# Partition statistics for cubic partition pairs

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**Abstract**

In this brief note, we give two partition statistics which explain the following partition congruences:

$$\begin{aligned} b(5n+4) &\equiv 0 \pmod 5, \\ b(7n+a) &\equiv 0 \pmod 7, \text{if } a = 2,\ 3,\ 4, \text{ or } 6. \end{aligned}$$

Here, $b(n)$ is the number of 4-color partitions of $n$ with colors $r$, $y$, $o$, and $b$ subject to the restriction that the colors $o$ and $b$ appear only in even parts.

## 1 Introduction

In a series of papers ([3], [4], [5]) H.-C. Chan studied congruence properties of a certain kind of partition function $a(n)$, which arises from Ramanujan's cubic continued fraction. This partition function $a(n)$ is defined by

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Here and in the sequel, we will use the following standard $q$-series notation:

$$(a;q)_\infty := \prod_{n=1}^{\infty} (1 - aq^{n-1}),\ |q| < 1.$$

Since a partition congruence for $a(n)$ is deduced from the equation for Ramanujan's cubic continued fraction

$$\nu(q) := \frac{q^{1/3}}{1} \, + \, \frac{q+q^2}{1} \, + \, \frac{q^2+q^4}{1} \, + \cdots, \qquad |q| < 1,$$

(see [3] for the details.), $a(n)$ is known as the number of cubic partitions. After Chan's works, many analogous partition functions have been studied. In particular, H. Zhao and Z. Zhong [7] investigated congruences for the partition function

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}.$$

Here $b(n)$ counts the number of partition pairs $(\lambda_1, \lambda_2)$, where $\lambda_1$ and $\lambda_2$ are cubic partition such that the sum of parts in $\lambda_1$ and $\lambda_2$ equals to $n$. In this sense, we will call $b(n)$ the number of cubic partition pairs. We can interpret $b(n)$ as the number of 4-color partitions of $n$ with colors $r$, $y$, $o$, and $b$ subject to the restriction that the colors $o$ and $b$ appear only in even parts. For example, there are 7 such partitions as follows:

$$2_r, \quad 2_y, \quad 2_o, \quad 2_b, \quad 1_r + 1_r. \quad 1_r + 1_y, \quad 1_y + 1_y.$$

Once congruence properties of a certain type of partition function are known, it is natural to seek a partition statistic to give a combinatorial explanation of the known congruences. In this paper, we will give two partition statistics for the cubic partitions to explain the following congruences [7, Theorem 3.2]:

$$b(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$
$$b(7n+a) \equiv 0 \pmod{7}, \text{if } a = 2,\, 3,\, 4, \text{ or } 6, \tag{1.2}$$

for all $n \geq 0$.

Our first partition statistic is a rank analog for $b(n)$, which explains the first congruence (1.1). For a given cubic partition pair $\lambda$, we define the cubic partition pair rank as

$$\#\lambda_r^e - \#\lambda_y^e + 2\#\lambda_o^e - 2\#\lambda_b^e,$$

where $\#\lambda_*^e$ is the number of even parts in $\lambda$ with color $*$. We define $N^*(m,n)$ as the number of cubic partition pairs of $n$ with cubic partition pair rank $= m$. Then, from the fact that $\frac{1}{(zq;q)_\infty} = \sum_{m=0}^{\infty} p(m,n) z^m q^n$, where $p(m,n)$ denotes the number of partitions of $n$ with the number of parts equals $m$, we can see that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N^*(m,n) z^m q^n = \frac{1}{(q;q^2)_\infty^2 (zq^2, z^{-1}q^2, z^2q^2, z^{-2}q^2; q^2)_\infty}, \tag{1.3}$$

where $(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty$. We are now ready to state our first result.

**Theorem 1.** *Let $N^*(m,A,n)$ be the number of cubic partition pairs of $n$ with cubic partition rank $\equiv m \pmod{A}$. Then, for all $n \geq 0$ and $0 \leq i \leq j \leq 4$,*

$$N^*(i,5,5n+4) \equiv N^*(j,5,n) \pmod{5}.$$

Since $b(n) = \sum_{m=0}^{4} N^*(m,5,n)$, the next corollary is immediate.

**Corollary 2.** *For all* $n \geq 0$,

$$b(5n+4) \equiv 0 \pmod 5.$$

To explain the second congruences (1.2), we define the following function $M^*(m,n)$.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M^*(m,n)z^m q^n = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2 (zq^2.z^{-1}q^2, z^2q^2, z^{-2}q^2, z^3q^2, z^{-3}q^2; q^2)_\infty}. \qquad (1.4)$$

The statistic $M^*(m,n)$ is a weighted count of extended cubic partition pairs. Since a combinatorial meaning of $M^*(m,n)$ is quite long, we will give it in the following section. Now we state our second theorem.

**Theorem 3.** *Let* $M^*(m,A,n)$ *be defined by*

$$\sum_{i \equiv m \pmod A} M^*(i,n).$$

*Then, for all* $n \geq 0$ *and* $0 \leq i \leq j \leq 6$,

$$M^*(i,7,7n+a) \equiv M^*(j,7,7n+a) \pmod 7,$$

*if* $a = 2,\ 3,\ 4$ *or* $6$.

Since $b(n) = \sum_{i=0}^{6} M^*(i,7,n)$, the following corollary is also immediate.

**Corollary 4.** *For all* $n \geq 0$,

$$b(7n+a) \equiv 0 \pmod 7, \text{ if } a = 2,\ 3,\ 4, \text{ or } 6.$$

## 2 combinatorial interpretation of $M^*(m,n)$

To give a combinatorial explanation of the famous Ramanujan partition congruences G.E. Andrews and F.G. Garvan [1] introduced the crank of a partition. For a given partition $\lambda$, the crank $c(\lambda)$ of a partition is defined as

$$c(\lambda) := \begin{cases} \ell(\lambda), & \text{if } r = 0, \\ \omega(\lambda) - r, & \text{if } r \geq 1, \end{cases}$$

where $r$ is the number of 1's in $\lambda$, $\omega(\lambda)$ is the number of parts in $\lambda$ that are strictly larger than $r$ and $\ell(\lambda)$ is the largest part in $\lambda$. If we let $M(m,n)$ be the number of ordinary partitions of $n$ with crank $m$, Andrews and Garvan showed that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n = (1-z)q + \frac{(q;q)_\infty}{(zq, z^{-1}q; q)_\infty}. \qquad (2.1)$$

By extending the set of partitions $\mathcal{P}$ to a new set $\mathcal{P}^*$ by adding two additional copies of the partition 1, say $1^*$ and $1^{**}$, we see that (for details, consult [6, Section 2])

$$\frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty} = \sum_{\lambda\in\mathcal{P}^*} wt(\lambda) z^{c^*(\lambda)} q^{\sigma^*(\lambda)}, \tag{2.2}$$

where $wt(\lambda)$, $c^*(\lambda)$, and $\sigma^*(\lambda)$ are defined as follows. We define the weight $wt(\lambda)$ for $\lambda \in \mathcal{P}^*$ by

$$wt(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \mathcal{P},\ \lambda = 1^*, \text{ or } \lambda = 1^{**}, \\ -1, & \text{if } \lambda = 1, \end{cases}$$

and we also define the extended crank $c^*(\lambda)$ by

$$c^*(\lambda) = \begin{cases} c(\lambda), & \text{if } \lambda \in \mathcal{P}, \\ 0, & \text{if } \lambda = 1, \\ 1, & \text{if } \lambda = 1^*, \\ -1, & \text{if } \lambda = 1^{**}. \end{cases}$$

Finally, we define the extended sum parts function $\sigma^*(\lambda)$ in the following way:

$$\sigma^*(\lambda) = \begin{cases} \sigma(\lambda), & \text{if } \lambda \in \mathcal{P}, \\ 1, & \text{otherwise}, \end{cases}$$

where $\sigma(\lambda)$ is the sum of parts in the partition $\lambda$.

We now extend the definition of cubic partition pairs. Note that we may identify a cubic partition pair of $n$ with an element of

$$(\lambda_r, \lambda_y, \lambda_o, \lambda_b) \in \mathcal{P}\times\mathcal{P}\times\mathcal{P}\times\mathcal{P}$$

such that $\sigma(\lambda_r)+\sigma(\lambda_y)+2\,\sigma(\lambda_o)+2\,\sigma(\lambda_b) = n$. We extend the definition of cubic partition pairs in a natural way by defining them to be elements of $\mathcal{P}\times\mathcal{P}\times\mathcal{P}^*\times\mathcal{P}^*$. For the set of extended cubic partition pairs we define the sum of parts function $\sigma_{cp}$, weight function $wt_{cp}$, and crank function $c_{cp}$ as follows: For $\lambda = (\lambda_r, \lambda_y, \lambda_o, \lambda_b) \in \mathcal{P}\times\mathcal{P}\times\mathcal{P}^*\times\mathcal{P}^*$, we define

$$\begin{aligned} \sigma_{cp}(\lambda) &= \sigma(\lambda_r) + \sigma(\lambda_y) + 2\,\sigma^*(\lambda_o) + 2\,\sigma^*(\lambda_b), \\ wt_{cp}(\lambda) &= wt(\lambda_o)\cdot wt(\lambda_b), \\ c_{cp}(\lambda) &= \#\lambda_r^e - \#\lambda_y^e + 2\,c^*(\lambda_o) + 3\,c^*(\lambda_b). \end{aligned}$$

We finally define $M^*(m,n)$ as the number of extended cubic partition pairs of $n$ with crank $m$ counted according to the weight $wt_{cp}$ as follows:

$$M^*(m,n) = \sum_{\substack{\lambda\in\mathcal{P}\times\mathcal{P}\times\mathcal{P}^*\times\mathcal{P}^* \\ c_{cp}=m,\sigma_{cp}=n}} wt_{cp}(\lambda).$$

In light of (2.2) and the definition of $M^*(m,n)$, we can deduce that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M^*(m,n)z^m q^n$$
$$= \sum_{(\lambda_r,\lambda_y)\in\mathcal{P}\times\mathcal{P}} z^{(\#\lambda_r^e-\#\lambda_y^e)} q^{\sigma(\lambda_r)+\sigma(\lambda_y)} \sum_{\lambda_o\in\mathcal{P}^*} wt(\lambda_o) z^{2c^*(\lambda_o)} q^{2\sigma^*(\lambda)} \sum_{\lambda_b\in\mathcal{P}^*} wt(\lambda_b) z^{3c^*(\lambda_b)} q^{2\sigma^*(\lambda_b)}$$
$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2 (zq^2.z^{-1}q^2, z^2q^2, z^{-2}q^2, z^3q^2, z^{-3}q^2; q^2)_\infty},$$

as desired.

# 3 Proofs of Theorems

In this section, we will give proofs for Theorems 1 and 3.

*Proof of Theorem 1.* First, recall that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N^*(m,n)z^m q^n = \frac{1}{(q;q^2)_\infty^2 (zq^2, z^{-1}q^2, z^2q^2, z^{-2}q^2; q^2)_\infty},$$

By setting $z=\zeta=\exp(\frac{2\pi i}{5})$, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N^*(m,n)\zeta^m q^n &= \sum_{n=0}^{\infty}\sum_{m=0}^{4} N^*(m,5,n)\zeta^m q^n \\ &= \frac{1}{(q;q^2)_\infty^2, (\zeta q^2, \zeta^{-1}q^2, \zeta^2 q^2, \zeta^{-2}q^2; q^2)_\infty},\end{aligned} \tag{3.1}$$

where $N^*(m,5,n)$ is the number of cubic partition pairs of $n$ with cubic partition rank $\equiv m \pmod 5$. Now,

$$\begin{aligned}&\frac{1}{(q;q^2)_\infty^2, (\zeta q^2, \zeta^{-1}q^2, \zeta^2 q^2, \zeta^{-2}q^2; q^2)_\infty} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2 (q^{10};q^{10})_\infty} \\ &= \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^{10};q^{10})_\infty} \\ &\equiv \frac{(q^2;q^2)_\infty^3 (q;q)_\infty^3}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty} \pmod 5 \\ &\equiv \frac{\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2}}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty} \pmod 5.\end{aligned} \tag{3.2}$$

Here we used the binomial theorem to see that $(1-x)^5 \equiv 1-x^5 \pmod 5$ for the first equivalence and applied the Jacobi's identity [2, Theorem 1.3.9] for the final equivalence.

From (3.2), we can see that the coefficient of $q^{5n+4}$ in (3.1) is a multiple of 5 for each natural number $n$. Since $1+\zeta+\cdots+\zeta^4$ is the minimal polynomial in $\mathbb{Z}[\zeta]$, we deduce the theorem. □

Before turning to the proof of Theorem 3, we need the following lemma.

**Lemma 5** (Corollary 1.3.21 of [2]). *If* $|q|<1$*, then*

$$\sum_{-\infty}^{\infty}(6n+1)q^{n^2+n}=(q^2;q^2)_\infty^3(q^2;q^4)_\infty^2.$$

Now we are ready to give the proof of Theorem 3.

*Proof of Theorem 3.* Note that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}M^*(m,n)\xi^m q^n &= \sum_{n=0}^{\infty}\sum_{m=0}^{6}M^*(m,7,n)\xi^m q^n \\ &= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2,(\xi q^2,\xi^{-1}q^2,\xi^2q^2,\xi^{-2}q^2,\xi^3q^2,\xi^{-3}q^2;q^2)_\infty},\end{aligned} \tag{3.3}$$

where $\xi$ is now a primitive seventh root of unity. Therefore, we deduce that

$$\begin{aligned}&\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2,(\xi q^2,\xi^{-1}q^2,\xi^2q^2,\xi^{-2}q^2,\xi^3q^2,\xi^{-3}q^2;q^2)_\infty}\\ &=\frac{(q^2;q^2)_\infty^3}{(q;q^2)_\infty^2(q^{14};q^{14})_\infty}\\ &=\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^{14};q^{14})_\infty}\\ &=\frac{(q^2;q^2)_\infty^7(q;q^2)_\infty^2(q;q)_i^3}{(q^7;q^7)_\infty(q^{14};q^{14})_\infty}\\ &\equiv\frac{(q;q)_\infty^3}{(-q;q)_\infty^2(q^7;q^7)_\infty} \pmod 7\\ &\equiv\frac{\sum_{n=-\infty}^{\infty}(6n+1)q^{n(3n+1)/2}}{(q^7;q^7)_\infty} \pmod 7,\end{aligned}$$

where we used the binomial theorem for the first equivalence and Lemma 5 for the last equivalence. Proceeding as in the proof of Theorem 1, we can conclude Theorem 3. □

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