166 | 61 761 576 |
| 56 | 37 | 93 | 1 460 | 130 | 794 787 | 167 | 66 960 296 |
| 57 | 9 | 94 | 1 941 | 131 | 925 609 | 168 | 72 455 628 |
| 58 | 16 | 95 | 1 999 | 132 | 1 080 365 | 169 | 78 205 211 |
| 59 | 6 | 96 | 2 809 | 133 | 1 256 516 | 170 | 84 194 952 |
| 60 | 72 | 97 | 2 861 | 134 | 1 462 493 | 171 | 90 422 800 |
| 61 | 5 | 98 | 3 569 | 135 | 1 691 178 | 172 | 96 907 778 |
| 62 | 22 | 99 | 3 832 | 136 | 1 960 531 | 173 | 103 579 676 |
| 63 | 16 | 100 | 5 157 | 137 | 2 262 445 | 174 | 110 428 354 |
| 64 | 65 | 101 | 5 644 | 138 | 2 612 802 | 175 | 117 487 564 |
| 65 | 16 | 102 | 7 012 | 139 | 3 008 486 | 176 | 124 638 538 |
| 66 | 77 | 103 | 7 868 | 140 | 3 455 009 | 177 | 131 927 624 |
| 67 | 19 | 104 | 9 735 | 141 | 3 958 995 | 178 | 139 275 613 |
| 68 | 71 | 105 | 11 111 | 142 | 4 536 189 | 179 | 146 638 317 |
| 69 | 32 | 106 | 13 806 | 143 | 5 178 047 | 180 | 154 028 623 |
| 70 | 81 | 107 | 15 906 | 144 | 5 903 381 | 181 | 161 359 146 |
| 71 | 55 | 108 | 19 655 | 145 | 6 715 687 | 182 | 168 619 294 |
| 72 | 173 | 109 | 22 619 | 146 | 7 629 172 | 183 | 175 716 385 |
| 73 | 65 | 110 | 27 800 | 147 | 8 645 817 | 184 | 182 665 320 |
| 74 | 144 | 111 | 32 269 | 148 | 9 772 477 | 185 | 189 374 242 |

Table 13: Number of crowns (cont.)

| Crown | # | Crown | # | Crown | # | Crown | # |
|---|---|---|---|---|---|---|---|
| 186 | 195 806 871 | 217 | 134 563 689 | 248 | 4 411 819 | 279 | 3 866 |
| 187 | 201 907 700 | 218 | 126 661 383 | 249 | 3 734 688 | 280 | 3 026 |
| 188 | 207 659 159 | 219 | 118 861 873 | 250 | 3 149 311 | 281 | 2 220 |
| 189 | 212 988 838 | 220 | 111 206 681 | 251 | 2 648 386 | 282 | 1 621 |
| 190 | 217 852 023 | 221 | 103 671 809 | 252 | 2 219 528 | 283 | 1 190 |
| 191 | 222 201 411 | 222 | 96 346 526 | 253 | 1 850 527 | 284 | 880 |
| 192 | 226 058 774 | 223 | 89 252 032 | 254 | 1 538 216 | 285 | 617 |
| 193 | 229 322 865 | 224 | 82 406 974 | 255 | 1 272 656 | 286 | 458 |
| 194 | 231 961 935 | 225 | 75 860 206 | 256 | 1 051 337 | 287 | 337 |
| 195 | 233 966 564 | 226 | 69 578 295 | 257 | 862 379 | 288 | 237 |
| 196 | 235 362 932 | 227 | 63 614 491 | 258 | 707 331 | 289 | 186 |
| 197 | 236 055 372 | 228 | 57 977 229 | 259 | 576 064 | 290 | 135 |
| 198 | 236 115 675 | 229 | 52 664 490 | 260 | 468 744 | 291 | 88 |
| 199 | 235 469 719 | 230 | 47 671 940 | 261 | 378 298 | 292 | 63 |
| 200 | 234 145 518 | 231 | 43 011 588 | 262 | 304 621 | 293 | 35 |
| 201 | 232 142 509 | 232 | 38 676 935 | 263 | 244 241 | 294 | 36 |
| 202 | 229 517 435 | 233 | 34 668 107 | 264 | 194 690 | 295 | 12 |
| 203 | 226 209 636 | 234 | 30 961 644 | 265 | 155 113 | 296 | 19 |
| 204 | 222 338 699 | 235 | 27 551 781 | 266 | 123 781 | 297 | 14 |
| 205 | 217 827 123 | 236 | 24 435 171 | 267 | 96 942 | 298 | 5 |
| 206 | 212 820 389 | 237 | 21 602 222 | 268 | 76 095 | 299 | 7 |
| 207 | 207 301 265 | 238 | 19 035 194 | 269 | 59 785 | 300 | 4 |
| 208 | 201 303 814 | 239 | 16 706 493 | 270 | 46 762 | 301 | 1 |
| 209 | 194 883 375 | 240 | 14 612 461 | 271 | 36 086 | 302 | 1 |
| 210 | 188 122 519 | 241 | 12 739 285 | 272 | 27 879 | 303 | 4 |
| 211 | 180 992 703 | 242 | 11 064 520 | 273 | 21 426 | 306 | 1 |
| 212 | 173 617 922 | 243 | 9 577 688 | 274 | 16 461 | 309 | 2 |
| 213 | 166 018 485 | 244 | 8 260 815 | 275 | 12 570 | 314 | 1 |
| 214 | 158 267 357 | 245 | 7 095 407 | 276 | 9 619 | | |
| 215 | 150 399 412 | 246 | 6 078 552 | 277 | 7 117 | | |
| 216 | 142 486 139 | 247 | 5 188 692 | 278 | 5 272 | | |