An extension of matroid rank submodularity and the Z-Rayleigh property

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Submitted: Mar 24, 2011; Accepted: May 6, 2011; Published: May 16, 2011
Mathematics Subject Classification: 05B35

Abstract

We define an extension of matroid rank submodularity called R-submodularity, and introduce a minor-closed class of matroids called extended submodular matroids that are well-behaved with respect to R-submodularity. We apply R-submodularity to study a class of matroids with negatively correlated multivariate Tutte polynomials called the Z-Rayleigh matroids. First, we show that the class of extended submodular matroids are Z-Rayleigh. Second, we characterize a minor-minimal non-Z-Rayleigh matroid using its R-submodular properties. Lastly, we use R-submodularity to show that the Fano and non-Fano matroids (neither of which is extended submodular) are Z-Rayleigh, thus giving the first known examples of Z-Rayleigh matroids without the half-plane property.

1 Introduction and background

One of the fundamental axioms of a matroid rank function is its submodular property. For a matroid, $M(E, \rho)$, this is stated as follows [11, p.23].

(SM) For all $X, Y \subseteq E$, $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

In this paper we define the following extension of submodularity. For mutually disjoint $P_1, P_2, R \subseteq E$, we say P_1 and P_2 are *R-submodular* in M if there exists a bijection $\pi : 2^R \to 2^R$ such that for all $C \subseteq R$, $\rho(P_1 \cup P_2 \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2 \cup R \setminus \pi C)$. Under our definition, the property **(SM)** is equivalent to the $\emptyset$-submodularity of sets $X \setminus Y$ and $Y \setminus X$ in the minor $M/X \cap Y$ for all $X, Y \subseteq E$. We further say the matroid $M(E, \rho)$ is *extended submodular* if for all mutually disjoint subsets P_1, $P_2, R \subseteq E$, the sets P_1 and P_2 are R-submodular in M and its minors. We show that the class of all extended submodular matroids is closed under some fundamental matroid operations, and includes the uniform matroids and series-parallel networks among others.

A primary application of extended submodularity is in the study of Rayleigh properties of the multivariate Tutte polynomial of matroids. Following Sokal [13], we define the *multivariate Tutte polynomial* of a matroid $M(E, \rho)$ to be

$$Z(M, q; \mathbf{y}) = \sum_{A \subseteq E} q^{-\rho(A)} \prod_{e \in A} y_e, \tag{1}$$

where q and $\mathbf{y} = (y_e : e \in E)$ are commuting indeterminates. (An equivalent function called the *Tugger polynomial* was defined earlier by Kung [7].) We refer to [13] for the many useful properties of $Z(M, q; \mathbf{y})$. The *Tutte polynomial* of $M(E, \rho)$ is the two variable polynomial

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)},$$

and is known to be a special case of $Z(M, q; \mathbf{y})$. For $e, f \in E$ let

$$\begin{aligned}\Delta Z\{e, f\}(M, q; \mathbf{y}) &= Z(M/e \setminus f, q; \mathbf{y}) \cdot Z(M \setminus e/f, q; \mathbf{y}) \\ &\quad - q^{\rho(\{e\})+\rho(\{f\})-\rho(\{e,f\})} Z(M/e/f, q; \mathbf{y}) \cdot Z(M \setminus e \setminus f, q; \mathbf{y}).\end{aligned} \tag{2}$$

Sokal [14] calls $M(E, \rho)$ *Z-Rayleigh* if for all distinct $e, f \in E$, $0 < q \leq 1$ and $\mathbf{y} > 0$, $\Delta Z\{e, f\}(M, q; \mathbf{y}) \geq 0$. Here we use the notation $\mathbf{y} > 0$ to denote $y_e > 0$ for all $e \in E$. We shall often consider $\Delta Z\{e, f\}(M, q; \mathbf{y})$ to be a multivariate polynomial, called a *Z-Rayleigh difference polynomial*, in the elements of $\mathbf{y}$. Note that our definition of $\Delta Z\{e, f\}(M, q; \mathbf{y})$ follows Wagner [16, Section 3].

The generating polynomial of the bases of a matroid $M(E, \rho)$ is

$$B(M; \mathbf{y}) = \sum_{\substack{A \subseteq E \\ A \in \mathcal{B}}} \prod_{e \in A} y_e,$$

where $\mathcal{B}$ is the collection of bases of M [7]. Choe and Wagner [4] initiated the discussion on Rayleigh matroids by calling $M(E, \rho)$ *Rayleigh* if for all distinct $e, f \in E$ and $\mathbf{y} > 0$, $\Delta B\{e, f\}(M; \mathbf{y}) = B(M/e \setminus f; \mathbf{y}) B(M \setminus e/f; \mathbf{y}) - B(M/e/f; \mathbf{y}) B(M \setminus e \setminus f; \mathbf{y}) \geq 0$. Matroids with analogous properties for the collection of independent and spanning sets were termed *independence correlated* and *spanning correlated* matroids by Semple and Welsh [12] and Cocks [5]. In the rest of this paper, we follow Wagner [16] and use the terms *B-Rayleigh*, *I-Rayleigh* and *S-Rayleigh* for Rayleigh, independence correlated and spanning correlated matroids, respectively. Wagner [16] also called $M(E, \rho)$ *Potts-Rayleigh* if $\Delta Z\{e, f\}(M, q; \mathbf{y}) \geq 0$ for all distinct $e, f \in E$ with q in some interval $0 < q \leq q_*(M) \leq 1$ and $\mathbf{y} > 0$. It is easy to show that all Z-Rayleigh matroids are Potts-Rayleigh, and all Potts-Rayleigh matroids are also B-Rayleigh, I-Rayleigh and S-Rayleigh. Semple and Welsh [12] further showed that the class B-Rayleigh includes both the I-Rayleigh and S-Rayleigh matroids. However, no other inclusion relationship is known among these classes.

It is known that all uniform matroids and series parallel networks are Z-Rayleigh [16]. In this paper, we show that every extended submodular matroid is Z-Rayleigh with the additional property that all coefficients of its Z-Rayleigh difference polynomials are non-negative for all q in the interval $0 < q \leq 1$. This provides a combinatorial explanation for the occurrences of only non-negative coefficients in the B-Rayleigh, I-Rayleigh and S-Rayleigh difference polynomials of some matroids as reported in [12], [5] and [16]. More generally, for any $e, f \in E$, we show that the R-submodularity of sets $\{e\}$ and $\{f\}$ in a minor of $M(E, \rho)$ is a sufficient condition for non-negativity of a corresponding coefficient in $\Delta Z\{e, f\}(M, q; \mathbf{y})$ whenever $0 < q \leq 1$. Even when a matroid is not extended submodular, this result allows us to quickly test the non-negativity of the coefficients of its Z-Rayleigh difference polynomials for all q in $0 < q \leq 1$, which in turn reduces the amount of computation required to verify if the matroid is Z-Rayleigh. We illustrate this by showing that the Fano and non-Fano matroids are Z-Rayleigh.

The remainder of the paper is organized as follows. The next section defines R-submodularity in matroids, and Section 3 introduces the class of extended submodular matroids. Section 4 discusses the relationship between R-submodularity and the Z-Rayleigh property of matroids. We conclude with a discussion of open problems in Section 5. Throughout the paper, we assume familiarity with fundamental matroid concepts and notations [11]. Additionally, if N is a minor of matroid $M(E, \rho)$, we use ρ_N to denote the rank function of N. Finally, we note that the term *extended submodular inequality* has been used previously with a meaning very different from ours by Greene and Magnanti [6, Section 3] for an inequality that is valid in all matroids, and by Bouchet [1, Section 2] as an axiom for multimatroids.

2 R-submodularity in matroids

The notion of R-submodularity is based on matroid rank dominations that were introduced in [9].

Given three mutually disjoint sets, P_1, P_2, R, we first define a set, $S(P_1, P_2, R)$, of disjoint pairs as follows.

$$S(P_1, P_2, R) = \{(P_1 \cup C, P_2 \cup R \setminus C) : C \subseteq R\}.$$

Equivalently, the set $S(P_1, P_2, R)$ is the collection of all partitions (X, Y) of the set $P_1 \cup P_2 \cup R$ subject to the constraints $P_1 \subseteq X$ and $P_2 \subseteq Y$. There is one such pair (X, Y) for every subset of R, and hence $|S(P_1, P_2, R)| = 2^{|R|}$.

For the sake of brevity, henceforth any reference to the set $S(P_1, P_2, R)$ will be understood to imply that the three sets P_1, P_2 and R are mutually disjoint.

Definition 1 (**Rank domination and equivalence [9]**)**.** Let $M(E, \rho)$ be a matroid and $P_1, P_2, Q_1, Q_2, R \subseteq E$ such that (P_1, P_2, R) and (Q_1, Q_2, R) are triples of mutually disjoint sets. We say the set $S(P_1, P_2, R)$ is *rank dominated* by $S(Q_1, Q_2, R)$ in M if there exists a bijection $\pi : 2^R \to 2^R$ such that for all $C \subseteq R$, $\rho(P_1 \cup C) + \rho(P_2 \cup R \setminus C) \leq \rho(Q_1 \cup \pi C) + \rho(Q_2 \cup R \setminus \pi C)$. We write $S(P_1, P_2, R) \leq_M S(Q_1, Q_2, R)$ to denote such a relationship, omitting the subscript M when the matroid is clear from the context, and call π a *rank dominating bijection* of the 4-tuple (P_1, P_2, Q_1, Q_2) in M.

Additionally, if for all $C \subseteq R$, $\rho(P_1 \cup C) + \rho(P_2 \cup R \setminus C) = \rho(Q_1 \cup \pi C) + \rho(Q_2 \cup R \setminus \pi C)$, we also say the set $S(P_1, P_2, R)$ is *rank equivalent* to $S(Q_1, Q_2, R)$ in M, and write $S(P_1, P_2, R) \equiv_M S(Q_1, Q_2, R)$ (omitting the subscript M if the matroid is clear from the context).

Note that despite a notational change in the definition of a rank dominating bijection from the one used in [9, Section 2], it is easy to check that the two definitions are equivalent. The following properties of rank domination and equivalence were shown in [9].

Proposition 2 (Mani [9])**.** *In any matroid $M(E, \rho)$ for all $P_1, P_2, Q_1, Q_2, T_1, T_2, R \subseteq E$:*

1. *(Reflexivity). $S(P_1, P_2, R) \equiv S(P_1, P_2, R)$. (Thus rank domination is also reflexive.)*
2. *(Transitivity). If $S(P_1, P_2, R) \leq S(Q_1, Q_2, R)$ and $S(Q_1, Q_2, R) \leq S(T_1, T_2, R)$, then $S(P_1, P_2, R) \leq S(T_1, T_2, R)$. (Hence rank equivalence is also transitive.)*
3. *(Symmetry of $\equiv$). If $S(P_1, P_2, R) \equiv S(Q_1, Q_2, R)$ then $S(Q_1, Q_2, R) \equiv S(P_1, P_2, R)$.*
4. *(Antisymmetry of $\leq$). $S(P_1, P_2, R) \leq S(Q_1, Q_2, R)$ and $S(Q_1, Q_2, R) \leq S(P_1, P_2, R)$ if and only if $S(P_1, P_2, R) \equiv S(Q_1, Q_2, R)$.*
5. *$S(P_1, P_2, R) \equiv S(P_2, P_1, R)$.*

Rank submodularity provides an useful example of rank dominations as shown in our next example.

Example 3. In any matroid $M(E, \rho)$, for all disjoint pairs $P_1, P_2 \subseteq E$, we have $S(P_1 \cup P_2, \emptyset, \emptyset) \leq S(P_1, P_2, \emptyset)$. This can be shown to be equivalent to the property **(SM)**.

2.1 Definition and properties

We now define R-submodularity as follows.

Definition 4 (**R-submodularity**)**.** Let $M(E, \rho)$ be a matroid and $R \subseteq E$. We say two disjoint sets $P_1, P_2 \subseteq E \setminus R$ are *R-submodular* in M if there exists a bijection $\pi : 2^R \to 2^R$ such that for all $C \subseteq R$, $\rho(P_1 \cup P_2 \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2 \cup R \setminus \pi C)$. We call π an *R-submodular bijection* of the ordered pair (P_1, P_2) in M.

Additionally, if for all $C \subseteq R$, $\rho(P_1 \cup P_2 \cup C) + \rho(R \setminus C) = \rho(P_1 \cup \pi C) + \rho(P_2 \cup R \setminus \pi C)$, we say P_1 and P_2 are *R-modular* in M, and π an *R-modular bijection* of the ordered pair (P_1, P_2).

Equivalently, P_1 and P_2 are R-submodular in M if $S(P_1 \cup P_2, \emptyset, R) \leq_M S(P_1, P_2, R)$, and R-modular if $S(P_1 \cup P_2, \emptyset, R) \equiv_M S(P_1, P_2, R)$. The next result is easy to establish for all $M(E, \rho)$.

Proposition 5. *1. All disjoint $P_1, P_2 \subseteq E$ are $\emptyset$-submodular in M. (See Example 3.)*

2. For all disjoint $P, R \subseteq E$, the sets P and $\emptyset$ are R-modular in M.

We further know of the following instances of R-submodularity for a matroid $M(E, \rho)$. The first of these was communicated to us by Noble [10]. Recall that the *closure* operator in M is the map $\mathrm{cl}_M : 2^E \to 2^E$ defined by $\mathrm{cl}_M(X) = \{e : \rho(X \cup \{e\}) = \rho(X)\}$.

Proposition 6 (Noble [10])**.** *All disjoint $P_1, P_2 \subseteq E$ are R-submodular in M whenever $R \subseteq \mathrm{cl}_M(P_1 \cup P_2) \setminus (P_1 \cup P_2)$.*

Proposition 7 (Mani [9])**.** *All disjoint $P_1, P_2 \subseteq E$ are R-submodular in M whenever $R \subseteq E \setminus (P_1 \cup P_2)$ and $|R| \leq 3$.*

However, as the following counterexamples demonstrate, in general Proposition 7 is false if $|R| > 3$.

Example 8. Consider the matroids W_3 and $\mathcal{W}_3$ with geometric representations as shown in Figure 1. W_3 and $\mathcal{W}_3$ are called the rank-three wheel and whirl matroids, respectively [11, p.293]. If we let $P_1 = \{1\}$, $P_2 = \{4\}$ and $R = \{2, 3, 5, 6\}$, then Table 1 shows that P_1 and P_2 are not R-submodular in W_3 and $\mathcal{W}_3$.

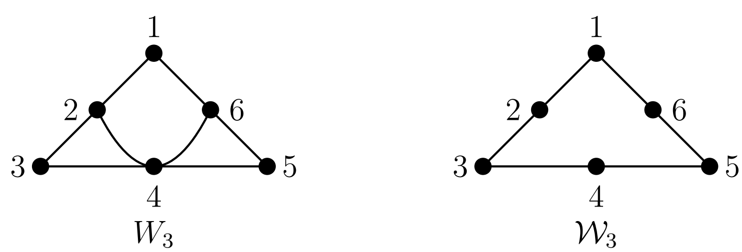

Figure 1: Rank 3-wheel and whirl

Nevertheless, in Section 3 we introduce a minor closed class of matroids where all disjoint $P_1, P_2 \subseteq E$ are R-submodular whenever $R \subseteq E \setminus (P_1 \cup P_2)$.

We next look at the effect of some common matroid operations on its R-submodularity. First, the deletion operation is easily seen to preserve R-submodularity.

Proposition 9. *Let $M(E, \rho)$ be a matroid. Then for all mutually disjoint $P_1, P_2, R \subseteq E$ and $F \subseteq E \setminus (P_1 \cup P_2 \cup R)$, $\pi : 2^R \to 2^R$ is an R-submodular bijection of the pair (P_1, P_2) in M if and only if π is an R-submodular bijection of (P_1, P_2) in $M \setminus F$.*

Proof. Use $\rho_{M \setminus F}(X) = \rho(X)$ for all $X \subseteq E \setminus F$ in Definition 4. $\square$

In contrast, the next counterexample demonstrates that a contraction operation (and by inference, also duality) does not necessarily preserve R-submodularity.

C	W_3		$\mathcal{W}_3$	
	$\rho(\{1,4\} \cup C) +$ $\rho(\{2,3,5,6\} \backslash C)$	$\rho(\{1\} \cup C) +$ $\rho(\{2,3,4,5,6\} \backslash C)$	$\rho(\{1,4\} \cup C) +$ $\rho(\{2,3,5,6\} \backslash C)$	$\rho(\{1\} \cup C) +$ $\rho(\{2,3,4,5,6\} \backslash C)$
$\emptyset$	5	4	5	4
{2}	6	5	6	5
{3}	6	5	6	5
{5}	6	5	6	5
{6}	6	5	6	5
{2,3}	5	5	5	5
{2,5}	5	6	5	6
{2,6}	5	5	5	5
{3,5}	5	5	5	6
{3,6}	5	6	5	6
{5,6}	5	5	5	5
{2,3,5}	4	5	4	5
{2,3,6}	4	5	4	5
{2,5,6}	4	5	4	5
{3,5,6}	4	5	4	5
{2,3,5,6}	3	4	3	4

Table 1: $\{1\}$ and $\{4\}$ are not $\{2,3,5,6\}$-submodular in W_3 and $\mathcal{W}_3$. (Submodular bijections in these two cases are impossible as columns two and four contain more number of 6's than columns three and five, respectively).

Example 10. Let $M(G)$ be the cycle matroid of graph G with edges labeled as shown in Figure 2. Also let $P_1 = \{1\}$, $P_2 = \{4\}$ and $R = \{2,3,5,6\}$. Then it can be checked from Proposition 5-2 and Lemma 14-1 below, that P_1 and P_2 are R-submodular in $M(G)$, while Example 8 implies that P_1 and P_2 are not R-submodular in $M(G)/7 = W_3$.

Despite this observation, our next result shows that an R-submodularity relationship in $M(E,\rho)$ manifests itself in a particular minor of its dual matroid. The *dual* of $M(E,\rho)$ is the matroid $M^*(E,\rho^*)$, where for all $X \subseteq E$,

$$\rho^*(X) = |X| - \rho(E) + \rho(E \setminus X). \tag{3}$$

Proposition 11. *Let $M(E,\rho)$ be a matroid. Then for all mutually disjoint $P_1, P_2, R \subseteq E$, $\pi : 2^R \to 2^R$ is an R-submodular bijection of the pair (P_1, P_2) in M if and only if π is an R-submodular bijection of (P_1, P_2) in M^*/F, where $F = E \setminus (P_1 \cup P_2 \cup R)$.*

Proof. By definition, $\pi : 2^R \to 2^R$ is an R-submodular bijection of (P_1, P_2) in M if and only if for all $C \subseteq R$,

$$\rho(P_1 \cup P_2 \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2 \cup R \setminus \pi C).$$

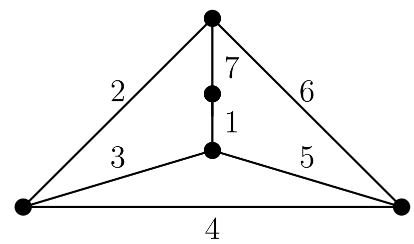

Figure 2: A Graph G such that $M(G)/7 = W_3$

Since $F \cup P_1 \cup P_2 \cup R = E$, we also have

$$P_1 \cup P_2 \cup C = E \setminus (F \cup R \setminus C), \quad R \setminus C = E \setminus (F \cup P_1 \cup P_2 \cup C)$$
$$P_1 \cup \pi C = E \setminus (F \cup P_2 \cup R \setminus \pi C), \text{ and } P_2 \cup R \setminus \pi C = E \setminus (F \cup P_1 \cup \pi C).$$

It follows that π is an R-submodular bijection of (P_1, P_2) in M if and only if for all $C \subseteq R$,

$$\rho(E\setminus(F\cup R\setminus C))+\rho(E\setminus(F\cup P_1\cup P_2\cup C)) \leq \rho(E\setminus(F\cup P_2\cup R\setminus\pi C))+\rho(E\setminus(F\cup P_1\cup\pi C)).$$

Now, applying (3) we see, equivalently, for all $C \subseteq R$,

$$\rho^*(F \cup P_1 \cup P_2 \cup C) + \rho^*(F \cup R \setminus C) \leq \rho^*(F \cup P_1 \cup \pi C) + \rho^*(F \cup P_2 \cup R \setminus \pi C).$$

That is, π is an R-submodular bijection of (P_1, P_2) in both M and M^*/F. □

When $P_1 \cup P_2 \cup R = E$, we get the following useful corollary.

Corollary 12. *Let $M(E, \rho)$ be a matroid, and $P_1, P_2, R \subseteq E$ be three mutually disjoint sets such that $P_1 \cup P_2 \cup R = E$. Then $\pi : 2^R \to 2^R$ is an R-submodular bijection of (P_1, P_2) in M if and only if π is an R-submodular bijection of (P_1, P_2) in M^*.*

If $M_1(E_1, \rho_1)$ and $M_2(E_2, \rho_2)$ are two matroids defined on disjoint sets E_1 and E_2, then their *direct sum* is the matroid $M_1 \oplus M_2(E, \rho)$, where $E = E_1 \cup E_2$, and $\rho(X) = \rho_1(X \cap E_1) + \rho_2(X \cap E_2)$ for all $X \subseteq E$. We now show that R-submodularity extends itself over the direct sum of matroids.

Proposition 13. *Let $M_1(E_1, \rho_1)$ and $M_2(E_2, \rho_2)$ be two matroids such that E_1 and E_2 are disjoint. Also let $P_{11}, P_{21}, R_1 \subseteq E_1$ and $P_{21}, P_{22}, R_2 \subseteq E_2$ such that P_{11} and P_{21} are R_1-submodular in M_1, and P_{12} and P_{22} are R_2-submodular in M_2. Then $P_{11} \cup P_{12}$ and $P_{21} \cup P_{22}$ are $R_1 \cup R_2$-submodular in the matroid $M_1 \oplus M_2$.*

Proof. For $i = 1, 2$, let $\pi_i : 2^{R_i} \to 2^{R_i}$ be an R_i-submodular bijection of the pair (P_{1i}, P_{2i}) in M_i. Also let $P_1 = P_{11} \cup P_{12}$, $P_2 = P_{21} \cup P_{22}$ and $R = R_1 \cup R_2$. It is straightforward to verify that the bijection $\pi : 2^R \to 2^R$ defined by, for all $C \subseteq R$, $\pi C = \pi_1(C \cap R_1) \cup \pi_2(C \cap R_2)$ is an R-submodular bijection of (P_1, P_2) in $M_1 \oplus M_2$. □

2.2 Useful R-submodularity constructions

Our next series of results shows R-submodularity in minors can sometimes be used to construct larger pairs of R-submodular sets in a matroid.

Lemma 14. *Let $M(E,\rho)$ be a matroid, $P_1, P_2, R \subseteq E$ and $p \in E$ such that P_1 and P_2 are R-submodular in $M \setminus p$.*

1. *If $p \in \mathrm{cl}_M(P_1 \cup P_2)$, then for $i, j \in \{1,2\}, i \neq j$, sets $P_i \cup \{p\}$ and P_j are R-submodular in M.*

2. *For $i, j \in \{1,2\}, i \neq j$, if $p \in E \setminus \mathrm{cl}_M(P_i \cup R)$ then:*

   (a) $P_i \cup \{p\}$ and P_j are R-submodular in M, and

   (b) $S(P_i \cup P_j, \{p\}, R) \leq_M S(P_i \cup \{p\}, P_j, R)$.

Proof. Let $\pi : 2^R \to 2^R$ be an R-submodular bijection of (P_1, P_2) in $M \setminus p$. Then, from Proposition 9, π is also an R-submodular bijection of (P_1, P_2) in M. Hence for all $C \subseteq R$,

$$\rho(P_1 \cup P_2 \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2 \cup R \setminus \pi C). \tag{4}$$

(1). We prove the case $i = 1, j = 2$ below, and skip the similar proof when $i = 2, j = 1$.

When $p \in \mathrm{cl}_M(P_1 \cup P_2)$, we know for all $C \subseteq R$, $\rho(P_1 \cup P_2 \cup \{p\} \cup C) = \rho(P_1 \cup P_2 \cup C)$. Since $\rho(X) \leq \rho(X \cup \{p\})$ for all $X \subseteq E$ (4) implies, for all $C \subseteq R$,

$$\rho(P_1 \cup P_2 \cup \{p\} \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \{p\} \cup \pi C) + \rho(P_2 \cup R \setminus \pi C).$$

In other words, π is also an R-submodular bijection of the pair $(P_1 \cup \{p\}, P_2)$.

(2). We prove the case $i = 1, j = 2$, and omit the similar proof for the case $i = 2, j = 1$.

When $p \in E \setminus \mathrm{cl}_M(P_1 \cup R)$, we know for all $C \subseteq R$, $p \notin \mathrm{cl}_M(P_1 \cup \pi C)$ and so $\rho(P_1 \cup \{p\} \cup \pi C) = \rho(P_1 \cup \pi C) + 1$. Also $\rho(X \cup \{p\}) \leq \rho(X) + 1$ for all $X \subseteq E$, and thus (4) implies for all $C \subseteq R$,

$$\rho(P_1 \cup P_2 \cup \{p\} \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \{p\} \cup \pi C) + \rho(P_2 \cup R \setminus \pi C),$$

and

$$\rho(P_1 \cup P_2 \cup C) + \rho(\{p\} \cup R \setminus C) \leq \rho(P_1 \cup \{p\} \cup \pi C) + \rho(P_2 \cup R \setminus \pi C).$$

Thus π is an R-submodular bijection of the pair $(P_1 \cup \{p\}, P_2)$, and a rank dominating bijection of the 4-tuple $(P_1 \cup P_2, \{p\}, P_1 \cup \{p\}, P_2)$ in M. □

Lemma 15. *Let $M(E,\rho)$ be a matroid, $P_1, P_2, R \subseteq E$ and $p \in E$ such that P_1 and P_2 are R-submodular in M/p.*

1. *If $p \in E \setminus \mathrm{cl}_M(R)$, then for $i, j \in \{1,2\}, i \neq j$, sets $P_i \cup \{p\}$ and P_j are R-submodular in M.*

2. *For $i, j \in \{1,2\}, i \neq j$, if $p \in \mathrm{cl}_M(P_i)$ then:*

(a) P_i *and* $P_j \cup \{p\}$ *are* R*-submodular in* M*, and*

(b) $S(P_i \cup P_j, \{p\}, R) \leq_M S(P_i, P_j \cup \{p\}, R)$.

Proof. Let $\pi : 2^R \to 2^R$ be an R-submodular bijection of (P_1, P_2) in M/p. Then, for all $C \subseteq R$,

$$\rho_{M/p}(P_1 \cup P_2 \cup C) + \rho_{M/p}(R \setminus C) \leq \rho_{M/p}(P_1 \cup \pi C) + \rho_{M/p}(P_2 \cup R \setminus \pi C).$$

Since $\rho_{M/p}(X) = \rho(X \cup \{p\}) - \rho(\{p\})$ for all $X \subseteq E \setminus \{p\}$, this can be rewritten as, for all $C \subseteq R$,

$$\rho(P_1 \cup P_2 \cup \{p\} \cup C) + \rho(\{p\} \cup R \setminus C) \leq \rho(P_1 \cup \{p\} \cup \pi C) + \rho(P_2 \cup \{p\} \cup R \setminus \pi C). \quad (5)$$

(1). We prove the case $i = 1, j = 2$. The proof for $i = 2, j = 1$ is similar.

When $p \in E \setminus \mathrm{cl}_M(R)$, for all $C \subseteq R$, $\rho(\{p\} \cup (R \setminus C)) = \rho(R \setminus C) + 1$. Using $\rho(X) \geq \rho(X \cup \{p\}) - 1$ for all $X \subseteq E$, we get from (5),

$$\rho(P_1 \cup P_2 \cup \{p\} \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \{p\} \cup \pi C) + \rho(P_2 \cup R \setminus \pi C),$$

for all $C \subseteq R$. Hence π is an R-submodular bijection of the pairs $(P_1 \cup \{p\}, P_2)$ and $(P_1, P_2 \cup \{p\})$ in M.

(2). Again we only prove the case $i = 1, j = 2$.

When $p \in \mathrm{cl}_M(P_1)$, for all $C \subseteq R$, $\rho(P_1 \cup \{p\} \cup \pi C) = \rho(P_1 \cup \pi C)$. Now applying $\rho(X) \leq \rho(X \cup \{p\})$ for all $X \subseteq E$ to (5) we obtain

$$\rho(P_1 \cup P_2 \cup \{p\} \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2 \cup \{p\} \cup R \setminus \pi C),$$

and

$$\rho(P_1 \cup P_2 \cup C) + \rho(\{p\} \cup R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2 \cup \{p\} \cup R \setminus \pi C),$$

for all $C \subseteq R$. Thus π is an R-submodular bijection of $(P_1, P_2 \cup \{p\})$ and a rank dominating bijection of the 4-tuple $(P_1 \cup P_2, \{p\}, P_1, P_2 \cup \{p\})$ in M. ☐

Lemma 16. *Let* $M(E, \rho)$ *be a matroid,* $P_1, P_2, R \subseteq E$ *and* $r \in E$ *such that* P_1 *and* P_2 *are* R*-submodular in minors* $M \setminus r$ *and* M/r*. If* $r \in \mathrm{cl}_M(P_i)$ *or* $r \in E \setminus \mathrm{cl}_M(P_i \cup R)$ *for some* $i \in \{1, 2\}$*, then* P_1 *and* P_2 *are* $R \cup \{r\}$*-submodular in* M*.*

Proof. Note that,

$$S(P_1 \cup P_2, \emptyset, R \cup \{r\}) = S(P_1 \cup P_2 \cup \{r\}, \emptyset, R) \cup S(P_1 \cup P_2, \{r\}, R), \quad (6a)$$

$$\text{and } S(P_1, P_2, R \cup \{r\}) = S(P_1 \cup \{r\}, P_2, R) \cup S(P_1, P_2 \cup \{r\}, R), \quad (6b)$$

where all unions are over disjoint sets. We look at two possible cases.

Case $r \in \mathrm{cl}_M(P_i), i \in \{1, 2\}$**:** We prove the case $r \in \mathrm{cl}_M(P_1)$, and omit the similar proof of the case $r \in \mathrm{cl}_M(P_2)$.

When $r \in \mathrm{cl}_M(P_1)$ we know from Lemma 14-1 that $S(P_1 \cup P_2 \cup \{r\}, \emptyset, R) \leq_M S(P_1 \cup \{r\}, P_2, R)$, and from Lemma 15-2(b) that $S(P_1 \cup P_2, \{r\}, R) \leq_M S(P_1, P_2 \cup \{r\}, R)$. Hence from (6a)-(6b), $S(P_1 \cup P_2, \emptyset, R \cup \{r\}) \leq_M S(P_1, P_2, R \cup \{r\})$.

Case $r \in E \setminus \mathrm{cl}_M(P_i \cup R), i \in \{1, 2\}$**:** We prove the case $r \in E \setminus \mathrm{cl}_M(P_1 \cup R)$, and the proof when $r \in E \setminus \mathrm{cl}_M(P_2 \cup R)$ is similar.

When $r \in E \setminus \mathrm{cl}_M(P_1 \cup R)$, we have from Lemma 14-2(b), $S(P_1 \cup P_2, \{r\}, R) \leq_M S(P_1 \cup \{r\}, P_2, R)$, and from Lemma 15-1, $S(P_1 \cup P_2 \cup \{r\}, \emptyset, R) \leq_M S(P_1, P_2 \cup \{r\}, R)$. The result then follows from (6a)-(6b). □

Lemma 17. *Let $M(E, \rho)$ be a matroid, $P_1, P_2, R \subseteq E$ and $r \in E$ such that P_1 and P_2 be R-submodular in $M \setminus r$ and M/r. If there exists an element $s \in R$ such that $\{r, s\}$ is a circuit or a cocircuit in M, then P_1 and P_2 are $R \cup \{r\}$-submodular in M.*

Proof. We consider the two cases separately.

Case 1 ($\{r, s\}$ is a circuit in M**):** Let $R' = R \setminus \{s\}$. Also let,

$$S_1 = S(P_1 \cup P_2 \cup \{r, s\}, \emptyset, R') \cup S(P_1 \cup P_2, \{r, s\}, R'), \tag{7a}$$
$$S_2 = S(P_1 \cup P_2 \cup \{r\}, \{s\}, R') \cup S(P_1 \cup P_2 \cup \{s\}, \{r\}, R'), \tag{7b}$$
$$S_3 = S(P_1 \cup \{r, s\}, P_2, R') \cup S(P_1, P_2 \cup \{r, s\}, R'), \text{ and} \tag{7c}$$
$$S_4 = S(P_1 \cup \{r\}, P_2 \cup \{s\}, R') \cup S(P_1 \cup \{s\}, P_2 \cup \{r\}, R'), \tag{7d}$$

where all unions are over disjoint sets. Clearly, then

$$S(P_1 \cup P_2, \emptyset, R \cup \{r\}) = S_1 \cup S_2, \text{ and}$$
$$S(P_1, P_2, R \cup \{r\}) = S_3 \cup S_4.$$

For $i, j \in \{1, 2, 3, 4\}$ we say $S_i \leq_M S_j$ if there exists a bijection $\sigma : S_i \to S_j$ such that $\rho(W) + \rho(Z) \leq \rho(X) + \rho(Y)$ whenever $\sigma(W, Z) = (X, Y)$. To prove $S(P_1 \cup P_2, \emptyset, R \cup \{r\}) \leq_M S(P_1, P_2, R \cup \{r\})$, note that it is enough to show that $S_1 \leq_M S_3$ and $S_2 \leq_M S_4$.

Since $\{r, s\}$ is a circuit in M, using the rank dominating bijection defined by $\pi C = C$ for all $C \subseteq R'$, we have

$$S(P_1 \cup P_2 \cup \{r, s\}, \emptyset, R') \equiv_M S(P_1 \cup P_2 \cup \{s\}, \emptyset, R'), \tag{8a}$$
$$S(P_1 \cup P_2, \{r, s\}, R') \equiv_M S(P_1 \cup P_2, \{s\}, R'), \tag{8b}$$
$$S(P_1 \cup \{r, s\}, P_2, R') \equiv_M S(P_1 \cup \{s\}, P_2, R'), \text{ and} \tag{8c}$$
$$S(P_1, P_2 \cup \{r, s\}, R') \equiv_M S(P_1, P_2 \cup \{s\}, R'). \tag{8d}$$

However, by definition,

$$S(P_1 \cup P_2, \emptyset, R) = S(P_1 \cup P_2 \cup \{s\}, \emptyset, R') \cup S(P_1 \cup P_2, \{s\}, R'), \text{ and}$$
$$S(P_1, P_2, R) = S(P_1 \cup \{s\}, P_2, R') \cup S(P_1, P_2 \cup \{s\}, R').$$

Since P_1 and P_2 are R-submodular in $M \setminus r$, it follows that,

$$S(P_1 \cup P_2 \cup \{s\}, \emptyset, R') \cup S(P_1 \cup P_2, \{s\}, R') \leq_M S(P_1 \cup \{s\}, P_2, R') \cup S(P_1, P_2 \cup \{s\}, R').$$

From (7a), (7c) and (8a)-(8d), this is equivalent to $S_1 \leq_M S_3$.

Also when $\{r, s\}$ is a circuit in M, s is a loop in M/r. Thus, if P_1 and P_2 are R-submodular in M/r, then they are also R'-submodular in M/r, and so

$$S(P_1 \cup P_2, \emptyset, R') \leq_{M/r} S(P_1, P_2, R'). \tag{9}$$

Using the facts $\rho_{M/r}(X) = \rho(X \cup \{r\}) - \rho(\{r\})$ for all $X \subseteq E \setminus \{r\}$, and $\{r, s\}$ is a circuit in M in (9), we obtain

$$\begin{aligned} S(P_1 \cup P_2 \cup \{r\}, \{s\}, R') \equiv_M S(P_1 \cup P_2 \cup \{s\}, \{r\}, R') \leq_M \\ S(P_1 \cup \{s\}, P_2 \cup \{r\}, R') \equiv_M S(P_1 \cup \{r\}, P_2 \cup \{s\}, R'). \end{aligned} \tag{10}$$

From (7b), (7d) and (10), it follows that $S_3 \leq_M S_4$, which proves this case.

Case 2 ($\{r, s\}$ is a cocircuit in M): Let $F = E \setminus (P_1 \cup P_2 \cup R)$, and consider the matroid $N = M^*/F$. Since P_1 and P_2 are R-submodular in $M \setminus r$ and M/r, from Proposition 11, they are also R-submodular in $N \setminus r$ and N/r.

Further, the set $\{r, s\}$ is a cocircuit in $M \setminus F$, and hence is a circuit in $N = M^*/F$. Thus, using the previous case, P_1 and P_2 are $R \cup \{r\}$-submodular in N. Finally, another application of Proposition 11 gives P_1 and P_2 are $R \cup \{r\}$-submodular in M. □

3 Extended submodular matroids

Considering Example 8, we may ask if there exist minor closed classes of matroids in which P_1 and P_2 are R-submodular for all mutually disjoint $P_1, P_2, R \subseteq E$. In this section we show that there are some well-known classes of matroids with this property.

3.1 Definition and properties

Definition 18 (**R-family of minors**)**.** Let $M(E, \rho)$ be a matroid. Given an $R \subseteq E$, we define the *R-family of minors* of M, $\mathcal{MF}(M, R)$, to be

$$\mathcal{MF}(M, R) = \{M/C \setminus (R \setminus C) : C \subseteq R\}. \tag{11}$$

That is, the R-family of minors is the set of all minors of M obtained by deleting or contracting every element in R.

Definition 19 (**Extended submodular matroids**)**.** A matroid $M(E, \rho)$ is *extended submodular* if for all mutually disjoint $P_1, P_2, R \subseteq E$ and minors $N \in \mathcal{MF}(M, E \setminus (P_1 \cup P_2 \cup R))$, P_1 and P_2 are R-submodular in N. We denote the class of extended submodular matroids by $\mathcal{ESM}$.

We first show that the class $\mathcal{ESM}$ is closed under some well-known matroid operations.

Proposition 20. *If $M(E, \rho) \in \mathcal{ESM}$ then $M^* \in \mathcal{ESM}$.*

Proof. Let $P_1, P_2, R \subseteq E$, and $N^* \in \mathcal{MF}(M^*, E \setminus (P_1 \cup P_2 \cup R))$. Clearly, if $N = (N^*)^*$, then $N \in \mathcal{MF}(M, E \setminus (P_1 \cup P_2 \cup R))$. By definition, we know P_1 and P_2 are R-submodular in N. Since the ground set of N is $P_1 \cup P_2 \cup R$, from Corollary 12, sets P_1 and P_2 are also R-submodular in N^*. □

Proposition 21. *If $M_1(E_1, \rho_1), M_2(E_2, \rho_2) \in \mathcal{ESM}$ then their direct sum $M_1 \oplus M_2 \in \mathcal{ESM}$.*

Proof. Let $E = E_1 \cup E_2$, and $M = M_1 \oplus M_2$. Now, let $P_1, P_2, R \subseteq E$, $T = E \setminus (P_1 \cup P_2 \cup R)$ and suppose $N \in \mathcal{MF}(M, T)$. By definition, $N = M/X \setminus (T \setminus X)$ for some $X \subseteq T$. If we let, for $i = 1, 2$,

$$P_{1i} = P_1 \cap E_i, \quad P_{2i} = P_2 \cap E_i, \quad R_i = R \cap E_i,$$
$$T_i = E_i \setminus (P_1 \cup P_2 \cup R), \text{ and } N_i = M_i/(X \cap E_i) \setminus (T_i \setminus X),$$

then it can be checked that $N_1 \in \mathcal{MF}(M_1, T_1)$, $N_2 \in \mathcal{MF}(M_2, T_2)$, and $N = N_1 \oplus N_2$.

As $M_1, M_2 \in \mathcal{ESM}$, for $i = 1, 2$, the sets P_{1i} and P_{2i} are R_i-submodular in N_i. Hence, from Proposition 13, P_1 and P_2 are R-submodular in N. □

We say a matroid M_p is a *parallel extension* of $M(E, \rho)$ if $M_p \setminus e = M$, and there exists an $f \in E$ such that $\{e, f\}$ is a circuit in M_p [11, p.155]. Similarly, M_s is a *series extension* of $M(E, \rho)$ if $M_s/e = M$, and there is an $f \in E$ such that $\{e, f\}$ is a cocircuit in M_s.

Proposition 22. *Let $M(E, \rho) \in \mathcal{ESM}$.*

1. *If M_p is a parallel extension of M, then $M_p \in \mathcal{ESM}$.*
2. *If M_s is a series extension of M, then $M_s \in \mathcal{ESM}$.*

Proof. **(1). $[M_p \in \mathcal{ESM}]$.** Let $M_p \setminus e = M$, $E_p = E \cup \{e\}$ and $f \in E$ such that $\{e, f\}$ is a circuit in M_p. Also, let $P_1, P_2, R \subseteq E_p$, $T = E_p \setminus (P_1 \cup P_2 \cup R)$ and $N \in \mathcal{MF}(M_p, T)$. Then, by definition, $N = M_p/C \setminus (T \setminus C)$ for some $C \subseteq T$.

We have the following two possible cases.

Case 1 ($e \notin P_1 \cup P_2 \cup R$): Then $e \in T$. There are three possibilities in this case.

Subcase 1a ($e \notin C$): Let $T_e = T \setminus \{e\}$. As $M = M_p \setminus e$, we see that

$$N = M_p/C \setminus (T \setminus C) = M/C \setminus (T_e \setminus C).$$

Thus, $N \in \mathcal{MF}(M, T_e)$ and since $M \in \mathcal{ESM}$, it follows that P_1 and P_2 are R-submodular in N.

Subcase 1b ($e \in C$, $f \notin P_1 \cup P_2 \cup R$): Then $f \in T$. As $\{e, f\}$ is a circuit in M_p, we know f is a loop in M_p/e. Let $C_f = C \setminus \{f\}$. Then,

$$N = M_p/C \setminus (T \setminus C) = M_p/C_f \setminus (T \setminus C_f).$$

However, as e and f are parallel elements in M_p and $f \notin C_f$, this is now similar to Subcase 1a.

Subcase 1c ($e \in C$, $f \in P_1 \cup P_2 \cup R$): Let $C_e^f = C \cup \{f\} \setminus \{e\}$, and $N_e^f = M_p / C_e^f \setminus (T \setminus C)$. Then $N_e^f \in \mathcal{MF}(M, T \cup \{f\})$.

Also, let $P_1^f = P_1 \setminus \{f\}, P_2^f = P_2 \setminus \{f\}$ and $R^f = R \setminus \{f\}$. Since $M \in \mathcal{ESM}$, the sets P_1^f and P_2^f are R^f-submodular in N_e^f. But e and f are parallel in M_p, and so P_1^f and P_2^f are also R^f-submodular in $N = M_p / C \setminus (T \setminus C)$. Finally, as f is a loop in N, this implies P_1 and P_2 are R-submodular in N as required in this case.

Case 2 ($e \in P_1 \cup P_2 \cup R$): There are again three different possibilities.

Subcase 2a ($f \notin P_1 \cup P_2 \cup R$): If $f \notin P_1 \cup P_2 \cup R$, then as e and f are parallel elements in M_p, this is equivalent to $e \notin P_1 \cup P_2 \cup R, f \in P_1 \cup P_2 \cup R$, which was handled in Subcases 1a and 1c above.

Subcase 2b ($e \in R$, $f \in P_1 \cup P_2 \cup R$): Let $R^e = R \setminus \{e\}$. Then, $N \setminus e, N/e \in \mathcal{MF}(M, T)$, and as $M \in \mathcal{ESM}$, we know P_1 and P_2 are R^e-submodular in $N \setminus e$ and N/e.

Now, if $f \in R$, since $\{e, f\}$ is a circuit in N, we can deduce P_1 and P_2 are R-submodular in N from Lemma 17.

Also, if $f \in P_1$ or $f \in P_2$, then $e \in \mathrm{cl}_N(P_i)$ for some $i \in \{1, 2\}$, and we can use Lemma 16 to obtain P_1 and P_2 are R-submodular in N.

Subcase 2c ($e \in P_1 \cup P_2$, $f \in P_1 \cup P_2 \cup R$): Without loss of generality assume $e \in P_1$, and let $P_1^e = P_1 \setminus \{e\}$. As $N \setminus e \in \mathcal{MF}(M, T)$ and $M \in \mathcal{ESM}$, we know P_1^e and P_2 are R-submodular in $N \setminus e$.

Now, if $f \in P_1 \cup P_2$, then $e \in \mathrm{cl}_N(P_1 \cup P_2)$ and hence, from Lemma 14-1, we deduce P_1 and P_2 are R-submodular in N.

Alternatively, if $f \in R$, as e and f are parallel elements in N, the case is analogous to $e \in R$ and $f \in P_1$, which was handled in Subcase 2b.

(2). $[M_s \in \mathcal{ESM}]$. Let $M_s / e = M$. Then its dual M_s^* is a parallel extension of M^*, the dual matroid of M, such that $M_s^* \setminus e = M^*$ [11, p.155]. We know from Proposition 20 that $M^* \in \mathcal{ESM}$, and consequently, from the previous case, its parallel extension $M_s^* \in \mathcal{ESM}$. Finally, applying Proposition 20 again, we find $M_s \in \mathcal{ESM}$. □

A *parallel class* of $M(E, \rho)$ is a maximal set $X \subseteq E$ without loops such that $\rho(X) = 1$. A loopless matroid is called *simple* if all its parallel classes are of size one. Every matroid M has an *associated simple matroid* $\widetilde{M}$ obtained by deleting its loops and all but one distinguished element from each parallel class of M [11, p.52]. The next result is an immediate consequence of Proposition 22.

Corollary 23. *A matroid $M(E, \rho) \in \mathcal{ESM}$ if and only if its associated simple matroid $\widetilde{M} \in \mathcal{ESM}$.*

3.2 Examples and characterization results

We now identify some well-known classes of matroids that are extended submodular.

Lemma 24. *Let $M(E,\rho) = U_{m,n}$ be a uniform matroid of rank m and size n. Then, for any disjoint $P, R \subseteq E$, there exists an R-modular bijection $\pi : 2^R \to 2^R$ of the ordered pair $(P, \emptyset)$ in M such that for all $C \subseteq R$, $|\pi C| \geq |R| - |P| - |C|$.*

Proof. When $|R| \leq |P|$, the map $\pi C = C$ will satisfy our requirement.

When $|R| > |P|$, let $k = |R| - |P|$. In this case, first note that when $0 \leq |C| < k/2$, the map $\pi C = C$ will not satisfy our lemma because then

$$|\pi C| = |C| < |R| - |P| - |C|.$$

We instead define π as follows.

Let $\mathcal{R}_i$ and $\mathcal{R}_i^k$ be the set of all subsets of R of size i and $k-i$ respectively. We make the following claim.

Claim 25. *If $0 \leq i < k/2$, then $|\mathcal{R}_i| \leq |\mathcal{R}_i^k|$.*

Proof. Let $r = |R|$ and $\Delta = |\mathcal{R}_i| - |\mathcal{R}_i^k|$. Clearly when $|P| = 0$, we have $k - i = r - i$, and so $\Delta = 0$. When $0 < |P| < r$ and $0 \leq i < k/2$,

$$\begin{aligned}\Delta &= \binom{r}{i} - \binom{r}{k-i} \\ &= \frac{r!}{i!(k-i)!}\left(\frac{1}{(k-i+1)\cdots(k-i+r-k)} - \frac{1}{(i+1)\cdots(i+r-k)}\right).\end{aligned}$$

Since $i < k/2$, for all $j \in \mathbb{Z}$ we have $k - i + j > i + j$, and thus $\Delta < 0$. □

It follows that for each $0 \leq i < k/2$ there is an injection $\sigma_i : \mathcal{R}_i \to \mathcal{R}_i^k$. Additionally, for each $0 \leq i < k/2$ we define a map $\mu_i : \mathcal{R}_i^k \to \mathcal{R}_i \cup \mathcal{R}_i^k$, which is really the inverse map of σ_i extended to all elements of $\mathcal{R}_i^k$ as follows. For all $X \in \mathcal{R}_i^k$,

$$\mu_i(X) = \begin{cases} X', & \text{if there is a } X' \in \mathcal{R}_i \text{ such that } \sigma_i X' = X \\ X, & \text{otherwise.} \end{cases} \tag{12}$$

We are finally ready to define the bijection π that satisfies our lemma. For all $C \subseteq R$, let

$$\pi C = \begin{cases} \sigma_{|C|}(C), & \text{if } 0 \leq |C| < k/2 \\ \mu_{k-|C|}(C), & \text{if } k/2 < |C| \leq k \\ C, & \text{otherwise.} \end{cases} \tag{13}$$

It remains to show that (13) satisfies our requirements.

When $0 \leq |C| < k/2$, clearly

$$|\pi C| = |\sigma_{|C|}(C)| = k - |C| = |R| - |P| - |C|.$$

Consequently, in this case $|P \cup C| = |R \setminus \pi C|$ and $|R \setminus C| = |P \cup \pi C|$, and as M is a uniform matroid, this implies $\rho(P \cup C) + \rho(R \setminus C) = \rho(P \cup \pi C) + \rho(R \setminus \pi C)$.

When $k/2 < |C| \leq k$, we have $|\pi C| = |\mu_{k-|C|}(C)|$. Hence, from (12) we have two possibilities. First, it can be that

$$|\pi C| = k - |C| = |R| - |P| - |C|,$$

and the proof that π satisfies our requirements are similar to the previous case. Alternatively, when $k/2 < |C| \leq k$, from (12), we can have

$$|\pi C| = k - (k - |C|) = |C| > |R| - |P| - |C|.$$

It also follows trivially in this case that $\rho(P \cup C) + \rho(R \setminus C) = \rho(P \cup \pi C) + \rho(R \setminus \pi C)$.

Finally when $|C| = k/2$ or $k < |C| \leq |R|$, from (13), we have

$$|\pi C| = |C| \geq |R| - |P| - |C|.$$

Also, trivially, $\rho(P \cup C) + \rho(R \setminus C) = \rho(P \cup \pi C) + \rho(R \setminus \pi C)$, in this case as required. $\square$

Lemma 26. *Let $M(E, \rho) = U_{m,n}$ be a uniform matroid of rank m and size n. Then, for any mutually disjoint $P_1, P_2, R \subseteq E$, there exists an R-submodular bijection $\pi : 2^R \to 2^R$ of the pair (P_1, P_2) in M such that for all $C \subseteq R$, $|\pi C| \geq |R| - |P_1| - |C|$.*

Proof. We use induction on $|P_2|$. When $|P_2| = 0$, the statement follows from Lemma 24. Let $k \in \mathbb{Z}_{>0}$, and assume the lemma is true whenever $|P_2| < k$.

Suppose $|P_2| = k$, and let $P_2' = P_2 \setminus \{p\}$ for some $p \in P_2$. By the inductive hypothesis, there exists an R-submodular bijection $\pi : 2^R \to 2^R$ of (P_1, P_2') in M such that for all $C \subseteq R$, $|\pi C| \geq |R| - |P_1| - |C|$. Equivalently, for all $C \subseteq R$, we have $|P_2' \cup R \setminus \pi C| \leq |P_1 \cup P_2' \cup C|$. Since M is uniform, this implies if $p \notin \mathrm{cl}_M(P_1 \cup P_2' \cup C)$ then $p \notin \mathrm{cl}_M(P_2' \cup R \setminus \pi C)$. Hence, for all $C \subseteq R$,

$$\rho(P_1 \cup P_2' \cup \{p\} \cup C) + \rho(R \setminus C) \leq \rho(P_1 \cup \pi C) + \rho(P_2' \cup \{p\} \cup R \setminus \pi C).$$

Equivalently, π is also an R-submodular bijection of the pair $(P_1, P_2' \cup \{p\})$ in M. $\square$

We note that it is a curious result of the proofs of Lemmas 24 and 26 that for any mutually disjoint $P_1, P_2, R \subseteq E$, there exists an R-submodular bijection of the pair (P_1, P_2) in the uniform matroid $U_{m,n}$ that depends only on the sizes of the sets P_2 and R. In particular, this bijection is entirely independent of the rank m and size n of the matroid.

More importantly, as the class of uniform matroids is minor closed, these results show that all such matroids are extended submodular.

Corollary 27. *All uniform matroids are extended submodular.*

Chaourar and Oxley [3] showed that the excluded minors for the class of uniform matroids and their series-parallel extensions are $W_3, \mathcal{W}_3, Q_6, P_6$ and R_6. We refer to [11, p.503–504] for a description of the matroids Q_6, P_6 and R_6. Among these excluded minors, we can see from Example 8 that the matroids W_3 and $\mathcal{W}_3$ are not extended submodular. However, a routine computational verification shows that $Q_6, P_6, R_6 \in \mathcal{ESM}$ [8, Appendix A]. Following Oxley [11, p.365], for a set $\{M_1, M_2, \dots\}$ of matroids, we use the notation $\mathrm{EX}(M_1, M_2, \dots)$ to denote the class of matroids with no minors isomorphic to any element of $\{M_1, M_2, \dots\}$. Then, from the previous discussion, we get the following result.

Proposition 28. *$EX(W_3, \mathcal{W}_3, Q_6, P_6, R_6) \subsetneq \mathcal{ESM}$.*

Since all matroids in the excluded minor list of the previous result are of rank three, and the class $\mathcal{ESM}$ is dual closed, we can say the following.

Corollary 29. *Every matroid of rank or corank at most two is extended submodular.*

Brylawski [2] showed that the series-parallel networks are precisely the class of binary matroids without a minor isomorphic to W_3. Since $\mathcal{W}_3, Q_6, P_6$ and R_6 are known to be non-binary matroids, from Proposition 28 and Example 8, we get our next corollary.

Corollary 30. *A binary matroid is extended submodular if and only if it is a series-parallel network.*

4 R-submodularity and the Z-Rayleigh property

We now show that R-submodularity in minors of a matroid enforce non-negativity of corresponding coefficients (explained below in Lemma 33) in its Z-Rayleigh difference polynomials defined in (2). We begin by listing some existing results for the class of Z-Rayleigh matroids.

Let $M_1(E_1, \rho_1)$ and $M_2(E_2, \rho_2)$ be two matroids such that $E_1 \cap E_2 = \{e\}$. Then their *2-sum* is the matroid $M_1 \oplus_2 M_2(E, \rho)$ with ground set $E = E_1 \cup E_2 \setminus \{e\}$ and for all $X \subseteq E$,

$$\rho(X) = \begin{cases} \rho_1(X \cap E_1) + \rho_2(X \cap E_2) - 1, & \text{if } e \in \mathrm{cl}_{M_1}(X \cap E_1) \cap \mathrm{cl}_{M_2}(X \cap E_2); \\ \rho_1(X \cap E_1) + \rho_2(X \cap E_2), & \text{otherwise.} \end{cases}$$

Proposition 31. *1. The class of Z-Rayleigh matroids is closed under duals, minors and 2-sums [16, Lemma 4.1, Theorem 5.8].*

2. All uniform matroids are Z-Rayleigh [16, Section 5].

Note that Theorem 5.8 in [16] states that the class of Potts-Rayleigh matroids is closed under 2-sums. Nevertheless, the same argument is readily seen to show that the restricted class of Z-Rayleigh matroids is also closed under 2-sums.

It is an immediate consequence of Proposition 31 that all 2-sum extensions of uniform matroids including the series-parallel networks are Z-Rayleigh. Our results in the first half of this section shows that uniform matroids and their series-parallel extensions also have the additional property that every coefficient of their Z-Rayleigh difference polynomials is non-negative for all q in the interval $0 < q \leq 1$.

4.1 The basic results

Let $M(E, \rho)$ be a matroid. For any subset $A \subseteq E$, we frequently use the convenient notation $\mathbf{y}^A = \prod_{e \in A} y_e$ with the convention $\mathbf{y}^A = 1$ if A is empty. For disjoint sets $A, B \subseteq$

$E \setminus \{e,f\}$, we denote the coefficient of the monomial $\mathbf{y}^A(\mathbf{y}^B)^2$ in $\Delta Z\{e,f\}(M,q;\mathbf{y})$ by $[\mathbf{y}^A(\mathbf{y}^B)^2]\Delta Z\{e,f\}(M,q;\mathbf{y})$.

The following result is useful to establish the connection between R-submodularity of $\{e\}$ and $\{f\}$, and the non-negativity of the coefficients in a $\Delta Z\{e,f\}(M,q;\mathbf{y})$.

Lemma 32. *Let $M(E,\rho)$ be a matroid and $e,f \in E$ be two distinct non-loop elements. Then, for any disjoint pair of sets $R,T \subseteq E \setminus \{e,f\}$ and a bijection $\sigma : 2^R \to 2^R$, $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) = q^2\beta_{R,T}(q)$, where*

$$\beta_{R,T}(q) = \sum_{C \subseteq R}\Big(q^{-\rho(\{e\}\cup T\cup\sigma C)-\rho(\{f\}\cup T\cup R\setminus\sigma C)} - q^{-\rho(\{e,f\}\cup T\cup C)-\rho(T\cup R\setminus C)}\Big).$$

Proof. Let $\overline{E} = E \setminus \{e,f\}$. Recall that for any disjoint $U,V \subseteq E$ and all $A \subseteq E \setminus (V \cup U)$,

$$\rho_{M/V\setminus U}(A) = \rho(A \cup V) - \rho(V). \tag{14}$$

Using (1), (2), (14), we get

$$\Delta Z\{e,f\}(M,q;\mathbf{y}) = q^2\alpha(q;\mathbf{y}), \tag{15}$$

where

$$\begin{aligned}\alpha(q;\mathbf{y}) = &\sum_{(X,Y)\in 2^{\overline{E}}\times 2^{\overline{E}}} q^{-\rho(X\cup\{e\})-\rho(Y\cup\{f\})}\mathbf{y}^{X\cup Y}\mathbf{y}^{X\cap Y}\\ &- \sum_{(W,Z)\in 2^{\overline{E}}\times 2^{\overline{E}}} q^{-\rho(W\cup\{e,f\})-\rho(Z)}\mathbf{y}^{W\cup Z}\mathbf{y}^{W\cap Z}.\end{aligned} \tag{16}$$

We can now split each of the two sums in (16) into smaller sums on pairs (X,Y) (and (W,Z), respectively) according to $X \cup Y$ and $X \cap Y$ (and $W \cup Z$ and $W \cap Z$, respectively). That is,

$$\alpha(q;\mathbf{y}) = \sum_{\substack{(A,B)\in 2^{\overline{E}}\times 2^{\overline{E}}\\ A\cap B=\emptyset}} \gamma_{A,B}(q)\cdot \mathbf{y}^A(\mathbf{y}^B)^2, \tag{17}$$

where

$$\gamma_{A,B}(q) = \sum_{\substack{(X,Y)\in 2^{\overline{E}}\times 2^{\overline{E}}\\ X\cap Y=B,\, X\cup Y=A\cup B}} q^{-\rho(X\cup\{e\})-\rho(Y\cup\{f\})} - \sum_{\substack{(W,Z)\in 2^{\overline{E}}\times 2^{\overline{E}}\\ W\cap Z=B,\, W\cup Z=A\cup B}} q^{-\rho(W\cup\{e,f\})-\rho(Z)}. \tag{18}$$

From (15) and (17), it is plain that

$$[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) = q^2\gamma_{R,T}(q). \tag{19}$$

Also, note that,

$$\begin{aligned}\{(X\cup\{e\}, Y\cup\{f\}) : (X,Y) \in 2^{\overline{E}}\times 2^{\overline{E}}, X\cap Y = T, X\cup Y = R\cup T\} =\\ \{(\{e\}\cup T\cup C, \{f\}\cup T\cup R\setminus C) : C \subseteq R\},\ \text{and}\end{aligned}$$

$$\{(W \cup \{e,f\}, Z) : (W,Z) \in 2^{\overline{E}} \times 2^{\overline{E}}, W \cap Z = T, W \cup Z = R \cup T\} = \\ \{(\{e,f\} \cup T \cup C, T \cup R \setminus C) : C \subseteq R\}.$$

Hence, given any bijection $\sigma : 2^R \to 2^R$, we can rewrite (18) as

$$\begin{aligned}\gamma_{R,T}(q) &= \sum_{C \subseteq R} \Big(q^{-\rho(\{e\} \cup T \cup \sigma C) - \rho(\{f\} \cup T \cup R \setminus \sigma C)} - q^{-\rho(\{e,f\} \cup T \cup C) - \rho(T \cup R \setminus C)} \Big) \\ &= \beta_{R,T}(q).\end{aligned}$$

The result now follows from (19). $\square$

One consequence of this observation is particularly relevant for our purposes.

Lemma 33. *Let $M(E,\rho)$ be a matroid, $e,f \in E$ be two distinct elements and $\overline{E} = E \setminus \{e,f\}$. Also for any disjoint $R,T \subseteq \overline{E}$, let $\overline{T} = \overline{E} \setminus (T \cup R)$. If the sets $\{e\}$ and $\{f\}$ are R-submodular in the minor $N = M/T \setminus \overline{T} \in \mathcal{MF}(M, \overline{E} \setminus R)$, then the coefficient $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) \geq 0$ for all q in the interval $0 < q \leq 1$.*

Furthermore, $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) = 0$ for all q in the interval $0 < q \leq 1$ if and only if the sets $\{e\}$ and $\{f\}$ are R-modular in N.

Proof. It is easy to check that if e or f is a loop in M then $\Delta Z\{e,f\}(M,q;\mathbf{y}) = 0$, and the lemma is vacuously true. Henceforth we assume that neither e nor f is a loop in M.

Let $\pi : 2^R \to 2^R$ be an R-submodular bijection for the pair $(\{e\},\{f\})$ in $N = M/T \setminus \overline{T}$. Then, we know from Lemma 32, $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) = q^2\beta_{R,T}(q)$, where

$$\beta_{R,T}(q) = \sum_{C \subseteq R} \Big(q^{-\rho(\{e\} \cup T \cup \pi C) - \rho(\{f\} \cup T \cup R \setminus \pi C)} - q^{-\rho(\{e,f\} \cup T \cup C) - \rho(T \cup R \setminus C)} \Big). \qquad (20)$$

Since $\rho_N(A) = \rho(A \cup T) - \rho(T)$ for all $A \subseteq R \cup \{e,f\}$ and π is an R-submodular bijection, we have for all $C \subseteq R$,

$$\rho(\{e,f\} \cup T \cup C) + \rho(T \cup R \setminus C) \leq \rho(\{e\} \cup T \cup \pi C) + \rho(\{f\} \cup T \cup R \setminus \pi C).$$

Hence, each term of the sum in (20) is at least zero whenever $0 < q \leq 1$, which gives $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) \geq 0$.

However, if π is an R-modular bijection of $(\{e\},\{f\})$ in N, then for all $C \subseteq R$,

$$\rho(\{e,f\} \cup T \cup C) + \rho(T \cup R \setminus C) = \rho(\{e\} \cup T \cup \pi C) + \rho(\{f\} \cup T \cup R \setminus \pi C),$$

and again from (20) it follows that $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) = 0$ whenever $0 < q \leq 1$.

Conversely, if $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) = 0$ whenever $0 < q \leq 1$, from Lemma 32, we must have $\beta_{R,T}(q) = 0$ whenever $0 < q \leq 1$. This implies $\beta_{R,T}(q)$ is zero everywhere, for otherwise, as a polynomial in q^{-1}, it can only have finitely many roots for q in the interval $0 < q \leq 1$. That is, there exists a bijection $\pi : 2^R \to 2^R$ such that every term of the sum in (20) is zero, which implies for all $C \subseteq R$,

$$\rho(\{e,f\} \cup T \cup C) + \rho(T \cup R \setminus C) = \rho(\{e\} \cup T \cup \pi C) + \rho(\{f\} \cup T \cup R \setminus \pi C),$$

In other words, π is an R-modular bijection of $(\{e\},\{f\})$ in N. $\square$

It is worth noting that while R-submodularity of sets $\{e\}$ and $\{f\}$ in a minor is sufficient for non-negativity of the corresponding coefficient in the Z-Rayleigh difference polynomial, we do not know if it is also a necessary condition.

Following Wagner [16], we use the notation $Z_1(\mathbf{y}) \gg Z_2(\mathbf{y})$ for two multivariate polynomials $Z_1(\mathbf{y}), Z_2(\mathbf{y})$, if $Z_1(\mathbf{y}) - Z_2(\mathbf{y})$ has only non-negative coefficients. The next result is an immediate consequence of Lemma 33, and is often useful in identifying cases where $\Delta Z\{e,f\}(M,q;\mathbf{y}) \gg 0$ for all q in the interval $0 < q \leq 1$.

Theorem 34. *Let $M(E,\rho)$ be a matroid, $e, f \in E$ be distinct elements and $\overline{E} = E \setminus \{e,f\}$. If for all $R \subseteq \overline{E}$ and minors $N \in \mathcal{MF}(M, \overline{E} \setminus R)$, the sets $\{e\}$ and $\{f\}$ are R-submodular in N, then $\Delta Z\{e,f\}(M,q;\mathbf{y}) \gg 0$ whenever $0 < q \leq 1$.*

In particular, all extended submodular matroids are also Z-Rayleigh because the R-submodularity conditions of Theorem 34 are satisfied by definition, for all distinct elements e, f in its ground set.

Proposition 35. *If $M(E,\rho) \in \mathcal{ESM}$ then M is Z-Rayleigh, and for all distinct $e, f \in E$, $\Delta Z\{e,f\}(M,q;\mathbf{y}) \gg 0$ whenever $0 < q \leq 1$.*

4.2 Beyond extended submodular matroids

We next apply R-submodularity to obtain some results on the Z-Rayleigh properties of matroids that are not extended submodular. We begin with the following result which shows that a Theorem proved by Semple and Welsh [12, Theorem 4.2] for the I-Rayleigh property also extends to the Z-Rayleigh property.

Theorem 36. *Let $M(E,\rho) = M(G)$ be the cycle matroid of a graph G. Also let e, f be distinct edges in E, and $\overline{E} = E \setminus \{e,f\}$. If $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) < 0$ for some disjoint $R, T \subseteq \overline{E}$, $0 < q \leq 1$ and $\mathbf{y} > 0$, then the graph $G/T \setminus \overline{T}$ contains a K_4 minor, where $\overline{T} = \overline{E} \setminus (R \cup T)$.*

Proof. Let $H = G/T \setminus \overline{T}$. Then, from Lemma 33 we know $\{e\}$ and $\{f\}$ are not R-submodular in $M(H)$, and thus, from Corollary 30, H has a K_4 minor. □

Wagner [16, Conjecture 5.4] proposed a stronger conjecture for the I-Rayleigh property of graphic matroids requiring the edges e and f in the above result to also be non-adjacent edges of the K_4 minor in G. It is likely that an analogous result is true for the Z-Rayleigh property of graphic matroids, and we offer the following conjecture.

Conjecture 37. *Let $M(E,\rho) = M(G)$ be the cycle matroid of a graph G. Also let e, f be distinct edges in E, and $\overline{E} = E \setminus \{e,f\}$. If $[\mathbf{y}^R(\mathbf{y}^T)^2]\Delta Z\{e,f\}(M,q;\mathbf{y}) < 0$ for some disjoint $R, T \subseteq \overline{E}$, $0 < q \leq 1$ and $\mathbf{y} > 0$, then the graph $G/T \setminus \overline{T}$ contains a K_4 minor with e and f as non-adjacent edges, where $\overline{T} = \overline{E} \setminus (R \cup T)$.*

We next give a characterization of minor-minimal non-Z-Rayleigh matroids using R-submodularity.

Lemma 38. *Let $M(E,\rho)$ be a matroid with distinct elements $e,f,g \in E$, and let $\overline{E} = E \setminus \{e,f,g\}$. If for all $R \subseteq \overline{E}$, the sets $\{e\}$ and $\{f\}$ are $R \cup \{g\}$-submodular in every minor $N \in \mathcal{MF}(M, \overline{E} \setminus R)$, then*

$$\Delta Z\{e,f\}(M,q;\mathbf{y}) \gg \Delta Z\{e,f\}(M \setminus g,q;\mathbf{y}) + y_g^2 \Delta Z\{e,f\}(M/g,q;\mathbf{y}),$$

whenever $0 < q \leq 1$.

Proof. Applying a routine manipulation, we can write

$$\Delta Z\{e,f\}(M,q;\mathbf{y}) = \Delta Z\{e,f\}(M \setminus g,q;\mathbf{y}) + y_g^2 \Delta Z\{e,f\}(M/g,q;\mathbf{y}) + \Delta_g(\mathbf{y}),$$

where $\Delta_g(\mathbf{y})$ contains precisely those monomials of $\Delta Z\{e,f\}(M,q;\mathbf{y})$ that are of the form $\mathbf{y}^{R\cup\{g\}}(\mathbf{y}^T)^2$ for disjoint $R,T \subseteq \overline{E}$. It follows then from Lemma 33 that $\Delta_g(\mathbf{y}) \gg 0$ for all q in the interval $0 < q \leq 1$, and hence the result. $\square$

Corollary 39. *Let $M(E,\rho)$ be a minor-minimal matroid that is not Z-Rayleigh, with $e,f \in E$ such that $\Delta Z\{e,f\}(M,q;\mathbf{y}) < 0$ for some q in the interval $0 < q \leq 1$ and $\mathbf{y} > 0$. Then for every $g \in E \setminus \{e,f\}$ there exists an $R \subseteq E_g$ and a minor $N \in \mathcal{MF}(M, E_g \setminus R)$ such that $\{e\}$ and $\{f\}$ are not $R \cup \{g\}$-submodular in N, where $E_g = E \setminus \{e,f,g\}$.*

Proof. Suppose for a contradiction that there exists a $g \in E \setminus \{e,f\}$ such that for all $R \subseteq E_g$, the sets $\{e\}$ and $\{f\}$ are $R \cup \{g\}$-submodular in every minor $N \in \mathcal{MF}(M, E_g \setminus R)$. Then, by Lemma 38

$$\Delta Z\{e,f\}(M,q;\mathbf{y}) \gg \Delta Z\{e,f\}(M \setminus g,q;\mathbf{y}) + y_g^2 \Delta Z\{e,f\}(M/g,q;\mathbf{y}).$$

However, since M is a minor-minimal non-Z-Rayleigh matroid, both $M \setminus g$ and M/g are Z-Rayleigh, which implies $\Delta Z\{e,f\}(M,q;\mathbf{y}) \geq 0$ whenever $0 < q \leq 1$ and $\mathbf{y} > 0$, a contradiction. $\square$

In particular, this leads us to a characterization of minor-minimal non-Z-Rayleigh matroids that is similar to a result by Wagner [15, Proposition 3.2] for B-Rayleigh matroids. We first need the following R-submodularity result, which despite notational differences, is really an extension of the argument used in the proof of [15, Lemma 3.1].

Lemma 40. *Let $M(E,\rho)$ be a simple and cosimple matroid with distinct elements $e,f,g \in E$ such that $\{e,f,g\}$ is a circuit or a cocircuit in M. Then for all $R \subseteq E \setminus \{e,f,g\}$, the sets $\{e\}$ and $\{f\}$ are $R \cup \{g\}$-submodular in M.*

Proof. Let $\overline{E} = E \setminus \{e,f,g\}$ and $R \subseteq \overline{E}$. We consider the two cases separately.
Case 1 ($\{e,f,g\}$ is a circuit in M): We define the $R \cup \{g\}$-submodular bijection of $(\{e\},\{f\})$ in M as follows. For all $C \subseteq R$, let

$$\pi C = \begin{cases} R \setminus C, & \text{if } e,g \notin \mathrm{cl}_M(R \setminus C); \\ \{g\} \cup C, & \text{otherwise,} \end{cases} \tag{21a}$$

$$\pi(\{g\} \cup C) = \begin{cases} \{g\} \cup C, & \text{if } e,g \notin \mathrm{cl}_M(R \setminus C); \\ R \setminus C, & \text{otherwise.} \end{cases} \tag{21b}$$

We first show that $\pi : 2^{\{g\}\cup R} \to 2^{\{g\}\cup R}$ is a bijection.

Suppose there exist $C_1, C_2 \subseteq R$ such that $\pi C_1 = \pi C_2$ or $\pi(\{g\} \cup C_1) = \pi(\{g\} \cup C_2)$, then since $g \notin R$, it is easy to see from (21a) and (21b) that $C_1 = C_2$.

Alternatively, suppose there exist C_1, C_2 such that $\pi C_1 = \pi(\{g\} \cup C_2)$. Then, if $g \in \pi C_1$, from (21a) and (21b), we get $C_1 = C_2$, at least one of $e, g \in \mathrm{cl}_M(R \backslash C_1)$, and both $e, g \notin \mathrm{cl}_M(R \backslash C_2)$, which is a contradiction. A very similar argument gives a contradiction when $g \notin \pi C_1$. Hence there cannot be $C_1, C_2 \subseteq R$ such that $\pi C_1 = \pi(\{g\} \cup C_2)$, and π is a bijection.

It remains to show that π is a $R \cup \{g\}$-submodular bijection of $(\{e\}, \{f\})$ in M. As $\{e, f, g\}$ is a circuit in M, we have for all $C \subseteq R$,

$$\rho(\{e, f, g\} \cup C) = \rho(\{e, f\} \cup C) = \rho(\{e, g\} \cup C) = \rho(\{f, g\} \cup C). \tag{22}$$

We have the following three possibilities for a subset $C \subseteq R$.

Subcase 1a ($g \in \mathrm{cl}_M(R \setminus C)$): Then $\rho(\{g\} \cup R \setminus C) \leq \rho(\{f\} \cup R \setminus C)$. Hence, from (21a), (21b) and (22),

$$\begin{aligned} \rho(\{e, f\} \cup C) + \rho(\{g\} \cup R \setminus C) &\leq \rho(\{e, g\} \cup C) + \rho(\{f\} \cup R \setminus C), \\ &= \rho(\{e\} \cup \pi C) + \rho(\{f, g\} \cup R \setminus \pi C), \text{ and} \end{aligned}$$

$$\begin{aligned} \rho(\{e, f, g\} \cup C) + \rho(R \setminus C) &\leq \rho(\{e\} \cup R \setminus C) + \rho(\{f, g\} \cup C) \\ &= \rho(\{e\} \cup \pi(\{g\} \cup C)) + \rho(\{f, g\} \cup R \setminus \pi(\{g\} \cup C)). \end{aligned}$$

Subcase 1b ($e, g \notin \mathrm{cl}_M(R \setminus C)$): Then $\rho(\{e\} \cup R \setminus C) = \rho(\{g\} \cup R \setminus C)$. Thus, from (21a), (21b) and (22),

$$\begin{aligned} \rho(\{e, f\} \cup C) + \rho(\{g\} \cup R \setminus C) &= \rho(\{e\} \cup R \setminus C) + \rho(\{f, g\} \cup C) \\ &= \rho(\{e\} \cup \pi C) + \rho(\{f, g\} \cup R \setminus \pi C), \text{ and} \end{aligned}$$

$$\begin{aligned} \rho(\{e, f, g\} \cup C) + \rho(R \setminus C) &\leq \rho(\{e, g\} \cup C) + \rho(\{f\} \cup R \setminus C) \\ &= \rho(\{e\} \cup \pi(\{g\} \cup C)) + \rho(\{f, g\} \cup R \setminus \pi(\{g\} \cup C)). \end{aligned}$$

Subcase 1c ($e \in \mathrm{cl}_M(R \setminus C), g \notin \mathrm{cl}_M(R \setminus C)$): Then observe that $f \notin \mathrm{cl}_M(R \setminus C)$. For otherwise, if both $e, f \in \mathrm{cl}_M(R \setminus C)$ then

$$g \in \mathrm{cl}_M(\{e, f\} \cup (R \setminus C)) = \mathrm{cl}_M(R \setminus C),$$

which is a contradiction. So we have $\rho(\{g\} \cup R \setminus C) = \rho(\{f\} \cup R \setminus C)$, and from (21a), (21b) and (22), it follows that

$$\begin{aligned} \rho(\{e, f\} \cup C) + \rho(\{g\} \cup R \setminus C) &= \rho(\{e, g\} \cup C) + \rho(\{f\} \cup R \setminus C), \\ &= \rho(\{e\} \cup \pi C) + \rho(\{f, g\} \cup R \setminus \pi C), \text{ and} \end{aligned}$$

$$\begin{aligned} \rho(\{e, f, g\} \cup C) + \rho(R \setminus C) &\leq \rho(\{e\} \cup R \setminus C) + \rho(\{f, g\} \cup C) \\ &= \rho(\{e\} \cup \pi(\{g\} \cup C)) + \rho(\{f, g\} \cup R \setminus \pi(\{g\} \cup C)). \end{aligned}$$

Hence, π is a $\{g\} \cup R$-submodular bijection of $(\{e\}, \{f\})$ in M in this case.
Case 2 ($\{e, f, g\}$ is a cocircuit in M): Consider the matroid $N^* = M^*/(\overline{E} \setminus R)$. Clearly, $\{e, f, g\}$ is a circuit in N^*, and hence by Case 1, we know $\{e\}$ and $\{f\}$ are $R \cup \{g\}$-submodular in N^*. Now, applying Proposition 11, we get $\{e\}$ and $\{f\}$ are also $R \cup \{g\}$-submodular in M as required. □

Corollary 41. *Let $M(E, \rho)$ be a simple and cosimple matroid with distinct elements $e, f, g \in E$ such that $\{e, f, g\}$ is a circuit or a cocircuit in M, and let $\overline{E} = E \setminus \{e, f, g\}$. Then for all $R \subseteq \overline{E}$, the sets $\{e\}$ and $\{f\}$ are $R \cup \{g\}$-submodular in every minor $N \in \mathcal{MF}(M, \overline{E} \setminus R)$.*

Proof. Clearly, for all $R \subseteq \overline{E}$, $\{e, f, g\}$ is a circuit or a cocircuit in every minor $N \in \mathcal{MF}(M, \overline{E} \setminus R)$, and the result follows from Lemma 40. □

We can now obtain the following characterization for a minor-minimal non-Z-Rayleigh matroid analogous to Proposition 3.2 in [15] for minon-minimal non-B-Rayleigh matroids.

Corollary 42. *Let $M(E, \rho)$ be a minor-minimal matroid that is not Z-Rayleigh. Then, for all $e, f \in E$ such that $\Delta Z\{e, f\}(M, q; \mathbf{y}) < 0$ for some q in the interval $0 < q \leq 1$ and $\mathbf{y} > 0$, the set $\{e, f\}$ is independent and closed in both M and M^*.*

Proof. From Proposition 31-1, we know that if M is a minor-minimal non-Z-Rayleigh matroid, then M is both simple and cosimple. Hence $\{e, f\}$ is independent in both M and M^*.

Let $e, f \in E$ be such that $\Delta Z\{e, f\}(M, q; \mathbf{y}) < 0$ for some q in the interval $0 < q \leq 1$ and $\mathbf{y} > 0$. To show that $\{e, f\}$ is closed in M and M^*, suppose for a contradiction that there exists an element $g \in E \setminus \{e, f\}$ such that $\{e, f, g\}$ is a circuit or a cocircuit in M. However, applying Corollaries 41 and 39, we note that this contradicts our choice of e and f. Thus $\{e, f\}$ must be closed in M and M^*. □

4.3 Rank-three matroids

When considering matroids of rank three, Lemma 38 gives us the following useful reduction lemma (see [15, Lemma 3.3] for an analogous result on B-Rayleigh matroids).

Corollary 43. *Let $M(E, \rho)$ be a rank-three simple and cosimple matroid with distinct elements $e, f, g \in E$, and let $\overline{E} = E \setminus \{e, f, g\}$. If for all $R \subseteq \overline{E}$, the sets $\{e\}$ and $\{f\}$ are $R \cup \{g\}$-submodular in every minor $N \in \mathcal{MF}(M, \overline{E} \setminus R)$, then $\Delta Z\{e, f\}(M, q; \mathbf{y}) \gg \Delta Z\{e, f\}(M \setminus g, q; \mathbf{y})$ for all q in the interval $0 < q \leq 1$.*

Proof. From Lemma 38, we have

$$\Delta Z\{e, f\}(M, q; \mathbf{y}) \gg \Delta Z\{e, f\}(M \setminus g, q; \mathbf{y}) + y_g^2 \Delta Z\{e, f\}(M/g, q; \mathbf{y}),$$

whenever $0 < q \leq 1$. Additionally, when M is of rank three, we know M/g is of rank two, and so by Corollary 29 and Proposition 35, $\Delta Z\{e, f\}(M/g, q; \mathbf{y}) \gg 0$ whenever $0 < q \leq 1$. □

Figure 3: The Fano and non-Fano matroids

We are now ready to show that some important rank three matroids are Z-Rayleigh.

Theorem 44. *The matroids W_3, $\mathcal{W}_3$, F_7 and F_7^- are Z-Rayleigh.*

Proof. We look at each of the matroids individually.
Case $M = W_3$: Using symmetry in Figure 1, it is enough to check $\Delta Z\{1,2\}(M, q; \mathbf{y})$ and $\Delta Z\{1,4\}(M, q; \mathbf{y})$ are non-negative whenever $0 < q \leq 1$ and $\mathbf{y} > 0$.

Since $\{1, 2, 3\}$ is a circuit, from Corollaries 41 and 43, we have $\Delta Z\{1,2\}(M, q; \mathbf{y}) \gg \Delta Z\{1,2\}(M \setminus 3, q; \mathbf{y}) \gg 0$ whenever $0 < q \leq 1$, as $M \setminus 3 \in \mathcal{ESM}$ by Proposition 28.

From Proposition 7 and Lemma 33, the only monomial whose coefficient can be non-negative in $\Delta Z\{1,4\}(M, q; \mathbf{y})$ for some q in the interval $0 < q \leq 1$ is $y_2y_3y_5y_6$. Hence, we can write

$$\Delta Z\{1,4\}(M, q; \mathbf{y}) = A_1(q)y_2y_3y_5y_6 + A_2(q)y_2^2y_5^2 + A_3(q)y_3^2y_6^2 + \Delta_1(\mathbf{y}), \tag{23}$$

where $A_1(q)$, $A_2(q)$ and $A_3(q)$ are the coefficients of the monomials $y_2y_3y_5y_6$, $y_2^2y_5^2$ and $y_3^2y_6^2$ in $\Delta Z\{1,4\}(M, q; \mathbf{y})$ respectively, and $\Delta_1(\mathbf{y}) \gg 0$ whenever $0 < q \leq 1$.

Using the identity map between columns two and three of Table 1 as the bijection in Lemma 32, we obtain

$$A_1(q) = 2(q^{-3} - q^{-4}) + 3(q^{-3} - q^{-2}) + (q^{-2} - q^{-1}). \tag{24}$$

Also, using Lemma 32, it is straightforward to get

$$A_2(q) = A_3(q) = q^{-4} - q^{-3}. \tag{25}$$

Using (24) and (25) in (23), we get

$$\Delta Z\{1,4\}(M, q; \mathbf{y}) = (q^{-4} - q^{-3})(y_2y_5 - y_3y_6)^2 + \\ \big(3(q^{-3} - q^{-2}) + (q^{-2} - q^{-1})\big)y_2y_3y_5y_6 + \Delta_1(\mathbf{y}),$$

which is non-negative whenever $0 < q \leq 1$ and $\mathbf{y} > 0$. Hence, W_3 is Z-Rayleigh.
Case $M = \mathcal{W}_3$: Using symmetry in Figure 1, it is enough to check $\Delta Z\{1,2\}(M, q; \mathbf{y})$, $\Delta Z\{2,6\}(M, q; \mathbf{y})$ and $\Delta Z\{1,4\}(M, q; \mathbf{y})$ are non-negative whenever $0 < q \leq 1$ and $\mathbf{y} > 0$.

As $\{1,2,3\}$ is a circuit, from Corollaries 41 and 43, we get $\Delta Z\{1,2\}(M,q;\mathbf{y}) \gg \Delta Z\{1,2\}(M\setminus 3,q;\mathbf{y}) \gg 0$ because $M\setminus 3 \in \mathcal{ESM}$ by Proposition 28.

Also $\{1,2,6\}$ is a cocircuit in M, and again from Corollaries 41 and 43, we get $\Delta Z\{2,6\}(M,q;\mathbf{y}) \gg \Delta Z\{2,6\}(M\setminus 1,q;\mathbf{y}) \gg 0$ whenever $0<q\le 1$, as $M\setminus 1 \in \mathcal{ESM}$ by Proposition 28.

Lastly, from Proposition 7 and Lemma 33, the only monomial whose coefficient can be non-negative in $\Delta Z\{1,4\}(M,q;\mathbf{y})$ is $y_2y_3y_5y_6$. Hence,

$$\Delta Z\{1,4\}(M,q;\mathbf{y}) = A_4(q)y_2y_3y_5y_6 + A_5(q)y_2^2y_5^2 + A_6(q)y_3^2y_6^2 + \Delta_2(\mathbf{y}), \tag{26}$$

where $A_4(q)$, $A_5(q)$ and $A_6(q)$ are the coefficients of the monomials $y_2y_3y_5y_6$, $y_2^2y_5^2$ and $y_3^2y_6^2$ in $\Delta Z\{1,4\}(M,q;\mathbf{y})$ respectively, and $\Delta_2(\mathbf{y}) \gg 0$ whenever $0<q\le 1$.

Using the identity map between columns four and five in Table 1 as the bijection in Lemma 32, we get

$$A_4(q) = (q^{-3}-q^{-4}) + 3(q^{-3}-q^{-2}) + (q^{-2}-q^{-1}). \tag{27}$$

Again using Lemma 32 we can get,

$$A_5(q) = A_6(q) = q^{-4}-q^{-3}. \tag{28}$$

Using (27) and (28) in (26), we find

$$\begin{aligned}\Delta Z\{1,4\}(M,q;\mathbf{y}) = (q^{-4}-q^{-3})(y_2y_5-y_3y_6)^2 + \\ \big((q^{-4}-q^{-3}) + 3(q^{-3}-q^{-2}) + (q^{-2}-q^{-1})\big)y_2y_3y_5y_6 + \Delta_2(\mathbf{y}),\end{aligned}$$

which is non-negative whenever $0<q\le 1$ and $\mathbf{y}>0$. Thus $\mathcal{W}_3$ is Z-Rayleigh.

Case $M = F_7$: Using symmetry in Figure 3, it is enough to check $\Delta Z\{1,4\}(M,q;\mathbf{y})$ is non-negative whenever $0<q\le 1$ and $\mathbf{y}>0$. Since $\{1,4,7\}$ is a circuit, from Corollaries 41 and 43, $\Delta Z\{1,4\}(M,q;\mathbf{y}) \gg \Delta Z\{1,4\}(M\setminus 7,q;\mathbf{y})$. But $M\setminus 7 = W_3$, and we showed in a previous case $\Delta Z\{1,4\}(M\setminus 7,q;\mathbf{y}) \ge 0$ whenever $0<q\le 1$ and $\mathbf{y}>0$. Hence, F_7 is Z-Rayleigh.

Case $M = F_7^-$: Using symmetry in Figure 3, it is enough to check $\Delta Z\{1,2\}(M,q;\mathbf{y})$ and $\Delta Z\{1,4\}(M,q;\mathbf{y})$ are non-negative whenever $0<q\le 1$ and $\mathbf{y}>0$.

Since $\{1,2,3\}$ is a circuit, from Corollaries 41 and 43, we know $\Delta Z\{1,2\}(M,q;\mathbf{y}) \gg \Delta Z\{1,2\}(M\setminus 3,q;\mathbf{y})$. However, $M\setminus 3$ is isomorphic to $\mathcal{W}_3$, and hence from a previous case, $\Delta Z\{1,2\}(M,q;\mathbf{y})$ is non-negative whenever $0<q\le 1$ and $\mathbf{y}>0$.

For $\Delta Z\{1,4\}(M,q;\mathbf{y})$, we first show that $\{1\}$ and $\{4\}$ are $R\cup\{7\}$-submodular in N for all $R\subseteq \overline{E}$ and minors $N \in \mathcal{MF}(M,\overline{E}\setminus R)$, where $\overline{E} = \{2,3,5,6\}$. First, it can be computationally checked that $\{1\}$ and $\{4\}$ are $\{2,3,5,6,7\}$-submodular in F_7^- [8, Appendix B]. Also $\{1,4,5\}$ is a cocircuit in $M\setminus 2$. Hence, from Lemma 40, $\{1\}$ and $\{4\}$ are $\{3,5,6,7\}$-submodular in $M\setminus 2$. A similar argument shows that for each $e\in\overline{E}$, the sets $\{1\}$ and $\{4\}$ are $R\cup\{7\}$-submodular in $M\setminus e$, where $R = \overline{E}\setminus\{e\}$. Since for every $e\in\overline{E}$, M/e is a rank-two matroid, by Corollary 29, $\{1\}$ and $\{4\}$ are also $R\cup\{7\}$-submodular in M/e, where $R=\overline{E}\setminus\{e\}$. Together with Proposition 7, these observations imply that $\{1\}$ and $\{4\}$ are $R\cup\{7\}$-submodular in N for all $R\subseteq\overline{E}$ and $N\in\mathcal{MF}(M,\overline{E}\setminus R)$.

Hence, using Corollary 43, $\Delta Z\{1,4\}(M,q;\mathbf{y}) \gg \Delta Z\{1,4\}(M \setminus 7,q;\mathbf{y})$ for all q in the interval $0 < q \leq 1$. But $M \setminus 7 = W_3$, and thus we get $\Delta Z\{1,4\}(M,q;\mathbf{y}) \geq 0$ whenever $0 < q \leq 1$ and $\mathbf{y} > 0$ from a previous case, and F_7^- is Z-Rayleigh. $\square$

From Proposition 31-1, we know that the class of Z-Rayleigh matroids is closed under duality, and we get the following corollary.

Corollary 45. *The matroids F_7^* and $(F_7^-)^*$ are Z-Rayleigh.*

5 Summary and open problems

We introduced R-submodularity as an extension of rank submodularity of matroids. While some instances of R-submodularity are known for all matroids, we also showed that there exists a class of well-behaved matroids with respect to R-submodularity, namely the class of extended submodular matroids, $\mathcal{ESM}$. Also the class of binary extended submodular matroids is identical to series-parallel networks. Yet a complete characterization of extended submodular matroids remains unresolved and is our first open problem. A positive step towards resolving this problem will be to find example matroids (if any) that are not extended submodular and without a W_3 or a $\mathcal{W}_3$ minor. The related question of whether the class $\mathcal{ESM}$ is closed under 2-sums is also unresolved.

As discussed in Section 4, the R-submodularity property has important consequences for the Rayleigh properties of matroids, and deserves further study. We suggest that R-submodularity can be a particularly effective tool in establishing the Rayleigh property for cases where the Rayleigh difference polynomials are expected to have only non-negative coefficients. As an example, we offer the following conjecture for graphs.

Conjecture 46. *Let $M(G)$ be the cycle matroid of a graph G with edge set E. If $e,f \in E$ are two distinct incident edges in G then for all $R \subseteq E \setminus \{e,f\}$, the sets $\{e\}$ and $\{f\}$ are R-submodular in $M(G)$.*

Since the class of graphic matroids is minor closed and the incidence property is preserved in minors, if true, this conjecture will show that $\Delta Z\{e,f\}(M,q;\mathbf{y}) \gg 0$ for all q in $0 < q \leq 1$ when M is graphic and e,f are incident edges of its graph representation. We note that it is a consequence of Corollary 42 that a minor-minimal counterexample of Conjecture 46 cannot have triangles including the edges e and f, and thus is not a complete graph.

We also illustrate in Theorem 44 that even in cases where $\Delta Z\{e,f\}(M,q;\mathbf{y})$ contains a few negative coefficients for some q in the interval $0 < q \leq 1$, R-submodularity can significantly reduce the computation required to verify if $\Delta Z\{e,f\}(M,q;\mathbf{y}) \geq 0$ whenever $0 < q \leq 1$ and $\mathbf{y} > 0$. Based on our empirical results with a few different rank-three matroids, we also pose the following open problem that is a significant strengthening of Theorem 1.1 in [15].

Conjecture 47. *Every matroid of rank or corank at most three is Z-Rayleigh.*

Lastly, an important unanswered question in the study of the various Rayleigh properties of matroids is if their respective classes are different from one another. For example, while it is likely there are matroids that are B-Rayleigh but not Z-Rayleigh, finding an example of such a matroid still remains open.

Acknowledgements

I am thankful to Alan Sokal for engaging discussions on the multivariate Tutte polynomial and the various Rayleigh properties of matroids that stimulated much of the research discussed in this paper. He also pointed out some relevant previous results, conjectured a stronger form of Proposition 28 and offered helpful advice on improving the notations. I would also like to thank Steve Noble for helpful insights on R-submodularity, and Graham Farr and David Wagner for advice and critical feedback on early drafts of this paper.

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