A Note on the Critical Group of a Line Graph

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Abstract

This note answers a question posed by Levine in [3]. The main result is Theorem 1 which shows that under certain circumstances a critical group of a directed graph is the quotient of a critical group of its directed line graph.

1 Introduction

Let G be a finite multidigraph with vertices V and edges E. Loops are allowed in G, and we make no connectivity assumptions. Each edge $e \in E$ has a tail e^- and a target e^+. Let $\mathbb{Z}V$ and $\mathbb{Z}E$ be the free abelian groups on V and E, respectively. The *Laplacian*[1] of G is the $\mathbb{Z}$-linear mapping $\Delta_G : \mathbb{Z}V \to \mathbb{Z}V$ determined by $\Delta_G(v) = \sum_{(v,u)\in E}(u - v)$ for $v \in V$. Given $w_* \in V$, define

$$\phi = \phi_{G,w_*} : \mathbb{Z}V \to \mathbb{Z}V$$
$$v \mapsto \begin{cases} \Delta_G(v) & \text{if } v \neq w_*, \\ w_* & \text{if } v = w_*. \end{cases}$$

The *critical group* for G with respect to w_* is the cokernel of ϕ:

$$K(G, w_*) := \operatorname{cok} \phi.$$

[1] The mapping $\Lambda\colon \mathbb{Z}^V \to \mathbb{Z}^V$ defined by $\Lambda(f)(v) = \sum_{(v,u)\in E}(f(v) - f(u))$ for $v \in V$ is often called the Laplacian of G. It is the negative $\mathbb{Z}$-dual (i.e., the transpose) of Δ_G.

The *line graph*, $\mathcal{L}G$, for G is the multidigraph whose vertices are the edges of G and whose edges are (e, f) with $e^+ = f^-$. As with G, we have the Laplacian $\Delta_{\mathcal{L}G}$ and the critical group $K(\mathcal{L}G, e_*) := \operatorname{cok} \phi_{\mathcal{L}G, e_*}$ for each $e_* \in E$.

If every vertex of G has a directed path to w_* then $K(G, w_*)$ is called the *sandpile group* for G with sink w_*. A *directed spanning tree* of G rooted at w_* is a directed subgraph containing all of the vertices of G, having no directed cycles, and for which w_* has out-degree 0 and every other vertex has out-degree 1. Let $\kappa(G, w_*)$ denote the number of directed spanning trees rooted at w_*. It is a well-known consequence of the matrix-tree theorem that the number of elements of the sandpile group with sink w_* is equal to $\kappa(G, w_*)$. For a basic exposition of the properties of the sandpile group, the reader is referred to [2].

In his paper, [3], Levine shows that if $e_* = (w_*, v_*)$, then $\kappa(G, w_*)$ divides $\kappa(\mathcal{L}G, e_*)$ under the hypotheses of our Theorem 1. This leads him to ask the natural question as to whether $K(G, w_*)$ is a subgroup or quotient of $K(\mathcal{L}G, e_*)$. In this note, we answer this question affirmatively by demonstrating a surjection $K(\mathcal{L}G, e_*) \to K(G, w_*)$. Further, in the case in which the out-degree of each vertex of G is a fixed integer k, we show the kernel of this surjection is the k-torsion subgroup of $K(\mathcal{L}G, e_*)$. These results appear as Theorem 1 and may be seen as analogous to Theorem 1.2 of [3]. In [3], partially for convenience, some assumptions are made about the connectivity of G which are not made in this note. For related work on the critical group of a line graph for an undirected graph, see [1].

2 Results

Fix $e_* = (w_*, v_*) \in E$. Define the modified target mapping

$$\begin{aligned} \tau\colon \mathbb{Z}E &\to \mathbb{Z}V \\ e &\mapsto \begin{cases} e^+ & \text{if } e \neq e_*, \\ 0 & \text{if } e = e_*. \end{cases} \end{aligned}$$

Also define

$$\begin{aligned} \rho\colon \mathbb{Z}E &\to \mathbb{Z}V \\ e &\mapsto \begin{cases} \Delta_G(w_*) - v_* - w_* + e^+ & \text{if } e \neq e_*, \\ 0 & \text{if } e = e_*. \end{cases} \end{aligned}$$

Let k be a positive integer. The graph G is *k-out-regular* if the out-degree of each of its vertices is k.

Theorem 1 *If* $\operatorname{indeg}(v) \geq 1$ *for all* $v \in V$ *and* $\operatorname{indeg}(v_*) \geq 2$*, then*

$$\rho\colon \mathbb{Z}E \to \mathbb{Z}V$$

descends to a surjective homomorphism $\overline{\rho}\colon K(\mathcal{L}G, e_*) \to K(G, w_*)$.

Moreover, if G is k-out-regular, the kernel of $\overline{\rho}$ is the k-torsion subgroup of $K(\mathcal{L}G, e_*)$.

Proof. Let $\rho_0 \colon \mathbb{Z}V \to \mathbb{Z}V$ be the homomorphism defined on vertices $v \in V$ by

$$\rho_0(v) := \Delta_G(w_*) - v_* - w_* + v$$

so that $\rho = \rho_0 \circ \tau$. The mapping ρ_0 is an isomorphism, its inverse being itself:

$$\begin{aligned}
\rho_0^2(v) &= \rho_0(\Delta_G(w_*) - v_* - w_* + v) \\
&= \sum_{e^- = w_*} (\rho_0(e^+) - \rho_0(w_*)) - \rho_0(v_*) - \rho_0(w_*) + \rho_0(v) \\
&= \Delta_G(w_*) - \rho_0(v_*) - \rho_0(w_*) + \rho_0(v) \\
&= v.
\end{aligned}$$

Let $\psi \colon \mathbb{Z}V \to \mathbb{Z}V$ be the homomorphism defined on vertices $v \in V$ by

$$\psi(v) := \begin{cases} \Delta_G(v) & \text{if } v \neq w_*, \\ \Delta_G(w_*) - v_* & \text{if } v = w_*. \end{cases}$$

Let ϕ_G and $\phi_{\mathcal{L}G}$ denote ϕ_{G,w_*} and $\phi_{\mathcal{L}G,e_*}$, respectively. We claim the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{Z}E & \xrightarrow{\phi_{\mathcal{L}G}} & \mathbb{Z}E \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\
\mathbb{Z}V & \xrightarrow{\psi} & \mathbb{Z}V \\
\| & & \downarrow{\scriptstyle\rho_0} \\
\mathbb{Z}V & \xrightarrow{\phi_G} & \mathbb{Z}V.
\end{array}$$

To prove commutativity of the top square of the diagram, first suppose $e \neq e_*$. Then

$$\tau(\phi_{\mathcal{L}G}(e)) = \tau(\Delta_{\mathcal{L}G}(e)) = \tau\left(\sum_{f^- = e^+} (f - e)\right).$$

If $e \neq e_*$ and $e^+ \neq w_*$, then

$$\tau\left(\sum_{f^- = e^+} (f - e)\right) = \sum_{f^- = e^+} (f^+ - e^+) = \Delta_G(e^+) = \psi(\tau(e)).$$

On the other hand, if $e \neq e_*$ and $e^+ = w_*$, then

$$\begin{aligned}
\tau\left(\sum_{f^- = e^+} (f - e)\right) &= \sum_{f^- = e^+, f \neq e_*} (f^+ - e^+) + \tau(e_* - e) \\
&= \sum_{f^- = e^+, f \neq e_*} (f^+ - e^+) - w_* \\
&= \Delta_G(w_*) - v_* = \psi(\tau(e)).
\end{aligned}$$

Therefore, $\tau(\phi_{\mathcal{L}G}(e)) = \psi(\tau(e))$ holds if $e \neq e_*$. Moreover, the equality still holds if $e = e_*$ since $\tau(e_*) = 0$. Hence, the top square of the diagram commutes.

To prove that the bottom square of the diagram commutes, there are two cases. First, if $v \neq w_*$, then

$$\rho_0(\psi(v)) = \sum_{(v,u)\in E} (\rho_0(u) - \rho_0(v)) = \sum_{(v,u)\in E} (u - v) = \Delta_G(v) = \phi_G(v).$$

Second, if $v = w_*$, then

$$\rho_0(\psi(v)) = \rho_0(\Delta_G(w_*) - v_*) = \Delta_G(w_*) - \rho_0(v_*) = w_* = \phi_G(v).$$

From the commutativity of the diagram, the cokernel of ψ is isomorphic to $K(G, w_*)$, and $\rho = \rho_0 \circ \tau$ descends to a homomorphism $\overline{\rho}\colon K(\mathcal{L}G, e_*) \to K(G, w_*)$ as claimed. The hypothesis on the in-degrees of the vertices assures that τ, hence $\overline{\rho}$, is surjective.

Now suppose that G, hence $\mathcal{L}$G, is k-out-regular. This part of our proof is an adaptation of that given for Theorem 1.2 in [3]. Since ρ_0 is an isomorphism, it suffices to show that the kernel of the induced map, $\overline{\tau}\colon K(\mathcal{L}G, e_*) \to \operatorname{cok}\psi$, has kernel equal to the k-torsion of $K(\mathcal{L}G, e_*)$. To this end, define the homomorphism $\sigma\colon \mathbb{Z}V \to \mathbb{Z}E$, given on vertices $v \in V$ by

$$\sigma(v) := \sum_{e^- = v} e.$$

We claim that the image of $\sigma \circ \psi$ lies in the image of $\phi_{\mathcal{L}G}$, so that σ induces a map, $\overline{\sigma}$, between $\operatorname{cok}\psi$ and $K(\mathcal{L}G, e_*)$. To see this, first note that for $v \in V$,

$$\begin{aligned}
\sigma(\Delta_G(v)) &= \sigma\left(\sum_{e^-=v} e^+ - kv\right) \\
&= \sum_{e^-=v} \sum_{f^-=e^+} f - k \sum_{e^-=v} e \\
&= \sum_{e^-=v} \Delta_{\mathcal{L}G}(e)
\end{aligned}$$

Therefore, for $v \neq w_*$, it follows that $\sigma(\psi(v))$ is in the image of $\phi_{\mathcal{L}G}$. On the other hand, using the calculation just made,

$$\begin{aligned}
\sigma(\Delta_G(w_*) - v_*) &= \sum_{e^-=w_*} \Delta_{\mathcal{L}G}(e) - \sum_{f^-=v^*} f \\
&= \sum_{e^-=w_*} \Delta_{\mathcal{L}G}(e) - \left(\sum_{f^-=v^*} f - k\, e_* + k\, e_*\right) \\
&= \sum_{e^-=w_*} \Delta_{\mathcal{L}G}(e) - \Delta_{\mathcal{L}G}(e_*) - k\, e_* \\
&= \sum_{e^-=w_*, e\neq e_*} \Delta_{\mathcal{L}G}(e) - k\, e_*,
\end{aligned}$$

which is also in the image of $\phi_{\mathcal{L}G}$.

We have established the mappings

$$\operatorname{cok}\psi \underset{\overline{\tau}}{\overset{\overline{\sigma}}{\rightleftarrows}} K(\mathcal{L}G, e_*)\ .$$

For $e \neq e_*$,

$$\overline{\sigma}(\overline{\tau}(e)) = \sum_{f^- = e^+} f = \Delta_{\mathcal{L}G}(e) + k\, e = k\, e \in K(\mathcal{L}G, e_*).$$

Thus, the kernel of $\overline{\tau}$ is contained in the k-torsion of $K(\mathcal{L}G, e_*)$, and to show equality it suffices to show that $\overline{\sigma}$ is injective.

The case where $k = 1$ is trivial since there are no G satisfying the hypotheses: if G is 1-out-regular and $\operatorname{indeg}(v) \geq 1$ for all $v \in V$, then $\operatorname{indeg}(v) = 1$ for all $v \in V$, including v_*. So suppose that $k > 1$ and that $\eta = \sum_{v \in V} a_v\, v$ is in the kernel of $\overline{\sigma}$. We then have

$$\sigma(\eta) = \sum_{v \in V} \sum_{e^- = v} a_v\, e = \sum_{e \neq e_*} b_e\, \Delta_{\mathcal{L}G}(e) + c\, e_* \tag{1}$$

for some integers b_e and c. Comparing coefficients in (1) gives

$$a_{e^-} = \sum_{f^+ = e^-, f \neq e_*} b_f - k\, b_e \qquad \text{for } e \neq e_*. \tag{2}$$

Define

$$F(v) = \frac{1}{k} \left(\sum_{f^+ = v, f \neq e_*} b_f - a_v \right).$$

From (2),

$$F(e^-) = b_e \qquad \text{for } e \neq e_*. \tag{3}$$

Since $k > 1$, for each vertex v, we can choose an edge $e_v \neq e_*$ with $e_v^- = v$. By (2) and (3), for all $v \in V$,

$$a_v = \sum_{f^+ = v, f \neq e_*} b_f - k\, b_{e_v} = \sum_{f^+ = v, f \neq e_*} F(f^-) - k\, F(v).$$

Therefore, as an element of $\operatorname{cok}\psi$,

$$\begin{aligned}
\eta = \sum a_v v &= \sum_{e \neq e_*} F\left(e^-\right) e^+ - \sum_{v \in V} kF(v)v \\
&= \sum_{v \in V, v \neq w_*} F(v) \left(\sum_{e^- = v} e^+ - kv \right) + F(w_*) \left(\sum_{e^- = w_*, e \neq e_*} e^+ - kw_* \right) \\
&= \sum_{v \in V, v \neq w_*} F(v)\Delta_G(v) + F(w_*)(\Delta_G(w_*) - v_*) \\
&= 0,
\end{aligned}$$

which shows that $\overline{\sigma}$ is injective. □

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