Commuting Involution Graphs for 3-Dimensional Unitary Groups

Alistaire Everett

School of Mathematics
The University of Manchester
Oxford Road, Manchester, M13 9PL, UK

a.everett@maths.manchester.ac.uk

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Abstract

For a group G and X a subset of G the commuting graph of G on X, denoted by $\mathcal{C}(G,X)$, is the graph whose vertex set is X with $x, y \in X$ joined by an edge if $x \neq y$ and x and y commute. If the elements in X are involutions, then $\mathcal{C}(G,X)$ is called a commuting involution graph. This paper studies $\mathcal{C}(G,X)$ when G is a 3-dimensional projective special unitary group and X a G-conjugacy class of involutions, determining the diameters and structure of the discs of these graphs.

1 Introduction

For a group G and a subset X of G, we define the commuting graph, denoted $\mathcal{C}(G,X)$, to be the graph whose vertex set is X with two distinct vertices $x, y \in X$ joined by an edge if and only if $xy = yx$. Commuting graphs first came to prominence in the groundbreaking paper of Brauer and Fowler [6], famous for containing a proof that only finitely many finite simple groups can contain a given involution centralizer. The commuting graphs employed in this paper had $X = G \setminus \{1\}$ – such graphs have played a vital role in recent results relating to the Margulis–Platanov conjecture (see [11]). When X is a conjugacy class of involutions, we call $\mathcal{C}(G,X)$ a commuting involution graph. This special case demonstrated its importance in the (mostly unpublished) work of Fischer [9], which led to the construction of three new sporadic simple groups. Aschbacher [1] also showed a necessary condition on a commuting involution graph for the presence of a strongly embedded subgroup in G. The detailed study of commuting involution graphs came to the fore in 2003 with the work of Bates, Bundy, Hart (nèe Perkins) and Rowley, which explored commuting involution graphs for G a symmetric group, or more generally a finite Coxeter group, a special linear group, or a sporadic simple group ([2], [3], [4], [5]).

Recently some of the remaining sporadic simple groups were addressed in Taylor [12] and Wright [14]. When G is a 4-dimensional projective symplectic group, the structure of $\mathcal{C}(G,X)$ was determined in [8].

We continue the study of $\mathcal{C}(G,X)$ when G is a finite simple group of Lie type of rank 1 and X is a G-conjugacy class of involutions. The case when G is a 2-dimensional projective special linear group was addressed in [4]. The well-known structures of $U_3(2^a)$ and $\mathrm{Sz}(2^{2a+1})$ where $a \in \mathbb{N}$ quickly reveal the commuting involution graphs are disconnected with the connected components are cliques. So the 3-dimensional projective unitary groups of odd characteristic and the Ree groups of characteristic 3 remain to be studied. This paper concentrates on the 3-dimensional unitary groups and from now on, we set $q = p^a$ for p an odd prime and $a \in \mathbb{N}$. Let $H = SU_3(q)$ and let X be the H-conjugacy class of involutions. For $t \in X$ we define the i^{th} disc to be $\Delta_i(t) = \{x \in X \,|\, d(t,x) = i\}$ where d is the standard distance metric on $\mathcal{C}(H,X)$. Our main theorem is as follows.

Theorem 1.1 *$\mathcal{C}(H,X)$ is connected of diameter 3, with disc sizes*

$$\begin{aligned} |\Delta_1(t)| &= q(q-1); \\ |\Delta_2(t)| &= q(q-2)(q^2-1); \ \textit{and} \\ |\Delta_3(t)| &= (q+1)(q^2-1). \end{aligned}$$

We remark that for $G = H/Z(H) \cong U_3(q)$ and $X_G = XZ(H)/Z(H)$, the graphs $\mathcal{C}(H,X)$ and $\mathcal{C}(G,X_G)$ are isomorphic. The proof of Theorem 1.1 is constructive, determining the graph structure as one "steps around the graph". With an appropriately chosen t, Lemma 2.3 shows that one can identify which disc a given involution $x \in X$ lies in, by inspection of its top-left entry. It is interesting to note that the third disc is a single $C_H(t)$-orbit if and only if $q \not\equiv 5 \pmod 6$, otherwise it splits into three $C_H(t)$-orbits. The collapsed adjacency graphs for both cases are given in [7]. Our group theoretic notation is standard, as given in [10].

2 The Structure of $\mathcal{C}(G,X)$

This section gives a proof of Theorem 1.1. Let V be the unitary $GF(q^2)H$-module with basis $\{e_i\}$ and define the unitary form on V by $(e_i, e_j) = \delta_{ij}$. Hence the Gram matrix of this form is the identity matrix, and H can be explicitly described as

$$H = \left\{ A \in SL_3(q^2) \,\middle|\, \overline{A}^T A = I_3 \right\} \cong SU_3(q).$$

For $\alpha \in GF(q^2)$ we set $\overline{\alpha} = \alpha^q$, and $\overline{(a_{ij})} = (\overline{a_{ij}})$. For a matrix g, define g_{ij} to be its $(i,j)^{\text{th}}$ entry. There is only one class of involutions in H, which we denote by X, and fix a representative $t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Lemma 2.1 *(i)* $C_H(t) = \left\{ \left(\begin{array}{c|cc} (ad-bc)^{-1} & & \\ \hline & a & b \\ & c & d \end{array}\right) \middle| \begin{array}{c} a,b,c,d \in GF(q^2) \\ \overline{a}a + \overline{c}c = \overline{b}b + \overline{d}d = 1 \\ ad-bc \neq 0 \\ \overline{a}b + \overline{c}d = \overline{b}a + \overline{d}c = 0 \end{array} \right\} \cong GU_2(q).$

(ii) $|X| = q^2(q^2-q+1)$.

(iii) $|\Delta_1(t)| = q(q-1)$.

(iv) If $x \in \Delta_1(t)$, *then* $|\Delta_1(t) \cap \Delta_1(x)| = 1$.

Proof Clearly

$$C_H(t) = \left\{ \left(\begin{array}{c|c} \det A^{-1} & \\ \hline & A \end{array}\right) \middle| A \in GU_2(q) \right\} \cong GU_2(q)$$

proving (i).
Part (ii) follows from the fact that $|H| = q^3(q^3+1)(q^2-1)$ and $|GU_2(q)| = q(q+1)(q^2-1)$. Let $x = \left(\begin{array}{c|c} \det A^{-1} & \\ \hline & A \end{array}\right) \in C_H(t) \cap X$. Using a result of Wall [13], there are two classes of involutions in $GU_2(q)$, represented by $-I_2$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. If $A = -I_2$, then $x = t$. Assume then that A is the latter choice, giving $\Delta_1(t) = x^{C_G(H)}$. By a routine calculation as in part (i), it is easy to see that

$$C_H(x) = \left\{ \left(\begin{array}{c|c} A & \\ \hline & \det A^{-1} \end{array}\right) \middle| A \in GU_2(q) \right\},$$

and so

$$C_H(\langle t,x \rangle) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \middle| a,b \in GF(q^2),\ \overline{a}a = \overline{b}b = 1 \right\}$$

with $|C_H(\langle t,x \rangle)| = (q+1)^2$. Hence $|\Delta_1(t)| = \frac{|C_H(t)|}{|C_H(\langle t,x \rangle)|} = q(q-1)$, proving (iii), while (iv) follows immediately from the structure of $C_H(\langle t,x \rangle)$. □

Henceforth, we set $x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_1(t)$.

Lemma 2.2 *(i) Let* $g,\ h \in \Delta_2(t)$. *If* $g_{11} \neq h_{11}$, *then* g *and* h *are not* $C_H(t)$*-conjugate.*

(ii) $\Delta_2(t) \cap \Delta_1(x) = \left\{ \left(\begin{array}{cc|c} a & b & \\ \overline{b} & -a & \\ \hline & & -1 \end{array}\right) \middle| \overline{b}b = 1 - a^2,\ a \in GF(q) \setminus \{\pm 1\} \right\}.$

(iii) For each $a \in GF(q) \setminus \{\pm 1\}$, *there are* $q+1$ *elements* g *of* $\Delta_2(t) \cap \Delta_1(x)$ *such that* $g_{11} = a$.

Proof By an analogous method to that in Lemma 2.1(i), it is clear that

$$\Delta_1(x) = \left\{ \left(\begin{array}{cc|c} a & b & \\ c & -a & \\ \hline & & -1 \end{array} \right) \middle| \, a, b, c \in GF(q^2), \ a^2 + bc = 1 \right\}.$$

Let

$$g = \left(\begin{array}{cc|c} a & b & \\ c & -a & \\ \hline & & -1 \end{array} \right) \in \Delta_1(x),$$

for $a, b, c \in GF(q^2)$, and $h \in C_H(t)$. Now $(h^{-1}gh)_{11} = h_{11}^{-1} a h_{11} = a$ and so any two $C_H(t)$-conjugate elements have the same top-left entry, so proving (i).

If $b = 0$ then $a^2 + bc = a^2 = 1$ and so $a = \pm 1$. But then $\overline{a}a = 1$ and thus $\overline{c}c = 0$ implying $c = 0$. Similarly, if $c = 0$ then $b = 0$. If $a = \pm 1$, then $1 + bc = 1$ and so $bc = 0$. Hence, either $b = 0$ or $c = 0$ and therefore both are 0. However, $a = 1$ implies $g = t$, and $a = -1$ implies $g \in \Delta_1(t)$. Therefore if $a = \pm 1$, then $g \notin \Delta_2(t)$. In particular, if $a \neq \pm 1$ then $g \in \Delta_2(t)$, since $d(t, x) = 1$ and $[g, x] = 1$. Suppose now $a \neq \pm 1$, so $b, c \neq 0$. Then by Lemma 2.1(i), we have $\overline{a}a + \overline{c}c = \overline{a}a + \overline{b}b = 1$ and $\overline{a}b = a\overline{c}$. Therefore $\overline{a}a + \overline{c}c = a^2\overline{c}b^{-1} + \overline{c}c = 1$ and so $a^2 b^{-1} + c = \overline{c^{-1}}$. It follows that $b\overline{c}^{-1} = a^2 + bc = 1$ and hence $b = \overline{c}$. However, this yields $\overline{a} = a$, implying $a \in GF(q) \setminus \{\pm 1\}$, proving (ii).

By combining parts (i) and (ii), $\Delta_1(x) \cap \Delta_2(t)$ is partitioned into $C_H(\langle t, x \rangle)$-orbits, with the action of $C_H(\langle t, x \rangle)$ leaving the diagonal entries unchanged. Since $a \neq \pm 1$, $\overline{b}b \neq 0$ and $\overline{b}b - (1 + a^2) = 0$. Since there are $q + 1$ solutions in $GF(q^2)$ to the equation $x^{q+1} = \lambda$ for any fixed $\lambda \in GF(q)$, there are $q + 1$ values of b that satisfy this equation. Therefore x is centralised by $q + 1$ involutions sharing a common top-left entry, proving (iii). □

Lemma 2.3 *There are exactly $(q - 2)$ $C_H(t)$-orbits in $\Delta_2(t)$.*

Proof By Lemma 2.2(i) and (ii), there are at least $(q - 2)$ $C_H(t)$-orbits in $\Delta_2(t)$. It suffices to prove that any two matrices commuting with x that share a common top-left entry are $C_H(\langle t, x \rangle)$-conjugate. Let $g \in \Delta_2(t) \cap \Delta_1(x)$, and $a \in GF(q) \setminus \{\pm 1\}$ be fixed such that $g_{11} = a$ and set $g_{12} = b$. By direct calculation, the diagonal entries of g remain unchanged under conjugation by $C_H(\langle t, x \rangle)$. Let

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^{-1} \end{pmatrix} \in C_H(\langle t, x \rangle)$$

where $\overline{\beta}\beta = 1$. Then

$$h^{-1}gh = \left(\begin{array}{cc|c} a & b\beta & \\ \beta^{-1}\overline{b} & -a & \\ \hline & & -1 \end{array} \right).$$

Clearly $b\beta$ takes $q+1$ different values for the $q+1$ different values of β. However, since there are only $q+1$ possible values for b, all such values are covered. That is to say, all matrices of the form

$$\left(\begin{array}{cc|c} a & b & \\ \overline{b} & -a & \\ \hline & & -1 \end{array}\right) \in \Delta_2(t) \cap \Delta_1(x),\ a \neq \pm 1,\ \overline{b}b = 1 - a^2$$

lie in the same $C_H(\langle t, x\rangle)$ orbit, and thus are all $C_H(t)$-conjugate. Therefore, all involutions that centralise x and share a common top-left entry are $C_H(t)$-conjugate and so the lemma follows. □

Lemma 2.4 $|\Delta_2(t)| = q(q^2-1)(q-2)$.

Proof Let

$$g = \left(\begin{array}{c|cc} -1 & & \\ \hline & a & b \\ & \overline{b} & -a \end{array}\right) \in \Delta_1(t) \text{ and } h = \left(\begin{array}{cc|c} \alpha & \beta & \\ \overline{\beta} & -\alpha & \\ \hline & & -1 \end{array}\right) \in \Delta_2(t) \cap \Delta_1(x)$$

for $\alpha \neq \pm 1$ and $\beta\overline{\beta} = 1 - \alpha^2$ fixed. Then

$$gh = \begin{pmatrix} -\alpha & a\beta & b\beta \\ -\overline{\beta} & -a\alpha & -b\alpha \\ 0 & -\overline{b} & a \end{pmatrix} \quad \text{and} \quad hg = \begin{pmatrix} -\alpha & -\beta & 0 \\ a\overline{\beta} & -a\alpha & -b \\ 0 & -\overline{b}\alpha & a \end{pmatrix}.$$

If $[g,h] = 1$ then $a\overline{\beta} = -\overline{\beta}$ and $b\beta = 0$ imply $a = -1$ and $b = 0$, since $\beta \neq 0$. Therefore, $g = x$ and thus h commutes with a single element of $\Delta_1(t)$. Since $\Delta_1(t)$ is a single $C_H(t)$-orbit, and combining Lemmas 2.1(iii) and 2.2(iii), all $C_H(t)$-orbits in $\Delta_2(t)$ have length $q(q-1)(q+1) = q(q^2-1)$. Hence $|\Delta_2(t)| = q(q^2-1)(q-2)$, since $\Delta_2(t)$ is a partition of $C_H(t)$-orbits. □

For each $\alpha \in GF(q) \setminus \{\pm 1\}$, define $\Delta_2^\alpha(t)$ to be the $C_H(t)$-orbit in $\Delta_2(t)$ consisting of matrices with top-left entry $\alpha \in GF(q) \setminus \{\pm 1\}$. By Lemmas 2.1(i) and 2.2(iii), $\Delta_2^\alpha(t)$ can be written explicitly as

$$\Delta_2^\alpha(t) = \left\{ \begin{pmatrix} \alpha & aD\beta & bD\beta \\ d\overline{\beta}D^{-2} & (-ad\alpha + bc)D^{-1} & bdD^{-1}(1-\alpha) \\ -c\overline{\beta}D^{-2} & acD^{-1}(\alpha - 1) & (bc\alpha - ad)D^{-1} \end{pmatrix} \;\middle|\; \begin{array}{c} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GU_2(q) \\ D = ad - bc \\ \overline{\beta}\beta = 1 - \alpha^2 \end{array} \right\}. \tag{2.1}$$

Lemma 2.5 *Suppose*

$$g = \left(\begin{array}{cc|c} \alpha & \beta & \\ \overline{\beta} & -\alpha & \\ \hline & & -1 \end{array}\right) \in \Delta_2^{\alpha}(t) \cap \Delta_1(x)$$

and

$$h = \begin{pmatrix} \gamma & aD\delta & bD\delta \\ d\overline{\delta}D^{-2} & (-ad\gamma + bc)D^{-1} & bdD^{-1}(1-\gamma) \\ -c\overline{\delta}D^{-2} & acD^{-1}(\gamma - 1) & (bc\gamma - ad)D^{-1} \end{pmatrix} \in \Delta_2^{\gamma}(t)$$

satisfy the conditions of (2.1)*. If* $[g,h] = 1$ *then*

(i) $d = a\overline{\beta}\beta^{-1}\overline{\delta^{-1}}\delta D^3$*;*

(ii) if $b, c \neq 0$ *then* $a = -(1+\alpha)(1-\gamma)^{-1}\overline{\beta^{-1}\delta}D^{-1}$ *and* $b = 2D\beta^{-1}(1-\gamma)^{-1}(\beta\gamma - a\alpha\delta D)c^{-1}$*; and*

(iii) if $b = c = 0$ *then* $\beta\gamma = a\alpha\delta D$*.*

Proof Recall that since $\alpha, \gamma \neq \pm 1$, we have $\beta, \delta \neq 0$. Direct calculation shows that

$$gh = \begin{pmatrix} \alpha\gamma + \beta d\overline{\delta}D^{-2} & \alpha a D\delta + \beta D^{-1}(bc - ad\gamma) & \alpha bD\delta + \beta bdD^{-1}(1-\gamma) \\ \overline{\beta}\gamma - \alpha d\overline{\delta}D^{-2} & \overline{\beta}aD\delta - \alpha D^{-1}(bc - ad\gamma) & \overline{\beta}bD\delta - \alpha bdD^{-1}(1-\gamma) \\ c\overline{\delta}D^{-2} & (1-\gamma)acD^{-1} & -D^{-1}(bc\gamma - ad) \end{pmatrix}$$

and

$$hg = \begin{pmatrix} \alpha\gamma + \overline{\beta}aD\delta & \beta\gamma - a\alpha D\delta & -bD\delta \\ \alpha d\overline{\delta}D^{-2} + \overline{\beta}D^{-1}(bc - ad\gamma) & \beta d\overline{\delta}D^{-2} - \alpha(bc - ad\gamma)D^{-1} & -bdD^{-1}(1-\gamma) \\ -\alpha c\overline{\delta}D^{-2} + \overline{\beta}(\gamma - 1)acD^{-1} & -c\beta\overline{\delta}D^{-2} - acD^{-1}\alpha(\gamma - 1) & -D^{-1}(bc\gamma - ad) \end{pmatrix}.$$

Now if $[g,h] = 1$ then we have the following relations from the (1,1), (1,2), (1,3) and (3,1) entries respectively:

$$\begin{aligned} \alpha\gamma + d\beta\overline{\delta}D^{-2} &= \alpha\gamma + \overline{\beta}a\delta D; \\ a\alpha\delta D + \beta D^{-1}(bc - ad\gamma) &= \beta\gamma - a\alpha\delta D; \\ b\alpha\delta D + bd\beta D^{-1}(1-\gamma) &= -b\delta D; \qquad \text{and} \\ -c\alpha\overline{\delta}D^{-2} + ac\overline{\beta}D^{-1}(\gamma - 1) &= c\overline{\delta}D^{-2}. \end{aligned}$$

The relations from the other entries are all equivalent to the four shown above. It is now a routine calculation to deduce parts (i)-(iii) from these relations. □

Lemma 2.6 *Let* $y_\alpha \in \Delta_2^{\alpha}(t)$ *for some* $\alpha \in GF(q) \setminus \{\pm 1\}$*. Then* $\left|\Delta_1(y_\alpha) \cap \Delta_2^{-\alpha}(t)\right| = 1$*.*

Proof Without loss of generality, choose y_α such that $[y_\alpha, x] = 1$, so $(y_\alpha)_{11} = \alpha$ and set $(y_\alpha)_{12} = \beta$. Let $y_{-\alpha} \in \Delta_2^{-\alpha}(t)$ be as in (2.1) for suitable $a, b, c, d \in GF(q^2)$. We remark that if $\alpha = 0$, we denote this element y_0' to distinguish it from y_0. Assuming $[y_{-\alpha}, y_\alpha] = 1$, we apply Lemma 2.5 by setting $\alpha = -\gamma$, and note that $\overline{\beta}\beta = \overline{\delta}\delta$. Suppose that $b, c \neq 0$, then a and b are as in Lemma 2.5(ii). Since $\alpha = -\gamma$, we have $a = -D^{-1}\overline{\beta^{-1}\delta}$, giving $b = 2D\beta^{-1}(1-\gamma)^{-1}(\beta\gamma - \overline{\beta^{-1}\delta}\delta\gamma)c^{-1}$. However, $\beta\gamma - \overline{\beta^{-1}\delta}\delta\gamma = \beta(\gamma - \overline{\beta^{-1}}\beta^{-1}\overline{\delta}\delta\gamma) = 0$ since $\overline{\beta^{-1}}\beta^{-1}\overline{\delta}\delta = 1$. This yields $b = 0$, contradicting our original assumption. Hence $b = c = 0$, giving a as in Lemma 2.5(iii) and thus $a\delta\alpha D = -\beta\alpha$. Hence either $\alpha = 0$ or $a = -\beta\delta^{-1}D^{-1}$.

If $\alpha \neq 0$, then $aD = -\beta\delta^{-1}$ and $dD^{-2} = -\overline{\beta\delta^{-1}}$ showing that

$$y_{-\alpha} = \left(\begin{array}{cc|c} -\alpha & -\beta^2\delta^{-1} & \\ -\overline{\beta^2\delta^{-1}} & \alpha & \\ \hline & & -1 \end{array}\right).$$

If $\alpha = \gamma = 0$, then both y_0 and y_0' commute with x, where $(y_0)_{12} = \beta$ and $(y_0')_{12} = \delta$. If y_0 and y_0' commute, then an easy calculation shows that $\delta = \pm\beta$. Since $y_0 \neq y_0'$, we must have $\delta = -\beta$.

Hence in both cases, y_α commutes with a single element of $\Delta_2^{-\alpha}(t)$. □

Lemma 2.7 *Let $y_\alpha \in \Delta_2^\alpha(t)$. Then $|\Delta_1(y_\alpha) \cap \Delta_2^\gamma(t)| = q + 1$ for $\alpha \neq -\gamma$.*

Proof As in Lemma 2.6, choose y_α such that $[y_\alpha, x] = 1$ with $(y_\alpha)_{11} = \alpha$ and set $(y_\alpha)_{12} = \beta$. Let $y_\gamma \in \Delta_2^\gamma(t)$ be as in (2.1) for suitable $a, b, c, d \in GF(q^2)$. For brevity we remark that if $\alpha = \gamma$, then y_α and y_γ will denote different elements. Assume $[y_\alpha, y_\gamma] = 1$, so the relevant relations from Lemma 2.5 hold for fixed $\alpha, \beta, \gamma, \delta$ satisfying $\alpha, \gamma \in GF(q) \setminus \{\pm 1\}$, $\overline{\beta}\beta = 1 - \alpha^2$ and $\overline{\delta}\delta = 1 - \gamma^2$.

Suppose $b = c = 0$, so Lemma 2.5(iii) holds. Since $\beta \neq 0$ and if $\alpha = 0$, then $\gamma = 0$, contradicting the assumption that $\alpha \neq -\gamma$. Hence $a = \beta\gamma\alpha^{-1}\delta^{-1}D^{-1}$. Using Lemma 2.5(i), we get $d = \overline{\beta\delta^{-1}}D^2\gamma\alpha^{-1}$ and so $ad = \overline{\beta}\beta\overline{\delta^{-1}}\delta^{-1}\gamma^2\alpha^{-2}D$. Combining the expressions for $\overline{\beta}\beta$, $\overline{\delta}\delta$ and D, we get

$$(\gamma^2 - \alpha^2\gamma^2)(\alpha^2 - \alpha^2\gamma^2)^{-1} = 1,$$

giving $\gamma^2 = \alpha^2$ resulting in $\gamma = \pm\alpha$. Since $\alpha \neq -\gamma$, we must have $\alpha = \gamma$. But then $aD\delta = \beta$ and so $y_\gamma = y_\alpha$. Therefore, we may assume $b, c \neq 0$.

By a long but routine check, substitutions of $\overline{\beta}\beta$, $\overline{\gamma}\gamma$ and the relations in Lemma 2.5 show that $ad - bc = D$ holds. These relations also clearly show that a, b, c and d are all non-zero. Hence by Lemma 2.1(i), we have $\overline{a}b = -\overline{c}d$ and so $\overline{c}c = -\overline{a}bcd^{-1}$, and there are $q + 1$ values of c that satisfy this equation.

It now suffices to check that the remaining conditions of Lemma 2.1(i) hold. Since $\alpha, \gamma \in GF(q)$, we have $\overline{(1-\alpha)}(1-\alpha)^{-1} = \overline{(1-\gamma)}(1-\gamma)^{-1} = 1$. Together with the relations already determined, we have $\overline{a}a + \overline{c}c = \overline{a}a - \overline{a}d^{-1}bc = \overline{D^{-1}}D^{-1}$. However

$\overline{D}D = 1$, so the conditions of Lemma 2.1(i) hold. By considering $\overline{\overline{a}a + \overline{c}c}$, we get a similar result for $\overline{b}b + \overline{d}d$. Hence there is only one possible value of each of a and d, there are $(q+1)$ different values of c with b depending on c, proving the lemma. $\square$

As a consequence, we have the following.

Corollary 2.8 *Let $y \in \Delta_2(t)$. Then $|\Delta_1(y) \cap \Delta_3(t)| = q+1$.*

Proof Since the valency of the graph is $q(q-1)$ and $|\Delta_1(y) \cap \Delta_1(t)| = 1$, Lemmas 2.6 and 2.7 give Corollary 2.8. $\square$

For the remainder of this paper, denote

$$y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \Delta_2^0(t)$$

and define

$$z_\gamma = \begin{pmatrix} 1 & -2 & \overline{\gamma} \\ -2 & 1 & -\overline{\gamma} \\ \gamma & -\gamma & -3 \end{pmatrix},$$

for $\overline{\gamma}\gamma = -4$. An easy check shows that $[z_\gamma, y] = 1$, $\overline{z_\gamma}^T = z_\gamma$ and z_γ is an involution, hence $z_\gamma \in X$ and $d(t, z_\gamma) \leq 3$. However, since t is the sole element with top-left entry 1 that is at most distance 2 from t, we have $d(t, z_\gamma) \geq 3$ and thus equality.

Lemma 2.9 $\Delta_1(y) \cap \Delta_3(t) = \{\, z_\gamma | \, \gamma \in GF(q^2),\ \overline{\gamma}\gamma + 4 = 0\}$.

Proof There are $q+1$ values of γ and z_γ centralises y for all such γ. By Corollary 2.8, $|\Delta_1(y) \cap \Delta_3(t)| = q+1$, and so the lemma follows. $\square$

Fix γ and let $g \in C_H(t)$ be of the form as described in Lemma 2.1(i) for suitable $a, b, c, d \in GF(q^2)$. Then

$$z_\gamma g = \begin{pmatrix} D^{-1} & -2a + c\overline{\gamma} & -2b + d\overline{\gamma} \\ -2D^{-1} & a - \overline{\gamma}c & b - d\overline{\gamma} \\ \gamma D^{-1} & -\gamma a - 3c & -b\gamma - 3d \end{pmatrix}$$

and

$$g z_\gamma = \begin{pmatrix} D^{-1} & -2D^{-1} & D^{-1}\overline{\gamma} \\ -2a + b\gamma & a - b\gamma & -a\overline{\gamma} - 3b \\ -2c + d\gamma & c - d\gamma & -c\overline{\gamma} - 3d \end{pmatrix}.$$

If $[z_\gamma, g] = 1$, then we equate the entries to get conditional relations. From the (2,2) entries, we see that $b = c\overline{\gamma}\gamma^{-1}$. This, combined with the (2,3) entry, gives $d = a + 4c\gamma^{-1}$.

The (3,1) entry shows that $c = -2^{-1}(D^{-1} - d)\gamma$, and so $d = 2D^{-1} - a$. Hence

$$b = -2^{-1}(a - D^{-1})\overline{\gamma};$$
$$c = -2^{-1}(a - D^{-1})\gamma; \quad \text{and}$$
$$d = 2D^{-1} - a$$

for $a \in GF(q^2)$. A routine check shows these relations are sufficient for $[z_\gamma, g] = 1$. These relations, together with the conditions of Lemma 2.1(i) and $\overline{D}D = 1$, give

$$a\overline{D^{-1}} + \overline{a}D^{-1} = 2. \tag{2.2}$$

Clearly, the number of possible such a is $|C_H(\langle t, z_\gamma\rangle)|$. Since $D = ad - bc$, we get $D^3 = 1$. Therefore $\overline{D}D = D^3 = 1$ which has a solution $D \neq 1$ if and only if $q \equiv 5 \pmod 6$.

Lemma 2.10 *If $q \not\equiv 5 \pmod 6$, then $|C_H(\langle t, z_\gamma\rangle)| = q$. Moreover, $\mathcal{C}(H, X)$ is connected of diameter 3 and $|\Delta_3(t)| = (q+1)(q^2-1)$.*

Proof Since $q \not\equiv 5 \pmod 6$, from (2.2) we have $D = 1$ and $\overline{a} + a - 2 = 0$. There are q distinct values of a satisfying this, so $|C_H(\langle t, z_\gamma\rangle)| = q$. Denote the $C_H(t)$-orbit containing z_γ by $\Delta_3^\gamma(t)$. Hence,

$$|\Delta_3^\gamma(t)| = \frac{|C_H(t)|}{|C_H(\langle t, z_\gamma\rangle)|} = (q+1)(q^2-1).$$

Combining Lemmas 2.1(ii)-(iii) and 2.4, we have

$$|X \setminus (\{t\} \cup \Delta_1(t) \cup \Delta_2(t))| = |\Delta_3^\gamma(t)|\,.$$

Hence $\mathcal{C}(H, X)$ is connected of diameter 3, and $\Delta_3^\gamma(t) = \Delta_3(t)$ as required. □

Remark Since $\Delta_3(t)$ is a single $C_H(t)$-orbit and the valency of the graph is $q(q-1)$, for $w \in \Delta_3(t)$ we have $|\Delta_1(w) \cap \Delta_3(t)| = q$. This proves Theorem 1.1 when $q \not\equiv 5 \pmod 6$.

We now turn our attention to the remaining case, when $q \equiv 5 \pmod 6$.

Lemma 2.11 *Suppose $q \equiv 5 \pmod 6$.*

(i) $|C_H(\langle t, z_\gamma\rangle)| = 3q$.

(ii) There are exactly three $C_H(t)$-orbits in $\Delta_3(t)$, each of length $\frac{1}{3}(q+1)(q^2-1)$.

(iii) $\mathcal{C}(H, X)$ is connected of diameter 3 and $|\Delta_3(t)| = (q+1)(q^2-1)$.

Proof From (2.2), we have $\overline{D}D = D^3 = 1$ and since $q \equiv 5 \pmod 6$, there are three possible values for D. Since $a\overline{D^{-1}} + \overline{a}D^{-1} - 2 = \overline{(a\overline{D^{-1}})} + a\overline{D^{-1}} - 2 = 0$ then for each value of D, there are q such values of $a\overline{D^{-1}}$. Hence there are $3q$ values of $a\overline{D^{-1}}$ in total, proving (i).
Fix γ, and let $\Delta_3^\gamma(t)$ be the $C_H(t)$-orbit containing z_γ. We have

$$|\Delta_3^\gamma(t)| = \frac{|C_H(t)|}{|C_H(\langle t, z_\gamma \rangle)|} = \frac{1}{3}(q+1)(q^2-1). \tag{2.3}$$

Let $h = \left(\begin{array}{c|cc} E & & \\ \hline & \lambda & \mu \\ & \sigma & \tau \end{array}\right) \in C_H(t)$ where $E = \lambda\tau - \mu\sigma$. Then

$$h^{-1}z_\gamma h = \begin{pmatrix} 1 & E(\overline{\gamma}\sigma - 2\lambda) & E(-2\mu + \tau\overline{\gamma}) \\ -E^{-2}(2\tau + \mu\gamma) & (\lambda\mu\gamma - \sigma\overline{\gamma}\tau + 4\mu\sigma)E^{-1} + 1 & (-\overline{\gamma}\tau^2 + \mu^2\gamma + 4\mu\tau)E^{-1} \\ E^{-2}(2\sigma + \lambda\gamma) & (-\lambda^2\gamma + \sigma^2\overline{\gamma} - 4\lambda\sigma)E^{-1} & (\lambda\mu\gamma - \sigma\overline{\gamma}\tau + 4\mu\sigma)E^{-1} - 3 \end{pmatrix}.$$

Suppose $h^{-1}z_\gamma h = z_\delta \in \Delta_3(t) \cap \Delta_1(y)$ for some $\delta \neq \gamma$. Hence $(h^{-1}z_\gamma h)_{21} = -2 = (h^{-1}z_\gamma h)_{12}$ gives $\tau = E^2 - 2^{-1}\mu\gamma$ and $\lambda = 2^{-1}\overline{\gamma}\sigma + E^{-1}$. Since $E = \lambda\tau - \mu\sigma$, we have $2^{-1}\overline{\gamma}\sigma E^2 - 2^{-1}\mu\gamma E^{-1} = 0$ and so $\mu = \overline{\gamma}\gamma^{-1}\sigma E^3$. Rewriting τ, we get $\tau = E^2 - 2^{-1}\overline{\gamma}\sigma E^3$. To summarise,

$$\begin{aligned} \lambda &= 2^{-1}\overline{\gamma}\sigma + E^{-1}; \\ \mu &= \overline{\gamma}\gamma^{-1}\sigma E^3; \qquad \text{and} \\ \tau &= E^2 - 2^{-1}\overline{\gamma}\sigma E^3. \end{aligned}$$

Using these relations and $\overline{\gamma}\gamma = -4$, a simple check shows that $(h^{-1}z_\gamma h)_{22} = 1$ and $(h^{-1}z_\gamma h)_{33} = -3$ hold, and $(h^{-1}z_\gamma h)_{31} = E^{-3}\gamma = \delta$. Easy substitutions and checks show that $(h^{-1}z_\gamma h)_{32} = -(h^{-1}z_\gamma h)_{31}$ and $\overline{(h^{-1}z_\gamma h)_{13}} = (h^{-1}z_\gamma h)_{31}$. Since $\overline{\delta}\delta = -4$, we have $\overline{E^3}E^3 = 1$. In particular, E^3 is a $(q+1)^{\text{th}}$ root of unity. There are $q+1$ such roots and only a third of them are cubes in $GF(q^2)^*$. Hence there are only $\frac{1}{3}(q+1)$ such values of $\delta = E^{-3}\gamma$. Therefore, we can pick γ_1, γ_2 and γ_3 such that $\overline{\gamma_i}\gamma_i = -4$ where the z_{γ_i} are not pairwise $C_H(t)$-conjugate. Hence there are at least 3 orbits in $\Delta_3(t)$, and by (2.3) they all have length $\frac{1}{3}(q+1)(q^2-1)$. But (as in the proof of Lemma 2.10), $|X \setminus (\{t\} \cup \Delta_1(t) \cup \Delta_2(t))| = (q+1)(q^2-1)$ and so this proves (ii), and (iii) follows immediately. □

This now completes the proof of Theorem 1.1.

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