Reduced Decompositions of Matchings

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Abstract

We give a characterization of matchings in terms of the canonical reduced decompositions. As an application, the canonical reduced decompositions of 12312-avoiding matchings are obtained. Based on such decompositions, we find a bijection between 12312-avoiding matchings and ternary paths.

1 Introduction

A *matching* on a set $[2n] = \{1, 2, \ldots, 2n\}$ is a graph on $[2n]$ in which every vertex has degree one. The set of matchings on $[2n]$ is denoted by $\mathcal{M}_n$. Note that $|\mathcal{M}_n| = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$. The *linear representation* of a matching is obtained by drawing $2n$ points in the plane lying on a horizontal line, and connecting them by n arcs such that each arc connects two of the points and lies above the points. Fig. 1 gives the linear representation of the matching $\{(1, 3), (2, 4), (5, 6)\}$.

In this paper, we always use the *canonical sequential form* [13] of a matching on the set $[2n]$, which is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ obtained in the following way. Draw the linear representation of the matching, and label the arcs with the numbers

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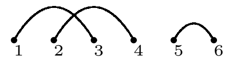

Figure 1: Linear representation.

$1, 2, \ldots, n$ ordered by their leftmost endpoints. Then label each endpoint with the label of the adjacent arc, and read the labels of the endpoints from left to right. For example, the matching in Fig. 1 can be also displayed by 121233.

Let π and τ be two sequences. We say π avoids τ or is τ-avoiding, whenever π does not contain a subsequence with all of the same pairwise comparisons as τ. For example, the sequence 12342143 is 12123-avoiding, but not 13132-avoiding since it has 14143 as a subsequence. In such a context τ is usually called a *pattern*. We denote the set of τ-avoiding matchings on $[2n]$ by $\mathcal{M}_n(\tau)$.

The systematic study of pattern avoiding permutations was initiated in 1985 [17]. Starting with the work of Billey, Jockusch and Stanley [3], there has been increasing interest in the connection between reduced decomposition and pattern avoiding permutation (see [1, 2, 16, 19] and references therein). Other results involving pattern avoiding matchings appeared in [5–7, 9–15, 20, 21].

Recently, by using generating functions, Chen, Mansour and Yan [5] show that the number of 12312-avoiding matchings is given by the 3-Catalan numbers. A combinatorial proof is also given in [5], which is based on a bijection between matchings and oscillating tableaux.

The aim of this paper is to give a new bijective proof for the cardinality of $\mathcal{M}_n(12312)$. The idea behind the proof is a new characterization of a matching, which we call the canonical reduced decomposition. In Section 2, we introduce the necessary notations, and describe an algorithm to generate the canonical reduced decomposition of a matching. The canonical reduced decompositions of 12312-avoiding matchings are studied in Section 3. Finally, in Section 4, we apply the canonical reduced decomposition to obtain a bijection between 12312-avoiding matchings and ternary paths. Note that a ternary path is a lattice path in the plane from $(0, 0)$ to $(2n, n)$ with $2n$ steps $E = (1, 0)$ and n steps $N = (0, 1)$ and never lying above the line $y = x/2$.

2 Canonical reduced decompositions of matchings

In this section, we characterize matchings in terms of their canonical reduced decompositions. Let $\mathfrak{S}_n^2$ denote the set of multiset permutations on $\{1, 1, 2, 2, \ldots, n, n\}$. We generalize the notion of reduced decompositions of permutations [19] to multiset permutations.

Definition 2.1. *For $1 \le i \le 2n-1$, define a map $s_i : \mathfrak{S}_n^2 \to \mathfrak{S}_n^2$ such that s_i acts on an element π in $\mathfrak{S}_n^2$ by interchanging the integers in positions i and $i+1$. We call s_i a simple transposition, and write the action of s_i on the right of π, denoted by πs_i. Therefore, $\pi(s_i s_j) = (\pi s_i)s_j$.*

For example, $231123s_4 = 231213$.

Definition 2.2. *A reduced decomposition of a multiset permutation $\pi \in \mathfrak{S}_n^2$ is a sequence of transpositions $s_{i_0}, s_{i_1}, \ldots, s_{i_t}$ such that $\pi = (1122\cdots nn)s_{i_0}s_{i_1}\cdots s_{i_t}$.*

Note that the reduced decomposition of a matching is not unique. For example, $123213 = 112233s_1s_2s_3s_4s_3 = 112233s_2s_3s_5s_4s_3$. To ensure the uniqueness of the decomposition, we give the following definition.

Definition 2.3. *A reduced decomposition of a matching Λ is canonical if it can be represented by*

$$\Lambda = (1122\cdots nn)\sigma_1\sigma_2\cdots\sigma_k,$$

where

$$\begin{aligned}
&\sigma_i = s_{h_i}s_{h_i+1}\cdots s_{t_i}, \quad h_i \le t_i \ (1 \le i \le k),\\
&h_i \in \{2, 4, \ldots, 2n-2\},\\
&h_1 > h_2 > h_3 > \cdots > h_k.
\end{aligned}$$

In particular, the canonical reduced decomposition of the matching $1122\cdots nn$ is empty, while the canonical reduced decomposition of the matching $12\cdots nn\cdots 21$ has the following form

$$12\cdots nn\cdots 21 = (1122\cdots nn)(s_{2n-2}s_{2n-1})(s_{2n-4}s_{2n-3}s_{2n-2}s_{2n-1})\cdots(s_2s_3\cdots s_{2n-1}).$$

Theorem 2.4. *The canonical reduced decomposition of a matching in $\mathcal{M}_n$ is unique.*

Proof. We prove the contrapositive: Suppose a matching Λ in $\mathcal{M}_n$ has two canonical reduced decompositions

$$\Lambda = (1122\cdots nn)\sigma_1\sigma_2\cdots\sigma_k = (1122\cdots nn)\widehat{\sigma}_1\widehat{\sigma}_2\cdots\widehat{\sigma}_m,$$

where $\sigma_i = s_{h_i}s_{h_i+1}\cdots s_{t_i}$ $(1 \le i \le k)$ and $\widehat{\sigma}_i = s_{\widehat{h}_i}s_{\widehat{h}_i+1}\cdots s_{\widehat{t}_i}$ $(1 \le i \le m)$.

We shall show $\sigma_i = \widehat{\sigma}_i$ for any i. The first step is to prove $\sigma_1 = \widehat{\sigma}_1$, equivalently, to prove $h_1 = \widehat{h}_1$ and $t_1 = \widehat{t}_1$. We consider the following three cases:

1. $h_1 > \widehat{h}_1$: The element of $1122\cdots nn$ in position h_1 will be transferred to position t_1+1 by the action of σ_1, that is to say,

$$(1122\cdots nn)\sigma_1 = 1122\cdots\left(\frac{h_1}{2}-1\right)\left(\frac{h_1}{2}-1\right)\ \underset{\substack{\uparrow\\ \textit{position}\ h_1-1}}{\frac{h_1}{2}}\ \left(\frac{h_1}{2}+1\right)\ \cdots\ \underset{\substack{\uparrow\\ \textit{position}\ t_1+1}}{\frac{h_1}{2}}\ \cdots.$$

   Since $h_1 > h_2 > \cdots > h_k$, the action of $\sigma_2\cdots\sigma_k$ on $(1122\cdots nn)\sigma_1$ preserves the relative order of integers $\frac{h_1}{2}, \frac{h_1}{2}+1, \ldots, n$. It implies $\Lambda = (1122\cdots nn)\sigma_1\sigma_2\cdots\sigma_k$ has the subsequence

$$\frac{h_1}{2}\left(\frac{h_1}{2}+1\right)\cdots\frac{h_1}{2}\cdots.$$

However, observing that $h_1 > \widehat{h}_1$ and $\widehat{h}_1 > \widehat{h}_2 > \cdots > \widehat{h}_m$, the matching $\Lambda = (1122\cdots nn)\widehat{\sigma}_1\widehat{\sigma}_2\cdots\widehat{\sigma}_m$ has the subsequence $\frac{h_1}{2}\frac{h_1}{2}\left(\frac{h_1}{2}+1\right)\cdots$, which gives a contradiction.

2. $h_1 < \widehat{h}_1$: The proof is similar as Case 1 and we omit it.

3. $h_1 = \widehat{h}_1$, $t_1 \neq \widehat{t}_1$: Similar analysis as Case 1, for $\Lambda = (1122\cdots nn)\sigma_1\sigma_2\cdots\sigma_k$, the subsequence composed of integers $\frac{h_1}{2}, \frac{h_1}{2}+1, \ldots, n$ has the form

$$\frac{h_1}{2}\left(\frac{h_1}{2}+1\right)\cdots\frac{h_1}{2}\cdots,$$

where there exist $t_1 - h_1 + 1$ elements between the two appearances of $\frac{h_1}{2}$. Meanwhile, for $\Lambda = (1122\cdots nn)\widehat{\sigma}_1\widehat{\sigma}_2\cdots\widehat{\sigma}_m$, the subsequence composed of integers $\frac{h_1}{2}, \frac{h_1}{2} + 1, \ldots, n$ has the form

$$\frac{h_1}{2}\left(\frac{h_1}{2}+1\right)\cdots\frac{h_1}{2}\cdots,$$

and there are $\widehat{t}_1 - h_1 + 1$ elements between the two appearances of $\frac{h_1}{2}$. This contradicts that $t_1 \neq \widehat{t}_1$.

It follows that $\sigma_1 = \widehat{\sigma}_1$. The proof of $\sigma_i = \widehat{\sigma}_i$ for $i \geq 2$ is analogous. □

Note that the product $s_i s_{i+1}\cdots s_j$ is equivalent to the cyclic permutation on the segment from position i to position $j+1$. For $\Lambda \in \mathcal{M}_n$, we describe an algorithm to generate the canonical reduced decomposition of Λ.

Algorithm:

1. Let $\Lambda_1 := \Lambda$. For $1 \leq i \leq n$, find the position, say ℓ, of the second appearance of i in Λ_i:

   (1.1) If $\ell = 2$, define σ_{n+1-i} to be the empty word;

   (1.2) If $\ell > 2$, define $\sigma_{n+1-i} = s_{2i}s_{2i+1}\cdots s_{2i+\ell-3}$;

   (1.3) Generate Λ_{i+1} by deleting the two elements i in Λ_i;

2. The canonical reduced decomposition of Λ is the product of non-empty words $\sigma_1, \sigma_2, \ldots, \sigma_n$.

For example,

$$\Lambda_1 = 12331442 \xrightarrow[\ell=5]{i=1} \sigma_4 = s_2s_3s_4, \Lambda_2 = 233442 \xrightarrow[\ell=6]{i=2} \sigma_3 = s_4s_5s_6s_7,$$
$$\Lambda_3 = 3344 \xrightarrow[\ell=2]{i=3} \sigma_2 \text{ is empty}, \Lambda_4 = 44 \xrightarrow[\ell=2]{i=4} \sigma_1 \text{ is empty}.$$

Thus, the canonical reduced decomposition of 12331442 is $(s_4s_5s_6s_7)(s_2s_3s_4)$.

Let Λ^* be the matching obtained by subtracting 1 from each element of Λ_2. It is constructive to notice the following corollary.

Corollary 2.5. *The canonical reduced decomposition of Λ^* is the product of non-empty words $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ after subtracting 2 from the index of each simple transposition.*

For example, for $\Lambda = 12331442$, we have $\Lambda^* = 122331$ and the canonical reduced decomposition of Λ^* is $(s_2s_3s_4s_5)$.

Extending the definition of the inversion on permutations [2, 4], an *inversion* of a matching $\pi_1\pi_2\cdots\pi_{2n}$ is a pair (π_i, π_j), where $1 \le i < j \le 2n$ and $\pi_i > \pi_j$.

Corollary 2.6. *If σ is the canonical reduced decomposition of a matching $\Lambda \in \mathcal{M}_n$, then Λ has k inversions if and only if σ has exactly k simple transpositions.*

3 Canonical reduced decompositions for $\mathcal{M}_n(12312)$

In this section, we restrict the canonical reduced decompositions to 12312-avoiding matchings. We present the following result by inheriting the notations of Λ and Λ^* in the preceding section.

Theorem 3.1. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be the canonical reduced decomposition of Λ, where $\sigma_i = s_{h_i}s_{h_i+1}\cdots s_{t_i}$ for $1 \le i \le k$. Then we have*

$$\Lambda \in \mathcal{M}_n(12312) \Leftrightarrow t_j \ge t_i \text{ or } t_j \le h_i - 2, \quad for\ 1 \le i < j \le k. \tag{3.1}$$

Proof. The cases for $k = 0, 1$ are trivial. Now we consider $k \ge 2$.

Observe that $\Lambda \in \mathcal{M}_n(12312)$ indicates $\Lambda^* \in \mathcal{M}_{n-1}(12312)$. We use induction on n. Clearly, the statement (3.1) is true for $n = 1, 2$. By induction hypothesis, we have

$$\Lambda^* \in \mathcal{M}_{n-1}(12312) \Leftrightarrow t_j^* \ge t_i^* \text{ or } t_j^* \le h_i^* - 2, \quad for\ 1 \le i < j \le m, \tag{3.2}$$

where $\sigma_1^*\sigma_2^*\cdots\sigma_m^*$ is the canonical reduced decomposition of Λ^* and $\sigma_i^* = s_{h_i^*}s_{h_i^*+1}\cdots s_{t_i^*}$.

For Λ, let ℓ denote the position of the second appearance of 1. Here are two cases:

1. If $\ell = 2$, then $m = k$, $h_i = h_i^* + 2$, and $t_i = t_i^* + 2$ for $1 \le i \le k$. Moreover, in this case, $\Lambda \in \mathcal{M}_n(12312)$ if and only if $\Lambda^* \in \mathcal{M}_{n-1}(12312)$. By (3.2), we have

$$\Lambda \in \mathcal{M}_n(12312) \Leftrightarrow t_j^* \ge t_i^* \text{ or } t_j^* \le h_i^* - 2 \Leftrightarrow t_j \ge t_i \text{ or } t_j \le h_i - 2,$$

for $1 \le i < j \le k$.

2. If $\ell > 2$, then $m = k - 1$, $h_i = h_i^* + 2$, $t_i = t_i^* + 2$, for $1 \le i \le k - 1$, and

$$\sigma_k = s_2s_3\cdots s_{\ell-1},$$

which gives $h_k = 2$ and $t_k = \ell - 1$. In this case, we prove (3.1) in two steps:

Step 1.($\Leftarrow$) By (3.2), we get $\Lambda^* \in \mathcal{M}_{n-1}(12312)$. So it is sufficient to show that Λ does not contain a subsequence

$$1, \ldots, i_1, \ldots, i_2, \ldots, 1, \ldots, i_1$$

where $i_2 > i_1 > 1$. Furthermore, we need only show that Λ does not have a subsequence

$$1, \ldots, \frac{h_{i_0}}{2}, \ldots, \frac{h_{i_0}}{2} + 1, \ldots, 1, \ldots, \frac{h_{i_0}}{2}, \tag{3.3}$$

where $\frac{h_{i_0}}{2} + 1$ is the first appearance in Λ. By contradiction, choose a subsequence of the form (3.3) such that h_{i_0} is minimal. This implies that the element $\frac{h_{i_0}}{2} + 1$ in (3.3) is in position $h_{i_0} - 1$ of Λ. Notice that the second appearance of 1 in (3.3) is in position $t_k + 1$ of Λ, and the position of the second appearance of $\frac{h_{i_0}}{2}$ in (3.3) is not after the position $t_{i_0} + 1$ in Λ. It follows that $t_k + 1 > h_{i_0} - 1$ and $t_k + 1 < t_{i_0} + 1$. Thus, we deduce that $t_k > h_{i_0} - 2$ and $t_k < t_{i_0}$, which is a contradiction to the right hand side of (3.1).

Step 2.($\Rightarrow$) By (3.2), we have $t_j^* \geq t_i^*$ or $t_j^* \leq h_i^* - 2$ for $1 \leq i < j \leq k-1$. This gives $t_j \geq t_i$ or $t_j \leq h_i - 2$ for $1 \leq i < j \leq k-1$. Then it suffices to prove that $t_k \geq t_i$ or $t_k \leq h_i - 2$ for $1 \leq i \leq k-1$. Otherwise, choose i_0 to be the maximal index such that $t_k < t_{i_0}$ and $t_k > h_{i_0} - 2$. This implies that the second appearance of $\frac{h_{i_0}}{2}$ in Λ is in position $t_{i_0} + 1$. Notice that the second appearance of 1 in Λ is in position $t_k + 1$, and the position of the first appearance of $\frac{h_{i_0}}{2} + 1$ is not after the position $h_{i_0} + 1$ in Λ. Therefore, there exists a subsequence of Λ with the following form

$$1, \ldots, \frac{h_{i_0}}{2}, \ldots, \frac{h_{i_0}}{2} + 1, \ldots, 1, \ldots, \frac{h_{i_0}}{2}, \ldots$$

which contradicts that Λ is 12312-avoiding. $\square$

4 Bijection between ternary paths and $\mathcal{M}_n(12312)$

Chen, Mansour and Yan [5] show that the number of 12312-avoiding matchings on $[2n]$ equals the 3-Catalan numbers [18, Sequence A001764], namely,

$$|\mathcal{M}_n(12312)| = \frac{1}{2n+1}\binom{3n}{n}.$$

Note that the 3-Catalan numbers also count ternary paths of length $3n$. A *ternary path* of length $3n$ is a lattice path in the plane from $(0,0)$ to $(2n, n)$ with $2n$ steps $E = (1,0)$ and n steps $N = (0,1)$ and never lying above the line $y = x/2$. For example, a ternary path $P = EEEEENEEENEENENENENN$ is shown in Fig. 2.

The purpose of this section is to establish a bijection between $\mathcal{M}_n(12312)$ and ternary paths of length $3n$. We follow the approach of some known results [2,8] to pattern avoiding permutations. Moreover, our bijection will rely on the canonical reduced decompositions of 12312-avoiding matchings.

By Definition 2.3 and Theorem 3.1, $\sigma_1\sigma_2\cdots\sigma_k$ is the canonical reduced decomposition of $\Lambda \in \mathcal{M}_n(12312)$, where $\sigma_i = s_{h_i}s_{h_i+1}\cdots s_{t_i}$ for $1 \leq i \leq k$, if and only if the set of

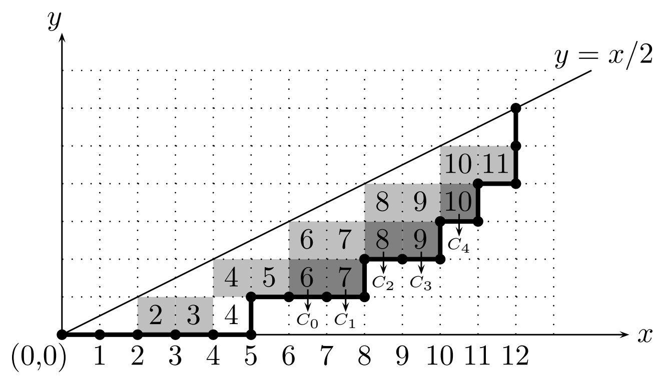

Figure 2: The strip decomposition.

parameters $\{(h_i, t_i) | 1 \leq i \leq k\}$ satisfies

$$h_1 > h_2 > \cdots > h_k, \tag{4.1}$$

$$t_i \geq h_i \in \{2, 4, \ldots, 2n-2\}, \quad (1 \leq i \leq k), \tag{4.2}$$

$$t_j \geq t_i \text{ or } t_j \leq h_i - 2, \quad (1 \leq i < j \leq k). \tag{4.3}$$

For a ternary path P, our bijection involves all the unit cells enclosed by P. Explicitly, a cell enclosed by P means that the cell is totally in the region surrounded by P and $y = x/2$. We give an x-labeling of these cells: Each cell with corner points (i, j), $(i+1, j)$, $(i+1, j+1)$ and $(i, j+1)$, receives a label i. We call a cell with an even (resp. odd) label an *even cell* (resp. *odd cell*) for short. A cell enclosed by P is *self-dependent* if the cell immediately to its South-West is not enclosed by P. We define the *ladder strip* of P as follows:

1. If $P = (EEN)^n$, that is, P is composed of n consecutive segments EEN, then P has no self-dependent cell. Define the ladder strip of P to be the empty set;

2. Otherwise, denote C_0 the even self-dependent cell enclosed by P, which is labeled with the maximal integer. Define the ladder strip of P to be the maximal sequence of cells $C_0, C_1, C_2, \ldots$, where C_{2i+1} is the adjacent cell to the East of C_{2i} and C_{2i+2} is the adjacent cell to the North-East of C_{2i+1} for each i.

Fig. 2 illustrates the x-labeling of a ternary path, whose ladder strip consists of the gray cells C_0, C_1, C_2, C_3, C_4 with labels $6, 7, 8, 9, 10$.

Suppose the ternary path P has k even self-dependent cells. We give the *strip decomposition* of P recursively by the following steps:

1. If $k = 0$, then the strip decomposition of P is the empty set;

2. Otherwise, decompose P into $P_1 L_1$, where L_1 is the ladder strip of P and P_1 is the ternary path obtained from P by deleting L_1. We can associate L_1 with a sequence of simple transpositions, say $\sigma_1 = s_i s_{i+1} \cdots s_j$, where $\{i, i+1, \ldots, j\}$ is the set of labels in L_1. Define $h_1 := i$, and $t_1 := j$;

3. Repeat the above procedures for the ternary path P_1, we will get σ_2. Furthermore, we can find $\sigma_3, \ldots, \sigma_k$ by applying this step recursively. Then a set of parameters $\{(h_i, t_i) | 1 \leq i \leq k\}$ is obtained;

4. The strip decomposition of P is $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$.

See Fig. 2 for an example, the ternary path

$$P = EEEEENEEENEENENENENN$$

can be decomposed into $P_1 L_1$, where $P_1 = EEEEENENEENEENEENN$ and L_1 is the ladder strip of P with labels $6, 7, 8, 9, 10$. Thus $\sigma_1 = s_6 s_7 s_8 s_9 s_{10}$. Moreover, the strip decomposition of P is

$$\sigma = \sigma_1 \sigma_2 \sigma_3 = (s_6 s_7 s_8 s_9 s_{10})(s_4)(s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11}).$$

Let $\Lambda = (1122 \cdots nn)\sigma$. Now we are led to the following results.

Lemma 4.1. *Λ is a matching, and σ is the canonical reduced decomposition of Λ.*

Proof. It suffices to show that σ satisfies the conditions (4.1) and (4.2). The condition (4.2) for σ is straightforward. To certify the condition (4.1) for σ, we first prove $h_1 > h_2$.

Recall that h_2 is the label of an even self-dependent cell, denoted by C, enclosed by P_1. Obviously, C is an even cell enclosed by P. We claim that C is also self-dependent in P: Otherwise, the adjacent cell, say $\widehat{C}$, to the South-West of C is enclosed by P but not by P_1. It implies that $\widehat{C}$ belongs to the ladder strip L_1 of P. Notice that $\widehat{C}$ is an odd cell. By the construction of L_1, $\widehat{C}$ is followed by the even cell C in L_1. This contradicts that C is enclosed by P_1.

By the above claim, h_2 is the label of an even self-dependent cell enclosed by P. Since h_1 is the maximal label of the even self-dependent cells in P, one sees that $h_1 \geq h_2$. Observing that all the even self-dependent cells enclosed by P have distinct labels, we deduce $h_1 > h_2$. Recursively, the condition (4.1) is true for σ. $\square$

Lemma 4.2. *Λ is a 12312-avoiding matching.*

Proof. By Lemma 4.1, it remains to show that σ satisfies the condition (4.3).

Let L_i and L_j denote two ladder strips derived by the strip decomposition of P. In addition, the associated sequences of simple transpositions are $\sigma_i = s_{h_i} s_{h_i+1} \cdots s_{t_i}$ and $\sigma_j = s_{h_j} s_{h_j+1} \cdots s_{t_j}$ respectively. Assume that $i < j$. We have the following cases.

If $t_j \leq h_i - 2$, the condition (4.3) follows immediately.

Otherwise, $t_j \geq h_i - 1$. According to (4.1) and (4.2), we obtain $t_j \geq h_i - 1 > h_j$. By the construction of L_j, there is a cell, say D, in L_j labeled with $h_i - 1$. It follows that D is an odd cell. Note that each cell enclosed by P and touching the line $y = x/2$ is even. Hence, D is not a cell touching $y = x/2$. This implies that the adjacent cell, say $\widehat{D}$, to the West of D is enclosed by P. Moreover, $\widehat{D}$ is a cell in L_j.

Let $B_0, B_1, B_2, \ldots$ and $\ldots, \widehat{D}, D, \ldots$ be the sequences of cells in L_i and L_j, respectively. Since the cell B_0 is self-dependent and labeled with h_i, we derive that B_0 is in a column adjacent to D and in a row not higher than D. See Figure 3 for the relative positions of cells in L_i and L_j. Clearly, for each cell B_k in L_i, there is a cell in L_j, denoted by A_k, which is in the same column as B_k. By the labeling rules, A_k and B_k have the same label. Therefore, the labels t_i and t_j of the ending cells in L_i and L_j must satisfy $t_j \geq t_i$. This completes the proof. $\square$

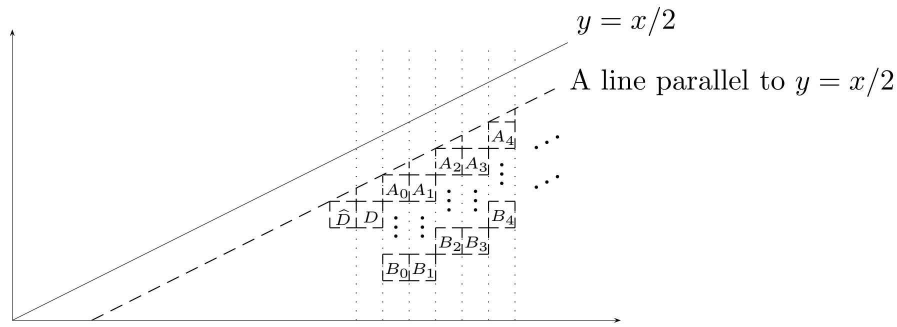

Figure 3: The relative positions of cells in L_i and L_j.

Conversely, for a set of parameters $\{(h_i, t_i) | 1 \leq i \leq k\}$ satisfying the conditions (4.1)–(4.3), one sees that the procedures are reversible to construct a ternary path. Therefore, we conclude with the following theorem.

Theorem 4.3. *There is a bijection between the set of ternary paths of length* $3n$ *and* $\mathcal{M}_n(12312)$.

By the strip decomposition and Corollary 2.6, we easily derive the following result.

Corollary 4.4. *For a ternary path* P*, the number of unit cells enclosed by* P *equals the number of inversions in the corresponding matching.*

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