Explicit Cayley covers of Kautz digraphs

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Abstract

Given a finite set V and a set S of permutations of V, the *group action graph* $\mathrm{GAG}(V, S)$ is the digraph with vertex set V and arcs (v, v^σ) for all $v \in V$ and $\sigma \in S$. Let $\langle S \rangle$ be the group generated by S. The Cayley digraph $\mathrm{Cay}(\langle S \rangle, S)$ is called a Cayley cover of $\mathrm{GAG}(V, S)$. We define the Kautz digraphs as group action graphs and give an explicit construction of the corresponding Cayley cover. This is an answer to a problem posed by Heydemann in 1996.

1 Introduction

The importance of graph symmetry from theoretical and applied points of view has been emphasized many times; see, for instance, [1, 2, 11, 12, 14]. Furthermore, the idea of associating a Cayley digraph to a non-symmetric digraph in such a way that the properties of one gives information about the other has been frequently used. For instance, Fiol et al. [7, 8, 9] have shown that, in the context of dynamic memory networks, the idea of associating a Cayley digraph on a permutation group on the set of vertices of the network plays a central role in a unified approach to the topic. The idea of symmetrization of a digraph is used by Espona and Serra in [6] to construct Cayley covers of the de Bruijn digraphs, and by Mansilla and Serra in the context of k-arc transitivity [15, 16]. The group action graphs defined by Annexstein et al. in [2], give a way to associate to each non-symmetric digraph a number of Cayley graphs.

The de Bruijn and Kautz digraphs are the iterated line digraphs from the complete digraph with and without loops, respectively [10]. They are dense digraphs, and they have high connectivity, and many other good properties [3]. But, in general, they are not symmetric. Serra and Fiol have calculated the permutation groups of the de Bruijn

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digraph [18]. From them, an explicit construction of the Cayley digraph associated to the de Bruijn digraphs as group action graphs is known. For Kautz digraphs $K(d,n)$, it is known for which values of d and n they are Cayley digraphs (see [4]), and, in the case that $d+1$ is a prime number, Mansilla and Serra [15] gave an explicit construction of the corresponding Cayley digraph.

The problem 47 posed by Heydemann in [13], consists in giving an explicit construction of the Cayley digraph associated to the Kautz digraphs $K(d,n)$ considered as group action graph. Our goal here is to solve this problem for all values of d and n.

The paper is organized as follows. In the next section we give definitions, known results and a suitable representation of the Kautz digraphs $K(d,n)$. Section 3 is devoted to an explicit construction of a group $\Sigma = \Sigma(d,n)$ and a generating system $\mathcal{G}$ for Σ. In Section 4 we show that the Cayley digraph $\mathrm{Cay}\,(\Sigma,\mathcal{G})$ is a Cayley regular cover of $K(d,n)$, the explicit construction asked by Heydemann.

For undefined concepts about group theory we refer to [17], and for undefined concepts about graph theory we refer to [5].

2 Group action graphs and Kautz digraphs

Given a (finite) set V and a set S of permutations of V, the *group action graph* (GAG for short) defined by V and S is the digraph with vertex set V and arcs (v,v^{σ}) for all $v \in V$ and $\sigma \in S$; it is denoted by $\mathrm{GAG}(V,S)$. If we admit multiple arcs, a GAG is regular, the degree being the cardinality of S. In a natural way, the group $\langle S\rangle$ generated by S acts on V, and $\mathrm{GAG}(V,S)$ is strongly connected if and only if the action of $\langle S\rangle$ on V is transitive. In this paper, all digraphs considered are strongly connected.

The concepts of arc-coloring and decompositions into permutations, are closely related to GAG. Let $\Gamma = (V,E)$ be a d-regular digraph. An *arc-coloring* of Γ is an assignment of an element in $\{0,\dots,d-1\}$ to each arc of E in such a way that, for all $v \in V$, the d arcs incident to v have different assignments, and the d arcs incident from v have different assignments as well. The element assigned to an arc is called the *color* of the arc. A set $S = \{\sigma_0,\dots,\sigma_{d-1}\}$ of d permutations of V is a *decomposition into permutations* of Γ if, for every vertex $v \in V$, (i) $(v,v^{\sigma_a}) \in E$ for all $a \in \{0,\dots,d-1\}$; (ii) $v^{\sigma_a} = v^{\sigma_b}$ implies $a = b$ for all $a,b \in \{0,\dots,d-1\}$.

The two concepts, arc-coloring and decomposition into permutations, are equivalent. Indeed, given an arc-coloring of Γ, for each color $a \in \{0,\dots,d-1\}$ and each $v \in V$, let v^{σ_a} be the unique vertex adjacent from v by an arc of color a. Then, the map $\sigma_a\colon V \to V$ defined by $v \mapsto v^{\sigma_a}$ is a permutation of V, and the set $S = \{\sigma_0,\dots,\sigma_{d-1}\}$ is a decomposition of Γ into permutations. Conversely, given a decomposition into permutations $S = \{\sigma_0,\dots,\sigma_{d-1}\}$ of Γ, if we assign to the arc (v,v^{σ_a}) the color a, then we obtain an arc-coloring of Γ. It is well known that, as a consequence of Hall's matching theorem, every d-regular digraph admits an arc-coloring (in fact, in general, it admits many arc-colorings).

A decomposition into permutations S of a d-regular digraph $\Gamma = (V,E)$ allows us to see Γ as a GAG. Indeed, Γ is just the GAG defined by V and S, that is, $\Gamma = \mathrm{GAG}\,(V,S)$.

As, in general, Γ admits many decompositions into permutations, the digraph Γ can be seen as a GAG in many ways.

Given a group Ω, and a generating system S for Ω, the *Cayley digraph* $\mathrm{Cay}\,(\Omega, S)$ is the digraph which has Ω as set of vertices and each vertex v is adjacent to the vertices vs, with $s \in S$. If Γ is a d-regular digraph and S is a decomposition into permutations of Γ, the Cayley graph $\mathrm{Cay}(\langle S\rangle, S)$ is called a *Cayley regular cover* of Γ. As pointed out in [2], the digraph Γ is a quotient digraph of each of its regular covers. Indeed, fixed a vertex v, the map $f\colon \mathrm{Cay}(\langle S\rangle, S) \to \Gamma$ defined by $f(\sigma) = v^\sigma$ is a digraph homomorphism onto Γ, and, for all $u \in V$, the set $f^{-1}(u)$ has the cardinality of the stabilizer in $\langle S\rangle$ of v.

The *de Bruijn digraph* $B(d,n)$ is the digraph with vertex set $\mathbb{Z}_d^n$ and each vertex $z_0 \dots z_{n-1}$ is adjacent to the d vertices $z_1 \dots z_n$ with $z_n \in \mathbb{Z}_d$. Clearly, $B(d,n)$ is d-regular. The digraph $B(d,1)$ is the complete digraph with loops on d vertices $K_d^+ = \mathrm{Cay}(\mathbb{Z}_d, \mathbb{Z}_d)$. For $n \geq 2$, the digraphs $B(d,n)$ are iterated line digraphs $B(d,n) = LB(d,n-1) = L^{n-1}B(d,1)$, see [10]. For $a \in \mathbb{Z}_d$, the map $\sigma_a\colon \mathbb{Z}_d^n \to \mathbb{Z}_d^n$ defined by $(z_0 \dots z_{n-1})^{\sigma_a} = z_1 \dots z_{n-1}(z_0+a)$ is a permutation of $\mathbb{Z}_d^n$, and the set $S = \{\sigma_0, \dots, \sigma_{d-1}\}$ is a decomposition of $B(d,n)$ into permutations. The Cayley regular cover $\mathrm{Cay}(\langle S\rangle, S)$ associated to these permutations is known, see [15].

The Kautz digraph $K(d,n)$ is the d-regular digraph with vertex set $V = \{z_0 \dots z_{n-1} \in \mathbb{Z}_{d+1}^n : z_i \neq z_{i+1} \text{ for } i = 0, \dots, n-2\}$, and each vertex $z_0 \dots z_{n-1}$ is adjacent to the d vertices $z_1 \dots z_{n-1} z_n$ with $z_n \in \mathbb{Z}_{d+1} \setminus \{z_{n-1}\}$. The digraph $K(d,1)$ is the complete digraph without loops on $d+1$ vertices $K_{d+1} = \mathrm{Cay}(\mathbb{Z}_{d+1}, \mathbb{Z}_{d+1} \setminus \{0\})$. For $n \geq 2$, the digraphs $K(d,n)$ are iterated line digraphs $K(d,n) = LK(d,n-1) = L^{n-1}K(d,1)$, see [10]. The trivial case $d = 1$ gives $K(1,n) \simeq K(1,1) \simeq \mathrm{Cay}(\mathbb{Z}_2, \{1\})$, so in what follows we assume $d \geq 2$. When $d+1$ is a prime number, Fiol et al. [8] give a representation of $K(d,n)$ and a decomposition of $K(d,n)$ into permutations for which the corresponding Cayley regular cover is explicitly obtained by Mansilla and Serra in [15]. Our explicit construction for all values of (d,n) is based in a similar description of the Kautz digraph and in a decomposition into permutations which uses this description.

To avoid inconsistencies of notation, for an integer $m \geq 2$, we take the set of integers $\{0, 1, \dots, m-1\}$ (and not equivalence classes of integers) as the elements of the cyclic group $\mathbb{Z}_m$ of order m generated by 1. In this way, as a set $\mathbb{Z}_d$ is a subset (but not a subgroup) of $\mathbb{Z}_{d+1}$ and, if $c \in \mathbb{Z}_d$ and and $z \in \mathbb{Z}_{d+1}$, the sum $z + c$ in $\mathbb{Z}_{d+1}$ has a non-ambigous meaning.

For $a \in \mathbb{Z}_d$, the map $\tau_a\colon \mathbb{Z}_{d+1} \to \mathbb{Z}_{d+1}$ defined by $x^{\tau_a} = x + a + 1$ is a permutation of $\mathbb{Z}_{d+1}$, and the set $S = \{\tau_0, \dots, \tau_{d-1}\}$ is a decomposition of K_{d+1} into permutations, and the arc (z, z^{τ_a}) is said to be of color a. As $K(d,n) = L^{n-1}K(d,1)$, each vertex in $K(d,n)$ is a walk $z_0 \dots z_{n-1}$ in $K(d,1)$, which is completely determined by the *initial vertex* z_0 and the *sequence of colors* $\mathbf{c} = c_0 \dots c_{n-2}$ of the successive arcs $(z_0, z_1), \dots, (z_{n-2}, z_{n-1})$ in $K(d,1)$. Denoting the vertex $z_0 \dots z_{n-1}$ of $K(d,n)$ by $(z_0, \mathbf{c}) = (z_0,\ c_0 \dots c_{n-2})$, the vertex set of $K(d,n)$ can be identified with $\mathbb{Z}_{d+1} \times \mathbb{Z}_d^{n-1}$, and each vertex $(z,\ c_0 \dots c_{n-2})$ is adjacent in $K(d,n)$ to the d vertices $(z + c_0 + 1,\ c_1 \dots c_{n-2} c_{n-1})$ with $c_{n-1} \in \mathbb{Z}_d$. From now on, we take this description for $K(d,n)$.

Let $V = \mathbb{Z}_{d+1} \times \mathbb{Z}_d^{n-1}$ be the vertex set of $K(d,n)$. For each $a \in \mathbb{Z}_d$, the map $\sigma_a\colon V \to V$

defined by $(z,\ c_0 \dots c_{n-2})^{\sigma_a} = (x+c_0+1,\ c_1 \dots c_{n-2}(c_0+a))$ is a permutation of V, and the set $S = \{\sigma_0, \dots, \sigma_{d-1}\}$ is a decomposition of $K(d,n)$ into permutations. Thus, $K(d,n)$ is the group action graph $K(d,n) = \mathrm{GAG}\,(V,S)$. In what follows we give an explicit description of the Cayley regular cover $\mathrm{Cay}(\langle S\rangle, S)$ of $K(d,n) = \mathrm{GAG}\,(V,S)$.

3 The group

Let ψ be the shift automorphism of $\mathbb{Z}_d^{n-1}$ defined by $\psi(a_0,\dots,a_{n-2}) = (a_1,\dots,a_{n-2},a_0)$. The map $i \mapsto \psi^i$ is a group homomorphism from $\mathbb{Z}_{n-1}$ to $\mathrm{Aut}\,\mathbb{Z}_d^{n-1}$, the automorphism group of $\mathbb{Z}_d^{n-1}$, so we can form the semidirect product $\mathbb{Z}_{n-1} \rtimes \mathbb{Z}_d^{n-1}$ with the operation defined by $(i,\mathbf{a})(j,\mathbf{b}) = (i+j,\ \psi^j(\mathbf{a}) + \mathbf{b})$.

Let H_{d+1}^d be the subgroup of $\mathbb{Z}_{d+1}^d$ formed by the elements with sum of coordinates equal to zero:

$$H_{d+1}^d = \{(x_0,\dots,x_{d-1}) \in \mathbb{Z}_{d+1}^d : x_0 + \cdots + x_{d-1} = 0\}.$$

The group H_{d+1}^d has order $|H_{d+1}^d| = (d+1)^{d-1}$. Let ϕ be the shift automorphism of H_{d+1}^d defined by $\phi(x_0,\dots,x_{d-1}) = (x_1,\dots,x_{d-1},x_0)$. If $\mathbf{a} = (a_0,\dots,a_{n-2}) \in \mathbb{Z}_d^{n-1}$, we define

$$\phi^{\mathbf{a}} = (\phi^{a_0}, \phi^{a_1}, \dots, \phi^{a_{n-2}}) \colon (H_{d+1}^d)^{n-1} \to (H_{d+1}^d)^{n-1}$$

by

$$\phi^{\mathbf{a}}(\mathbf{x}_0,\dots,\mathbf{x}_{n-2}) = (\phi^{a_0}(\mathbf{x}_0),\dots,\phi^{a_{n-2}}(\mathbf{x}_{n-2})).$$

Clearly, $\phi^{\mathbf{a}}$ is an automorphism of $(H_{d+1}^d)^{n-1}$. Denote by ψ the shift automorphism of $(H_{d+1}^d)^{n-1}$ defined by $\psi(\mathbf{x}_0,\dots,\mathbf{x}_{n-2}) = (\mathbf{x}_1,\dots,\mathbf{x}_{n-2},\mathbf{x}_0)$. (Note that we use the same symbol ψ for the shift automorphism of $\mathbb{Z}_d^{n-1}$ and for the shift automorphism of $(H_{d+1}^d)^{n-1}$. In both cases ψ is a shift of vectors of length $n-1$, while ϕ is applied to vectors of length d.) For each $(i,\mathbf{a}) \in \mathbb{Z}_{n-1} \rtimes \mathbb{Z}_d^{n-1}$, the map $\psi^{-i}\phi^{-\mathbf{a}}$ is an automorphism of $(H_{d+1}^d)^{n-1}$. Moreover, from the fact that $\phi^{\mathbf{a}}\psi = \psi\phi^{\psi^{-1}(\mathbf{a})}$, the map $f \colon \mathbb{Z}_{n-1} \rtimes (\mathbb{Z}_d)^{n-1} \to \mathrm{Aut}(H_{d+1}^d)^{n-1}$, defined by $f(i,\mathbf{a}) = \psi^{-i}\phi^{-\mathbf{a}}$ is a group homomorphism. Indeed, we have

$$\begin{aligned}
f(i,\mathbf{a})f(j,\mathbf{b}) &= (\psi^{-i}\phi^{-\mathbf{a}})(\psi^{-j}\phi^{-\mathbf{b}}) \\
&= \psi^{-i}(\phi^{-\mathbf{a}}\psi^{-j})\phi^{-\mathbf{b}} \\
&= \psi^{-i}\psi^{-j}\phi^{\psi^j(-\mathbf{a})}\phi^{-\mathbf{b}} \\
&= \psi^{-(i+j)}\phi^{-(\psi^j(\mathbf{a})+\mathbf{b})} \\
&= f(i+j,\ \psi^j(\mathbf{a}) + \mathbf{b}) \\
&= f((i,\mathbf{a})(j,\mathbf{b})).
\end{aligned}$$

Consider the semidirect product $\Sigma'(d,n) = \left(\mathbb{Z}_{n-1} \rtimes \mathbb{Z}_d^{n-1}\right) \rtimes (H_{d+1}^d)^{n-1}$, with the operation defined by

$$(i,\mathbf{a},\mathbf{X})(j,\mathbf{b},\mathbf{Y}) = \left(i+j,\ \psi^j(\mathbf{a}) + \mathbf{b},\ \mathbf{X} + \psi^{-i}\phi^{-\mathbf{a}}(\mathbf{Y})\right).$$

Let $\Sigma(d,n)$ be the direct product $\Sigma'(d,n)\times\mathbb{Z}_2$ if d is odd and n is even, and $\Sigma(d,n)=\Sigma'(d,n)$ otherwise. The order of $\Sigma(d,n)$ is

$$|\Sigma(d,n)|=\begin{cases}2(n-1)d^{n-1}(d+1)^{(d-1)(n-1)}, & \text{if } d \text{ is odd and } n \text{ is even;}\\ (n-1)d^{n-1}(d+1)^{(d-1)(n-1)}, & \text{otherwise.}\end{cases}$$

Now, we shall show a generating system for $\Sigma(d,n)$. First, let us introduce some notation to make it easier to write some elements in $\Sigma(d,n)$. Let $\mathbf{e}_0=10\dots0,\ \dots,\ \mathbf{e}_{n-2}=0\dots01$, be the vectors of the canonical base of $\mathbb{Z}_d^{n-1}$. If $a\in\mathbb{Z}_d$, then $a\mathbf{e}_i$ is the vector $0\dots0a0\dots0$ with a in the i-th position (positions are counted from 0 to $n-2$). Both the zero vector of $\mathbb{Z}_d^{n-1}$ and the zero vector of H_{d+1}^d are denoted by $\mathbf{0}$, while the zero vector of $(H_{d+1}^d)^{n-1}$ is denoted by $\mathbf{O}$. With this notation, the neutral element of $\Sigma(d,n)$ is $O=(0,\mathbf{0},\mathbf{O},0)$ if d is odd and n even and $O=(0,\mathbf{0},\mathbf{O})$ otherwise. We define $\mathbf{v}=1\dots12\in H_{d+1}^d$ and $\mathbf{X_v}=(\mathbf{v},\mathbf{0},\dots,\mathbf{0})\in(H_{d+1}^d)^{n-1}$. For each $j\in\{0,\dots,d-2\}$, let $\mathbf{g}_j=0\dots010\dots d$ be the vector of H_{d+1}^d with the j-th coordinate equal to 1, the last coordinate equal to $d=-1$, and the remaining coordinates equal to zero. Note that the vectors $\mathbf{g}_0,\dots,\mathbf{g}_{d-2}$ form a generating system for H_{d+1}^d.

For each $a\in\mathbb{Z}_d$ define $G(a)\in\Sigma(d,n)$ by $G(a)=(1,a\mathbf{e}_{n-2},\mathbf{X_v},1)$ if d is odd and n is even, and $G(a)=(1,a\mathbf{e}_{n-2},\mathbf{X_v})$ otherwise.

Proposition 1 *The set $\{G(0),\ G(1)\}$ is a generating system for $\Sigma(d,n)$.*

Proof It is sufficient to define elements U, $E(r)$ $(0\le r\le n-2)$, and $F(r,s)$ $(0\le r\le n-2$ and $0\le s\le d-2)$, in the subgroup $\langle G(0),G(1)\rangle$ of $\Sigma=\Sigma(d,n)$ generated by $G(0)$ and $G(1)$ and to show that the elements U, $E(r)$ and $F(r,s)$ form a generating system for Σ.

First consider the case when d is even or n is odd, i.e., when Σ does not have the factor $\mathbb{Z}_2$. Direct calculations give

$$\begin{aligned} G(1)^{n-2} &= (n-2,\ 01\dots1,\ (\mathbf{v},\dots,\mathbf{v},\mathbf{0}))\,,\\ G(0)G(1)^{n-2} &= (0,\ 01\dots1,\ (\mathbf{v},\dots,\mathbf{v}))\,,\\ \left(G(0)G(1)^{n-2}\right)^d &= (0,\ \mathbf{0},\ (-\mathbf{v},\mathbf{0},\dots,\mathbf{0}))=(0,\ \mathbf{0},\ -\mathbf{X_v}). \end{aligned}$$

Define $W=(G(0)G(1)^{n-2})^d=(0,\ \mathbf{0},\ -\mathbf{X_v})$ and $U=WG(0)=(1,\mathbf{0},\mathbf{O})$. Clearly, $U\in\langle G(0),G(1)\rangle$, and for any element $(i,\mathbf{a},\mathbf{X})\in\Sigma$, we have

$$(i,\mathbf{a},\mathbf{X})U=(i+1,\mathbf{a},\mathbf{X}). \tag{1}$$

Let $E(0)=WG(1)U^{-1}=(0,\mathbf{e}_0,\mathbf{O})$ and, for $1\le r\le n-2$, define

$$E(r)=U^rE(0)U^{-r}=(0,\mathbf{e}_r,\mathbf{O}).$$

Then, $E(r)\in\langle G(0),G(1)\rangle$ and, for any element $(i,\mathbf{a},\mathbf{X})\in\Sigma$, we have

$$(i,\mathbf{a},\mathbf{X})E(r)=(i,\mathbf{a}+\mathbf{e}_r,\mathbf{X}). \tag{2}$$

Now, for $0 \le r \le n-2$ and $0 \le s \le d-2$, we define

$$\begin{aligned} F(r,s) &= U^r\, W\, E(0)^{s+1}\, W^{-1} E(0)^{-(s+1)} \\ &= U^r\ (0,\ (s+1)\mathbf{e}_0,\ -\mathbf{X_v})\,(0,\ -(s+1)\mathbf{e}_0,\ \mathbf{X_v}) \\ &= (r,\ \mathbf{0},\ \mathbf{O})\,(0,\ \mathbf{0},\ (\mathbf{g}_s, \mathbf{0}, \ldots, \mathbf{0})) \\ &= \big(r,\ \mathbf{0},\ \psi^{-r}(\mathbf{g}_s, \mathbf{0}, \ldots, \mathbf{0})\big)\,. \end{aligned}$$

Clearly, $F(r,s) \in \langle G(0), G(1)\rangle$, and, for any element $(0, \mathbf{0}, \mathbf{X}) \in \Sigma$, we have

$$(0,\ \mathbf{0},\ \mathbf{X})F(r,s) = (r,\ \mathbf{0},\ \mathbf{X} + \psi^{-r}(\mathbf{g}_s, \mathbf{0}, \ldots, \mathbf{0})). \tag{3}$$

Note that if $\mathbf{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_{n-2})$, then $\mathbf{X} + \psi^{-r}(\mathbf{g}_s, \mathbf{0}, \ldots, \mathbf{0})$ is the vector obtained from $\mathbf{X}$ by adding $\mathbf{g}_s$ to $\mathbf{x}_r$, that is, by adding 1 to the s-th coordinate of $\mathbf{x}_r$ and -1 to the last coordinate of $\mathbf{x}_r$.

Let $(i, \mathbf{a}, \mathbf{X}) \in \Sigma$. Since $\mathbf{g}_0, \ldots, \mathbf{g}_{d-2}$ is a generating system for H^d_{d+1}, according to (3), an appropriate product of $F(r_1, s_1) \cdots F(r_k, s_k)$ gives an element of the form $(j, \mathbf{0}, \mathbf{X})$. Because of (2), multiplying on the right by an appropriate product $E(r_1) \cdots E(r_\ell)$ we can obtain $(j, \mathbf{a}, \mathbf{X})$. Finally, because of (1), a product on the right by U^{i-j} gives $(i, \mathbf{a}, \mathbf{X})$. We conclude that the elements U, $E(r)$, and $F(r,s)$ in $\langle G(0), G(1)\rangle$ form a generating system for Σ. Hence, $\langle G(0), G(1)\rangle = \Sigma$.

Now consider the case when d is odd and n is even; in this case, both $n-1$ and $d(n-1)$ are odd. Therefore,

$$\begin{aligned} G(1)^{n-1} &= (0,\ 1 \ldots 1,\ (\mathbf{v}, \ldots, \mathbf{v}), 1), \\ G(1)^{(n-1)d} &= (0, \mathbf{0}, \mathbf{O}, 1). \end{aligned}$$

Define U, $E(r)$ and $F(r,s)$ in terms of $G(0)$ and $G(1)$ in the same way as before. Given any element $A = (i, \mathbf{a}, \mathbf{X}, \alpha) \in \Sigma$, a suitable product of $F(r,s)$'s, $E(r)$'s and U's gives an element $(i, \mathbf{a}, \mathbf{X}, \beta)$. Now, multiplying by $G(1)^{(n-1)d}$ if necessary, we obtain A. So, in this case, $\Sigma = \langle G(0), G(1)\rangle$, as well. $\square$

By Proposition 1, it is clear that the set $S = \{G(0), \ldots, G(d-1)\}$ is a generating system for $\Sigma(d,n)$, too.

4 The Cayley cover

Recall that we have defined $K(d,n)$ as the digraph with $V = \mathbb{Z}_{d+1} \times \mathbb{Z}_d^{n-1}$ as vertex set and each vertex $(z,\ c_0 \ldots c_{n-2})$ adjacent to the vertices $(z + c_0 + 1,\ c_1 \ldots c_{n-2}c_{n-1})$ with $c_{n-1} \in \mathbb{Z}_d$. Moreover, we consider the arc-coloring corresponding to the permutations σ_a defined by $(z, c_0 \ldots c_{n-2})^{\sigma_a} = (z + c_0 + 1,\ c_1 \ldots c_{n-2}(c_0 + a))$.

To define an action of the group $\Sigma = \Sigma(d,n)$ constructed in the previous section on V, the following notation will be used. Let $\mathbf{h} = (1, \ldots, d-2, d-1, 0) \in \mathbb{Z}^d_{d+1}$. For $\mathbf{x} = (x_0, \ldots, x_{d-1}) \in H^d_{d+1}$, define $\mathbf{h} \cdot \mathbf{x} = x_0 + 2x_1 + \cdots + (d-1)x_{d-2}$ with operations in $\mathbb{Z}_{d+1}$. Moreover, when d is odd, we denote by m the element $m = (d+1)/2$ of $\mathbb{Z}_{d+1}$.

The action is defined depending on the parity of d and n. Define $\rho : V \times \Sigma \to V$, $\rho(v, A) = vA$ as follows:

- if d is even,

$$(z,\mathbf{c})\,(i,\mathbf{a},\mathbf{X}) = \left(z + \mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell),\ \mathbf{a}+\psi^i(\mathbf{c})\right);$$

- if d and n are odd,

$$(z,\mathbf{c})\,(i,\mathbf{a},\mathbf{X}) = \left(z + \mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell) + \varepsilon(i)m,\ \mathbf{a}+\psi^i(\mathbf{c})\right),$$

  where $\varepsilon(i) = 0$ if i is even and $\varepsilon(i) = 1$ if i is odd.

- if d is odd and n is even,

$$(z,\mathbf{c})\,(i,\mathbf{a},\mathbf{X},\alpha) = \left(z + \mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell) + \alpha m,\ \mathbf{a}+\psi^i(\mathbf{c})\right).$$

Then, we have:

Proposition 2 *The map ρ is a faithful action of Σ on V.*

Proof First, we show that ρ is an action. In all cases it is easy to check that if O is the neutral element in Σ, then $\rho(v,O) = v$ for every vertex v. We shall check that $\rho(\rho(v,A),B) = \rho(v,AB))$ for every vertex v and all $A,B\in\Sigma$. Consider first the case when d is even, and put $v = (z,\mathbf{c})$, $A = (i,\mathbf{a},\mathbf{X})$ and $B = (j,\mathbf{b},\mathbf{Y})$. We have,

$$\begin{aligned}
[(z,\mathbf{c})\,(i,\mathbf{a},\mathbf{X})]\,(j,\mathbf{b},\mathbf{Y}) &= \Big(z+\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell),\ \mathbf{a}+\psi^i(\mathbf{c})\Big)\,(j,\mathbf{b},\mathbf{Y})\\
&= \Big(z+\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell)+\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-a_\ell-c_{\ell+i}}(\mathbf{y}_\ell),\\
&\qquad \psi^j(\mathbf{a}+\psi^i(\mathbf{c}))+\mathbf{b}\Big).\\
&= \Big(z+\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell)+\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-a_{\ell-i}-c_\ell}(\mathbf{y}_{\ell-i}),\\
&\qquad \psi^j(\mathbf{a})+\psi^{i+j}(\mathbf{c})+\mathbf{b}\Big).\\
&= \Big(z+\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell+\phi^{-a_{\ell-i}}(\mathbf{y}_{\ell-i})),\\
&\qquad \psi^{i+j}(\mathbf{c})+\psi^j(\mathbf{a})+\mathbf{b}\Big)\\
&= (z,\mathbf{c})\Big(i+j,\ \psi^j(\mathbf{a})+\mathbf{b},\ \mathbf{X}+\psi^{-i}\phi^{-\mathbf{a}}(\mathbf{Y})\Big)\\
&= (z,\mathbf{c})\,[(i,\mathbf{a},\mathbf{X})\,(j,\mathbf{b},\mathbf{Y})]\,.
\end{aligned}$$

Consider the second case, when both d and n are odd. By the same argument as in the case d even, now one must add $\varepsilon(i)m+\varepsilon(j)m$ to the initial vertex of $[(z,\mathbf{c})A]B$. Since m has order 2, we have $m\varepsilon(i)+m\varepsilon(j)=m\varepsilon(i+j)$, which is what must be added to the initial vertex of $(z,\mathbf{c})(AB)$. Thus, in this case, ρ is an action, as well.

An analogous argument applies to the third case, when d is odd and n is even. If $A=(i,\mathbf{a},\mathbf{X},\alpha)$ and $B=(j,\mathbf{B},\mathbf{Y},\beta)$, one must add $m\alpha+m\beta$ to the initial vertex of $[(z,\mathbf{c})A]B$, and $m(\alpha+\beta)=m\alpha+m\beta$, to the initial vertex of $(z,\mathbf{c})(AB)$.

Now we shall see that the action is faithful. Consider first the case when d is even. Assume that $(z,\mathbf{c})(i,\mathbf{a},\mathbf{X})=(z,\mathbf{c})$ for all vertices $(z,\mathbf{c})$. By taking $\mathbf{c}=\mathbf{0}$, we get $\mathbf{0}=\mathbf{c}=\mathbf{a}+\psi^i(\mathbf{c})=\mathbf{a}+\mathbf{0}=\mathbf{a}$. For $\mathbf{c}=\mathbf{e}_0$, we have $\mathbf{e}_0=\mathbf{a}+\psi^i(\mathbf{e}_0)=\psi^i(\mathbf{e_0})$, so $i=0$. Finally, take $z=0$ and $\mathbf{c}=q\mathbf{e}_j$. Then

$$0=\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell)=\mathbf{h}\cdot\left(\phi^{-q}(\mathbf{x}_\ell)+\sum_{\ell\neq j}\mathbf{x}_\ell\right). \tag{4}$$

Analogously, for $z=0$ and $\mathbf{c}=(q+1)\mathbf{e}_j$, we have

$$0=\mathbf{h}\cdot\sum_{\ell=0}^{n-2}\phi^{-c_\ell}(\mathbf{x}_\ell)=\mathbf{h}\cdot\left(\phi^{-q-1}(\mathbf{x}_\ell)+\sum_{\ell\neq j}\mathbf{x}_\ell\right). \tag{5}$$

Subtracting (5) from (4), we obtain

$$\mathbf{h}\cdot\left(\phi^{-q}(\mathbf{x}_j)-\phi^{-q-1}(\mathbf{x}_j)\right)=0. \tag{6}$$

If $\mathbf{y}=(y_0,\ldots,y_{d-1})=\phi^{-q}(\mathbf{x}_j)$, since $\mathbf{y}\in H^d_{d+1}$, we have $y_{d-1}=-(y_0+\cdots+y_{d-2})$. Then,

$$\begin{aligned}
0 &= \mathbf{h}\cdot(\mathbf{y}-\phi^{-1}(\mathbf{y}))\\
&= 1(y_0-y_{d-1})+2(y_1-y_0)+\cdots+(d-2)(y_{d-1}-y_{d-2})\\
&= -y_0-y_1-\cdots-y_{d-2}-(d-1)y_{d-1}\\
&= -y_{d-1}-(d-1)y_{d-1}\\
&= y_{d-1}.
\end{aligned}$$

Thus, the $(d-1)$-rst coordinate of $\mathbf{y}=\phi^{-q}(\mathbf{x}_j)$ is 0, that is, the $(j-q)$-th coordinate of $\mathbf{x}_j$ is zero. Since this is for all $j\in\{0,\ldots,n-1\}$ and $q\in\{0,\ldots,d-1\}$, we obtain that $\mathbf{X}=\mathbf{O}$.

Consider now the case when d is odd. By the same argument as in the d even case, the values $i=0$ and $\mathbf{a}=\mathbf{0}$ are deduced. If n is also odd, then $m\varepsilon(i)=m\varepsilon(0)=0$, and equalities (4) and (5) can be obtained. By the same argument we get $\mathbf{X}=\mathbf{O}$. If n is even, equations (4) and (5) must be changed to

$$0=\mathbf{h}\cdot\left(\phi^{-q}(\mathbf{x}_\ell)+\sum_{\ell\neq j}\mathbf{x}_\ell\right)+m\alpha,\quad\text{and}\quad 0=\mathbf{h}\cdot\left(\phi^{-q-1}(\mathbf{x}_\ell)+\sum_{\ell\neq j}\mathbf{x}_\ell\right)+m\alpha,$$

and, by subtraction, the equality (6) is obtained. In the same way, we get $\mathbf{X} = \mathbf{O}$. Then, we have $(0, \mathbf{0}) = (0, \mathbf{0})(i, \mathbf{a}, \mathbf{O}, \alpha) = (0, \mathbf{0})(0, \mathbf{0}, \mathbf{O}, \alpha) = (m\alpha, \mathbf{0})$. As $\alpha \in \{0, 1\}$ and m is of order 2, we obtain $\alpha = 0$. $\square$

Next, we check that the permutations on V defined by the elements $G(a)$ in Σ act correctly.

Proposition 3 *For each $a \in \mathbb{Z}_d$ and each vertex $(z, \mathbf{c})$ of $K(d, n)$, the vertex adjacent from $(z, \mathbf{c})$ by an arc of color a is $(z, \mathbf{c})^{\sigma_a} = (z, \mathbf{c})G(a)$.*

Proof Let $\mathbf{c} = c_0 \ldots c_{n-2} \in \mathbb{Z}_d^{n-1}$ and $\mathbf{X_v} = (\mathbf{x}_0, \ldots, \mathbf{x}_{n-2}) = (\mathbf{v}, \mathbf{0}, \ldots, \mathbf{0})$. First, we calculate $\mathbf{h} \cdot \sum_{\ell=0}^{n-2} \phi^{-c_\ell}(\mathbf{x}_\ell)$.

For even d,

$$\begin{aligned} \mathbf{h} \cdot \sum_{\ell=0}^{n-2} \phi^{-c_\ell}(\mathbf{x}_\ell) &= \mathbf{h} \cdot \phi^{-c_0}(\mathbf{v}) \\ &= (1 + 2 + \cdots + (d-1)) + c_0 \\ &= (d-1)\frac{d}{2} + c_0 \\ &= (d+1)\frac{d-2}{2} + 1 + c_0 \\ &= 1 + c_0. \end{aligned}$$

If d and n are odd,

$$\begin{aligned} \mathbf{h} \cdot \sum_{\ell=0}^{n-2} \phi^{-c_\ell}(\mathbf{x}_\ell) + \varepsilon(1)m &= (1 + 2 + \cdots + (d-1)) + c_0 + m \\ &= d\frac{d-1}{2} + c_0 + \frac{d+1}{2} \\ &= (d+1)\frac{d-1}{2} + 1 + c_0 \\ &= 1 + c_0. \end{aligned}$$

Analogously, if d is odd and n is even,

$$\mathbf{h} \cdot \sum_{\ell=0}^{n-2} \phi^{-c_\ell}(\mathbf{x}_\ell) + m = 1 + c_0.$$

Thus, in any case,

$$(z, \mathbf{c})G(a) = (z + c_0 + 1,\ a\mathbf{e}_{n-2} + \phi(\mathbf{c}) = (z + c_0 + 1,\ c_1 \ldots c_{n-2}(c_0 + a)) = (z, \mathbf{c})^{\sigma_a}. \quad \square$$

Putting together the description of $K(d, n)$ as a GAG, and Propositions 1, 2, and 3, we have the main theorem:

Theorem 1 (i) *The Kautz digraph is the group action graph $K(d,n) = GAG(V,S)$ where $V = \mathbb{Z}_{d+1} \times \mathbb{Z}_d^{n-1}$, and S is the set $S = \{\sigma_a : a \in \mathbb{Z}_d\}$ of permutations of V defined by $(z,\ c_0 \dots c_{n-2})^{\sigma_a} = (z + c_0 + 1,\ c_1 \dots c_{n-2}(c_0 + a))$.*

(ii) *The group generated by S is $\Sigma(d,n)$, and*

$$\mathrm{Cay}\,(\Sigma(d,n),\ \{G(0), \dots, G(d-1)\})$$

is a Cayley regular cover of $K(d,n) = \mathrm{GAG}(V,S)$.

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