Asymptotic enumeration of labelled graphs by genus

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Submitted: Mar 14, 2010; Accepted: Dec 17, 2010; Published: Jan 12, 2011
Mathematics Subject Classification: 05A16, 05C30

Abstract

We obtain asymptotic formulas for the number of rooted 2-connected and 3-connected surface maps on an orientable surface of genus g with respect to vertices and edges simultaneously. We also derive the bivariate version of the large face-width result for random 3-connected maps. These results are then used to derive asymptotic formulas for the number of labelled k-connected graphs of orientable genus g for $k \leq 3$.

1 Introduction

The exact enumeration of various types of maps on the sphere (or, equivalently, the plane) was carried out by Tutte [26, 27, 28] in the 1960s via his device of rooting. (Terms in this paragraph are defined below.) Building on this, explicit results were obtained for some maps on low genus surfaces, e.g., as done by Arqués on the torus [1]. Beginning in the 1980s, Tutte's approach was used for the asymptotic enumeration of maps on general surfaces [3, 12, 4]. A matrix integral approach was initiated by ′t Hooft (see [21]). The enumerative study of graphs embeddable in surfaces began much more recently. Asymptotic results on the sphere were obtained in [8, 22, 20] and cruder asymptotics for general surfaces in [22]. In this paper, we will derive asymptotic formulas for the number of labelled graphs on an orientable surface of genus g for the following families: 3-connected and 2-connected with respect to vertices and edges, and 1-connected and all with respect to vertices. Along the way we also derive results for 2-connected and 3-connected maps with respect to vertices and edges. The result for all graphs as well as various parameters for these graphs was announced earlier by Noy [24] and appears in [15].

*Research supported by NSERC

Definition 1 (Maps and Embeddable Graphs) *A* map $\mathcal{M}$ *is a connected graph* $\mathcal{G}$ *embedded in a surface* Σ *(a closed 2-manifold) such that all components of* $\Sigma - \mathcal{G}$ *are simply connected regions, which are called* faces*.* $\mathcal{G}$ *is called the underlying graph of* $\mathcal{M}$*, and is denoted by* $G(\mathcal{M})$*. Loops and multiple edges are allowed in* $\mathcal{G}$*.*

- *A map is* rooted *if an edge is distinguished together with a direction on the edge and a side of the edge.*

In this paper, all maps are rooted and unlabeled.

- *A graph without loops or multiple edges is* simple.
- *A graph* $\mathcal{G}$ *is* embeddable in a surface *if it can be drawn on the surface without edges crossing.*
- *A graph has (orientable) genus* g *if it is embeddable in an orientable surface of genus* g *and none of smaller genus.*

Definition 2 (Generating Functions for Maps and Graphs) *Let* $\hat{M}_g(n,m;k)$ *be the number of (rooted, unlabeled)* k*-connected maps with* n *vertices and* m *edges, on an orientable surface of genus* g*. Let* $G_g(n,m;k)$ *be the number of (vertex) labelled, simple,* k*-connected graphs with* n *vertices and* m *edges, which are embeddable in an orientable surface of genus* g*. Let* $G_g(n;k) = \sum_m G_g(n,m;k)$*, the number of labelled, simple,* k*-connected graphs with* n *vertices. Let*

$$\hat{M}_{g,k}(x,y) = \sum_{n,m} \hat{M}_g(n,m;k)x^n y^m \quad \text{and} \quad G_{g,k}(x,y) = \sum_{n,m} G_g(n,m;k)(x^n/n!)y^m.$$

In the following theorem, $\rho(r)$ and $A_g(r)$ have the same definition in terms of r, but the definition of r varies.

Theorem 1 (Maps on Surfaces) *Define*

$$\begin{aligned} \rho(r) &= \frac{r^3(2+r)}{1+2r}, \\ A_g(r) &= \frac{1}{2\sqrt{\pi}} \frac{r^6(2+r)^{3/2}}{(1+2r)^2} \left(\frac{12(1+r)^3(1+2r)^4}{r^{12}(2+r)^5} \right)^{g/2} t_g, \end{aligned}$$

where t_g *is the map asymptotics constant defined in [3]. For* $k = 1, 2, 3$*, there are algebraic functions* $r = r_k(m/n)$*,* $C_k(r)$*, and* $\eta_k(r)$ *such that for any fixed* $\epsilon > 0$ *and fixed genus* g

$$\hat{M}_g(n,m;k) \;\sim\; C_k(r)A_g(r)(2+r)^{(k-1)(5g-3)/2}\, n^{5g/2-3}\rho(r)^{-n}\eta_k(r)^{-m},$$

uniformly as $n, m \to \infty$ *such that* $r_k(m/n) \in [\epsilon, 1/\epsilon]$*. The relevant functions are as follows:*

(i) $r = r_1(m/n)$ *satisfies* $\dfrac{(1+r)(1+r+r^2)}{r^2(2+r)} = \dfrac{m}{n}$, $\eta_1(r) = \dfrac{1+2r}{4(1+r+r^2)^2}$ *and*

$$C_1(r) \;=\; (2+r)\sqrt{\frac{1+r+r^2}{(1+2r)(4+7r+4r^2)}};$$

(ii) $r_2(m/n) = \dfrac{1}{m/n-1}$, $\eta_2(r) = \dfrac{4}{(1+2r)(2+r)^2}$ *and* $C_2(r) = \dfrac{1}{\sqrt{(1+2r)(2+r)}}$;

(iii) $r_3(m/n) = \dfrac{3-m/n}{2(m/n)-3}$, $\eta_3(r) = \dfrac{3}{4r(2+r)}$, *and* $C_3(r) = \dfrac{1}{\sqrt{r(2+r)^3}}$.

Theorem 2 (Embeddable Graphs) *For the ranges of m and n considered here, the number of graphs embeddable in an orientable surface of genus g is asymptotic to the number of such graphs of orientable genus g.*

(i) **(3-connected)** *For any fixed $\epsilon > 0$ and genus g,*

$$\frac{G_g(n,m;3)}{n!} \;\sim\; \frac{\hat{M}_g(n,m;3)}{4m}$$

uniformly as $n, m \to \infty$ such that $\frac{m}{n} \in [(3/2)+\epsilon, 3-\epsilon]$.

(ii) **(2-connected)** *Let $\alpha(t), \beta(t)$, $\rho_2(t)$, $\lambda_2(t)$, $\mu(t)$ and $\sigma(t)$ be functions of t defined in Section 6 (see also [8]). Let*

$$B_g(t) = \left(\frac{8}{9(1+t)(1-t)^6}\left(\frac{\beta(t)}{\alpha(t)}\right)^{5/2}\right)^{g-1}.$$

Fix $\epsilon > 0$ and genus g. Let $0 < t < 1$ satisfy $\mu(t) = m/n$. Then

$$\frac{G_g(n,m;2)}{n!} \;\sim\; \frac{B_g(t)t_g}{4\sigma(t)\sqrt{2\pi}} n^{5g/2-4}\rho_2(t)^{-n}\lambda_2(t)^{-m}$$

uniformly as $n, m \to \infty$ such that $m/n \in [1+\epsilon, 3-\epsilon]$.

(iii) **(vertices only)** *For $0 \le k \le 3$ and fixed g, there are positive constants x_k, α_k and β_k such that*

$$\frac{G_g(n;k)}{n!} \;\sim\; \alpha_k \beta_k^g\, t_g\, n^{5g/2-7/2} x_k^{-n},$$

where

$x_3 \doteq 0.04751$,	$x_2 \doteq 0.03819$,	$x_1 \doteq 0.03673$,	$x_0 = x_1$,
$\beta_3 \doteq 1.48590 \cdot 10^5$,	$\beta_2 \doteq 7.61501 \cdot 10^4$.	$\beta_1 \doteq 6.87242 \cdot 10^4$,	$\beta_0 = \beta_1$,
$\alpha_3 = \frac{1}{4\beta_3}$,	$\alpha_2 = \frac{1}{4\beta_2}$,	$\alpha_1 = \frac{1}{4\beta_1}$,	$\alpha_0 \doteq 3.77651 \cdot 10^{-6}$.

More accurate values of these constants can be computed by using the formulas in those sections where the theorem is proved.

Remark (t_g). It is known [18] that

$$t_g \;=\; \frac{-a_g}{2^{g-2}\Gamma\left(\frac{5g-1}{2}\right)}$$

where $a_0 = 1$ and, for $g > 0$,

$$a_g \;=\; \frac{(5g-4)(5g-6)}{48} a_{g-1} - \frac{1}{2}\sum_{h=1}^{g-1} a_h a_{g-h}. \tag{1}$$

Hence all the numbers in Theorems 1 and 2 can be computed efficiently to any desired accuracy for any given g and r.

Remark (Sharp Concentration). As noted in Comment 4 of Section 3, our methods for obtaining bivariate results show that the number of edges is sharply concentrated. To find the mean number of edges asymptotically, set $\eta_k(r) = 1$ in Theorem 1, $\eta_3(r) = 1$ in Theorem 2(i), and $\lambda_2(t) = 1$ in Theorem 2(ii). For r the asymptotic value of the mean is then the value of m for which $r(m/n)$ has that value of r; for t it is simply $\mu(t)n$.

The paper proceeds as follows.

Section 2 Maps on a fixed surface were enumerated in [4] with respect to vertices and faces. We convert this result to quadrangulations and then obtain results for other types of quadrangulations.

Section 3: We recall a local limit theorem and discuss some analytic methods used in subsequent sections.

Section 4: We then apply the techniques in [12] and [7] to obtain asymptotics for generating functions for k-connected maps, proving Theorem 1. The calculations for $A_g(r)$ are postponed to Section 9.

Section 5: Applying the techniques in [5], we show that almost all 3-connected maps have large face-width when counted by vertices and edges. Hence almost all 3-connected graphs of genus g have a unique embedding [25]. This leads to Theorem 2 for 3-connected graphs.

Section 6: Using the construction of 2-connected graphs from 3-connected graphs and polygons as in [8] we obtain Theorem 2 for 2-connected graphs.

Sections 7 and 8: We obtain Theorem 2 for 1-connected graphs from the 2-connected result and for all graphs from 1-connected by methods like those in [20].

Section 9: We derive the expression for $A_g(r)$ in terms of t_g.

Section 10: We make some comments on the number of labeled graphs of a given nonorientable genus.

2 Enumerating Quadrangulations

We begin with some definitions:

Definition 3 (Cycles) *A* cycle *in a map is a simple closed curve consisting of edges of the map.*

- *A cycle is called a k-*cycle *if it contains k edges.*
- *A cycle is called* separating *if deleting it separates the underlying graph.*
- *A cycle is called* facial *if it bounds a face of the map.*
- *A cycle is called* contractible *if it is homotopic to a point, otherwise it is called* non-contractible.
- *A contractible cycle in a nonplanar map separates the map into a planar piece and a nonplanar piece. The planar piece is called the* interior *of the cycle and we also say that the cycle* contains *anything that is in its interior. Since we usually draw a planar map such that the root face is the unbounded face, we define the interior of a cycle in a planar map to be the piece which does not contain the root face.*
- *A 2-cycle or 4-cycle is called* maximal *(*minimal*) if it is contractible and its interior is maximal (minimal).*

Definition 4 (Widths) *The* edge-width *of a map $\mathcal{M}$, written* $\mathrm{ew}(\mathcal{M})$*, is the length of a shortest non-contractible cycle of $\mathcal{M}$. The* face-width *(also called* representativity *of $\mathcal{M}$, written* $\mathrm{fw}(\mathcal{M})$*, is the minimum of $|G(\mathcal{M}) \cap C|$ taken over all non-contractible closed curves C on the surface.*

Definition 5 (Quadrangulations) *A quadrangulation is a map all of whose faces have degree 4.*

- *A* bipartite quadrangulation *is a quadrangulation whose underlying graph is bipartite. (All quadrangulations on the sphere are bipartite, but those on other surfaces need not be.)*
- *A quadrangulation is* near-simple *if it has no contractible 2-cycles and no contractible nonfacial 4-cycles.*
- *A quadrangulation is* simple *if it has no 2-cycles and all 4-cycles are facial.*

The following lemma, contained in [12] and [7], connects maps with bipartite quadrangulations.

Lemma 1 *By convention, we bicolor a bipartite quadrangulation so that the head of the root edge is black. There is a bijection ϕ between rooted maps and rooted bipartite quadrangulations, such that the following hold.*

(a) $\mathrm{fw}(\mathcal{M}) = \mathrm{ew}(\phi(\mathcal{M}))/2$.

(b) $\mathcal{M}$ *has* n *vertices and* m *edges if and only if* $\phi(\mathcal{M})$ *has* n *black vertices and* m *faces.*

(c) $\phi(\mathcal{M})$ *has no 2-cycle implies* $\mathcal{M}$ *is 2-connected which implies* $\phi(\mathcal{M})$ *has no contractible 2-cycle.*

(d) $\phi(\mathcal{M})$ *is simple implies* $\mathcal{M}$ *is 3-connected which implies* $\phi(\mathcal{M})$ *is near-simple.*

In this section we enumerate quadrangulations with no contractible 2-cycles and near-simple quadrangulations. Except that black vertices were not counted, this is done in [7]. In what follows, we reproduce that argument nearly verbatim, adding a second variable to count black vertices.

We define the generating functions $Q_g(x,y)$, $\hat{Q}_g(x,y)$ and $Q_g^\star(x,y)$ as follows.

$$Q_g(x,y) = \sum_{i,j\geq 1} Q(i,j;g)x^{i-1}y^j$$

where $Q(i,j;g)$ is the number of (rooted, bicolored) quadrangulations with i black vertices and j faces on an orientable surface of genus g. Similarly define $\hat{Q}_g(x,y)$ for quadrangulations without contractible 2-cycles and $Q_g^\star(x,y)$ for near-simple quadrangulations.

By Lemma 1, we have

$$Q_g(x,y) = x^{-1}\hat{M}_{g,1}(x,y) - \delta_{0,g}, \tag{2}$$

where the Kronecker delta occurs because of the convention that counts a single vertex as a map on the sphere.

In [4] the generating function $\hat{M}_g(u,v)$ counts maps by vertices and faces. Thus

$$\hat{M}_{g,1}(x,y) \;=\; y^{2g-2}\hat{M}_g(xy,y). \tag{3}$$

It is known [1, 4] that $\hat{M}_0(xy,y) = \frac{rs}{(1+r+s)^3}$ where $r(x,y)$ and $s(x,y)$ are power series uniquely determined by

$$x \;=\; \frac{r(2+r)}{s(2+s)} \quad\text{and}\quad y \;=\; \frac{s(2+s)}{4(1+r+s)^2}. \tag{4}$$

Thus

$$Q_0(x,y) \;=\; \frac{4(1+r+s)}{(2+r)(2+s)} - 1 \;=\; \frac{2r+2s-rs}{(2+r)(2+s)}, \tag{5}$$

and

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{s(2+s)(1+r+rs)}{2(1-rs)}, & \frac{\partial r}{\partial y} &= \frac{2r(2+r)(1+s)(1+r+s)^3}{s(2+s)(1-rs)}, \\ \frac{\partial s}{\partial x} &= \frac{s^2(2+s)^2}{2(1-rs)}, & \frac{\partial s}{\partial y} &= \frac{2(1+r)(1+r+s)^3}{1-rs}. \end{aligned} \tag{6}$$

Throughout the rest of the paper, we use $N(\epsilon)$ to denote the set

$$N(\epsilon) \;=\; \{re^{i\theta} : \epsilon\leq r\leq 1/\epsilon,\ |\theta|\leq\epsilon\}.$$

Theorem 3 (Quadrangulations) *Fix $g>0$ and let $q(x,y)$ be any of $Q_g(x,y)$, $\hat{Q}_g(x,y)$ and $Q_g^\star(x,y)$. The values of x and y are parameterized by r and s in the following manner.*

(i) For all (bipartite) quadrangulations ($q=Q_g$), x and y are given by (4).

(ii) For no contractible 2-cycles ($q=\hat{Q}_g$), x is given by (4) and $y=\dfrac{4s}{(2+s)(2+r)^2}$.

(iii) For near simple ($q=Q_g^\star$), x is given by (4) and $y=\dfrac{s(4-rs)}{4(2+r)}$.

The following are true.

(a) The function $q(x,y)$ is a rational function of r and s and hence an algebraic function of x and y.

(b) If r and s are positive reals such that $rs=1$, then (x,y) is in the singular set of $q(x,y)$.

(c) If (x',y') is another singularity of q, then either $|x'|>x$ or $|y'|>y$.

(d) Let $\rho(r)=\frac{r^3(2+r)}{1+2r}$, the value of x on the singular curve $rs=1$, and let y be its value on the singular curve at r. Fix $\epsilon>0$ and $g>0$. Uniformly for $r\in N(\epsilon)$

$$xq(x,y) \;\sim\; C(r)\left(1-\frac{x}{\rho(r)}\right)^{(3-5g)/2} \tag{7}$$

as $x\to\rho(r)$,

$$C(r)=\begin{cases}\sqrt{\dfrac{\pi}{3(1+r)}}\,\dfrac{(1+r+r^2)A_g(r)\,\Gamma\left(\frac{5g-3}{2}\right)}{r^2} & \text{for } q=Q_g,\\[2ex] \sqrt{\dfrac{\pi}{3(1+r)}}\,\dfrac{A_g(r)\,\Gamma\left(\frac{5g-3}{2}\right)}{r}(2+r)^{(5g-3)/2} & \text{for } q=\hat{Q}_g,\\[2ex] \sqrt{\dfrac{3\pi}{1+r}}\,\dfrac{A_g(r)\,\Gamma\left(\frac{5g-3}{2}\right)}{(2+r)(1+2r)}(2+r)^{5g-3} & \text{for } q=Q_g^\star,\end{cases}$$

and some function $A_g(r)$ whose value is determined in Section 9.

Proof: Theorem 3 of [4] shows that $\hat{M}_g(x,y)$ of that paper is a rational function of r and s and hence algebraic when $g>0$. (The theorem contains the misprint $9>0$ which should be $g>0$.) Use (2)–(5) to establish (a) for Q_g.

We now derive equations for $\hat{Q}$ and $Q^\star$ based on Q. This will easily imply (a) for $\hat{Q}$ and $Q^\star$.

It is important to note that, in any quadrangulation, all maximal 2-cycles have disjoint interiors, and that, in any nonplanar quadrangulation without contractible 2-cycles, all maximal 4-cycles have disjoint interiors. (This is simpler than the planar case [23, p. 260].) Therefore, we can close all maximal 2-cycles in quadrangulations to obtain quadrangulations without contractible 2-cycles and remove the interior of each maximal contractible 4-cycle to obtain near-simple quadrangulations. The process can be reversed and used to construct quadrangulations from near-simple quadrangulations.

Enumerating $\hat{Q}_g(x,y)$: The following argument is essentially from [7], by paying extra attention to the number of black vertices. All quadrangulations of genus $g > 0$ can be divided into two classes according as the root face lies in the interior of some contractible 2-cycle or not.

For any quadrangulation in the first class, let C be the minimal contractible 2-cycle containing the root face. Cutting along C, filling holes with disks and closing those two 2-cycles, we obtain a general quadrangulation of genus g and a planar quadrangulation with a distinguished edge. Taking the latter quadrangulation and cutting along all its maximal 2-cycles and closing as before gives a quadrangulation without contractible 2-cycles, together with a set of planar quadrangulations extracted from within the maximal 2-cycles. Remembering that y counts faces and that the number of edges is twice the number of faces, it follows that the generating function for the first class is

$$\frac{Q_g(x,y)}{1+Q_0(x,y)}\,\frac{2\hat{y}\,\partial\,\hat{Q}_0(x,\hat{y})}{\partial\hat{y}},$$

where

$$\hat{y} \;=\; y(1+Q_0(x,y))^2 \;=\; \frac{4s}{(2+s)(2+r)^2}. \tag{8}$$

For any quadrangulation in the second class, closing all maximal contractible 2-cycles gives quadrangulations without contractible 2-cycles. Thus the generating function for this class is $\hat{Q}_g(x,\hat{y})$. For the planar case, only the second class applies and so

$$\hat{Q}_0(x,\hat{y}) = Q_0(x,y). \tag{9}$$

Combining the two classes when $g > 0$, we have

$$Q_g(x,y) \;=\; \hat{Q}_g(x,\hat{y}) + \frac{Q_g(x,y)}{1+Q_0(x,y)}\,\frac{2\hat{y}\,\partial\,\hat{Q}_0(x,\hat{y})}{\partial\hat{y}}.$$

It follows that

$$\hat{Q}_g(x,\hat{y}) \;=\; \left(1-\frac{2\hat{y}}{1+Q_0(x,y)}\,\frac{\partial\,\hat{Q}_0(x,\hat{y})}{\partial\hat{y}}\right)Q_g(x,y) \tag{10}$$

for $g > 0$. Note that

$$1-\frac{2\hat{y}}{1+Q_0(x,y)}\,\frac{\partial\,\hat{Q}_0(x,\hat{y})}{\partial\hat{y}} \;=\; \frac{1}{1+r+s} \tag{11}$$

and so is bounded on the singular curve when r is near the positive real axis.

Enumerating $Q_g^\star(x,y)$: We now use a similar argument to derive $Q_g^\star(x,y^\star)$ from $\hat{Q}_g(x,\hat{y})$ when $g>0$. For any quadrangulation without contractible 2-cycles, let C be the maximal contractible 4-cycle containing the root face. Cutting along C and filling holes with disks, we obtain

1. a planar quadrangulation which has no 2-cycles and has a distinguished face other than the root face, and
2. a quadrangulation of genus g which, after the removal of the interiors of all maximal 4-cycles, gives a near-simple quadrangulation.

Note that

$$y^\star \;=\; \frac{\hat{Q}_0(x,\hat{y})-x\hat{y}-\hat{y}}{x\hat{y}} \;=\; \frac{s(4-rs)}{4(2+r)} \tag{12}$$

enumerates planar quadrangulations having at least one interior face and having no 2-cycles such that x marks the number of black vertices minus 2 and $\hat{y}$ marks the number of non-root faces. It follows from the construction that

$$\frac{\hat{Q}_g(x,\hat{y})}{\hat{y}} \;=\; \frac{Q_g^\star(x,y^\star)}{y^\star}\frac{\partial y^\star}{\partial \hat{y}}.$$

which gives

$$Q_g^\star(x,y^\star) \;=\; \frac{y^\star}{\partial y^\star/\partial\hat{y}}\frac{\hat{Q}_g(x,\hat{y})}{\hat{y}} \;=\; \frac{4-rs}{(2+s)(2+r)(1+r+s)}Q_g(x,y). \tag{13}$$

This completes the proof of Theorem 3(a).

Singularities: These must arise from poles due to the vanishing of the denominator of $q(x,y)$ or from branch points caused by problems with the Jacobian $\frac{\partial(x,y)}{\partial(r,s)}$. For the former, it can be seen from (10) and (13) that either $1+r+s=0$ or $2+r=0$ or $2+s=0$. By (4), each of these implies that either x or y vanishes or is infinite, which do not matter since the radius of convergence is nonzero and finite. Using the formulas in Theorem 3, one can compute Jacobians. One finds that the only singularity that matters is $1-rs=0$.

Conclusion (c) follows for Q from [4]. We now consider $\hat{Q}$ and $Q^\star$. Suppose

- x and y are positive reals on the singular curve,
- x' and y' are on the singular curve,
- $|x'|\le x$ and $|y'|\le y$.

To prove (c) it suffice to show that $x'=x$ and $y'=y$. Since we are dealing with generating functions with nonnegative coefficients, no singularity can be nearer the origin the that at the positive reals. Hence $|x'| = x$ and $|y'| = y$. As was done in [10], one easily verifies that on the singular curve $rs = 1$ one has

$$16x'y'^2\Big(16(y'+1)(x'y'+1)+2\Big) \;=\; 27 \tag{14}$$

for $Q^\star$. Taking absolute values in this equation one easily finds that $|y'+1| = |y+1|$ and $|x'y'+1| = |xy+1|$. Thus $y' = y$ and $x'y' = xy$ and we are done. For $\hat{Q}$, a look at the equations for x and y on the singular curve shows that we need only replace y' in (14) with $(3/4)(y'/4x')^{1/3}$ and argue as for $Q^\star$. This completes the proof of (c).

Asymptotics: We now turn to (d). The case $q = Q_g$ is contained implicitly in [4] for some function $A_g(r)$.

We now use (10) to derive the singular expansion for $\hat{Q}_g(\hat{x},\hat{y})$ at $\hat{x} = \rho(r)$ where r is determined by $\hat{y} = \eta_2(r)$. It is important to note that, with $\hat{y}$ fixed, (8) defines y as an analytic function in $x = \hat{x}$. Thus in (7), with $q(x,y) = Q_g(x,y)$, we should treat r as a function in y and consequently as a function in x. Using implicit differentiation, we obtain from (8) and (6) that

$$\frac{dy}{dx} = -\frac{\partial\hat{y}/\partial x}{\partial\hat{y}/\partial y} \;=\; -\frac{(\partial\hat{y}/\partial r)(\partial r/\partial x)+(\partial\hat{y}/\partial s)(\partial s/\partial x)}{(\partial\hat{y}/\partial r)(\partial r/\partial y)+(\partial\hat{y}/\partial s)(\partial s/\partial y)} \;=\; \frac{-s^2(2+s)^2}{4(2+r)(1+r+s)^3}. \tag{15}$$

Hence

$$\frac{d}{dx}\left(1-\frac{x}{\rho(r)}\right) \;=\; \frac{-1}{\rho(r)}+\frac{x}{\rho^2(r)}\frac{d\rho}{dx} \;=\; \frac{-1}{\rho(r)}+\frac{x}{\rho^2(r)}\frac{\rho'(r)}{\eta_1'(r)}\frac{dy}{dx}.$$

Using (15) and the expressions for $\rho(r)$ and $\eta_1(r)$ given in Theorem 1, we obtain

$$\frac{d}{dx}\left(1-\frac{x}{\rho(r)}\right)\bigg|_{x=\rho(r),s=1/r} \;=\; \frac{-1}{\rho(r)(2+r)},$$

and hence

$$1-\frac{x}{\rho(r)} \;\sim\; \frac{-1}{\rho(r)(2+r)}(\hat{x}-\rho(r)) \;=\; \frac{1}{2+r}\left(1-\frac{\hat{x}}{\rho(r)}\right).$$

Substituting this into (7), we obtain

$$\left(1-\frac{x}{\rho(r)}\right)^{(3-5g)/2} \sim\; (2+r)^{(5g-3)/2}\left(1-\frac{\hat{x}}{\rho(r)}\right)^{(3-5g)/2},$$

as $\hat{x}\to\rho(r)$ for each fixed $\hat{y}$. The factor (11) can simply be evaluated at $s = 1/r$ since it converges to a constant. This establishes (7) for $\hat{Q}_g(\hat{x},\hat{y})$.

Expansion (7) for $Q_g^\star(x^\star,y^\star)$ can be obtained similarly using (13). We note that fixing $y^\star$ defines y, and hence $\rho(r)$, as a function of $x = x^\star$. Using (12) and (6), we obtain

$$1-\frac{x}{\rho(r)} \;\sim\; \frac{1}{(2+r)^2}\left(1-\frac{x^\star}{\rho(r)}\right),$$

as $x^\star\to\rho(r)$ for each fixed $y^\star$.

This completes the proof of the theorem, except for the formula for $A_g(r)$ which will be derived in Section 9.

3 Some Technical Lemmas

The following lemma is the essential tool for our asymptotic estimates. It is based on the case $d=1$ of [9, Theorem 2], from which it follows immediately.

Lemma 2 *Suppose that $a_{n,k} \geq 0$. Define $a_n(v) = \sum_k a_{n,k} v^k$ and $a(u,v) = \sum_n a_n(v) u^n$. Let $R(c)$ be the radius of convergence of $a(u,c)$. Suppose that I is a closed subinterval of $(0,\infty)$ on which $0 < R < \infty$. For $v \in I$ define*

$$\mu(v) = \frac{-d\,\log \rho(v)}{d\,\log v}, \quad \sigma^2(v) = \frac{-d^2\,\log \rho(v)}{(d\,\log v)^2}, \quad K_n = \{n\mu(v) \mid v \in I\} \cap \mathbb{Z}$$

and $N(I,\delta) = \{z \mid |z| \in I \text{ and } |\arg z| < \delta\}$. Suppose there are $f(n)$, $g(v)$ and $\rho(v)$ such that in $N(I,\delta)$

(a) $a_n(v) \sim f(n)g(v)\rho(v)^{-n}$ uniformly as $n \to \infty$;

(b) $g(v)$ is uniformly continuous;

(c) $\rho(v) \neq 0$ has a uniformly continuous third derivative;

(d) $\sigma^2(v) > 0$ for $v > 0$.

Suppose also that

(e) $R(c) > R(|c|)$ whenever $c \neq |c| \in I$.

Then, as $n \to \infty$, we have, uniformly for $k \in K_n$,

$$a_{n,k} \;\sim\; \frac{a_n(v)v^{-k}}{\sqrt{2\pi n \sigma^2(v)}},$$

where $v \in I$ is given by $k/n = \mu(v)$.

Of course $|\rho(v)|$ is simply the radius of convergence $R(v)$ and $\rho(v) = R(v)$ when $v \in I$.

We now make some comments on applying this lemma. We will generally use these ideas without explicit mention.

Comment 1. We can simply apply the lemma directly. For example, we can apply it to (7) to obtain asymptotics. The only condition that is not immediate is the verification that $\sigma^2(v) > 0$ in condition (d). This is a straightforward but somewhat tedious calculation. Unless needed later, we omit the values of $\sigma^2(v)$ that we compute.

Comment 2. Consider adding and multiplying various $a(u,v)$, all with the same $\rho(v)$ (and hence $\mu(v)$) that satisfy the lemma. The result will be a function that again satisfies the lemma with the same $\rho(v)$.

To see this, note that the lemma is essentially a local limit theorem for random variables where $\Pr(X_n = k) = a_{n,k}v^k/a_n(v)$ and use [11, Lemma 5]. We also need the observation that multiplying $a(u,v)$ by functions with nonnegative coefficients and larger radii of convergence results in a function having the same $\rho(v)$ and so the lemma applies. In fact, it suffices to simply evaluate the new function at the singularity and multiply the resulting constant by $a(u, v)$.

Comment 3. Condition (a) will follow if $a(u, v)$ is algebraic and $a(u, s)$ has no other singularities on its circle of convergence when $s \in I$. In general, condition (a) is established using the "transfer theorem" [16, Sec. VI.3]. Thus, for example, Theorem 3(a,c) implies Lemma 2(a,e).

Comment 4. The values $n\mu(v)$ and $n\sigma^2(v)$ are asymptotic to the mean and variance of a random variable $X_n(v)$ with $\Pr(X_n(v)=k) = a_{n,k}v^k/a_n(v)$. Chebyshev's inequality then gives a sharp concentration result for $X_n(v)$ about its mean. When this is applied to maps or graphs with $v = 1$, it gives a sharp concentration for the edges about the mean. (The lemma is based on a local limit theorem, which could be used to give a sharper result.)

Since we will be bounding coefficients of generating functions, the following definition and lemma will be useful.

Definition 6 ($\tilde{O}$) *Let $A(x, y)$ and $B(x, y)$ be generating functions and let $B(x, y)$ have nonnegative coefficients. We write $A(x, y) = \tilde{O}(B(x, y))$ if there is a constant K such that*

$$\big|[x^iy^j]\, A(x, y)\big| \le K[x^iy^j]\, B(x, y) \quad \textit{for all } i, j.$$

Lemma 3 *Let $A(x, y)$, $B(x, y)$, $C(x, y)$, $D(x, y)$ and $H(x, y)$ be generating functions, and $C(x, y)$, $D(x, y)$ and $H(x, y)$ have nonnegative coefficients. Suppose further that $A(x, y) = \tilde{O}(C(x, y))$ and $B(x, y) = \tilde{O}(D(x, y))$. Then*

(i) **differentiation:** $A_x(x, y) = \tilde{O}(C_x(x, y))$ *and* $A_y(x, y) = \tilde{O}(C_y(x, y))$;

(ii) **integration:** $\displaystyle\int_0^x A(x, y)dx = \tilde{O}\left(\int_0^x C(x, y)dx\right)$ *and*
$\displaystyle\int_0^y A(x, y)dy = \tilde{O}\left(\int_0^y C(x, y)dy\right)$;

(iii) **product:** $A(x, y)B(x, y) = \tilde{O}(C(x, y)D(x, y))$;

(iv) **substitution:** $A(H(x, y), y) = \tilde{O}(C(H(x, y), y)$ *and*
$A(x, H(x, y)) = \tilde{O}(C(x, H(x, y))$
provided that the compositions as formal power series are well defined.

The proof follows immediately from the definition of $\tilde{O}$.

Obviously the definition of $\tilde{O}$ and Lemma 3 can be stated for any number of variables.

We want to apply Lemma 2 to $a(u, v) = A(u, v)+E(u, v)$ or $a(u, v) = A(u, v)-E(u, v)$ when A is a function and we know E only approximately. Of course, this cannot be done directly since derivatives are involved.

The lemma will apply to $A(u, v)$ for $v \in I$. We could attempt to estimate coefficients of $E(u, v)$ by some crude method, but this fails because the order of growth of $E(u, v)$ is not sufficiently smaller than that of $A(u,v)$. What we will have is that $E(u,v) = \tilde{O}(F(u,v))$ where F is a function built from functions to which the lemma applies and which have dominant singularities only where A has them. Thus both functions have the same $\rho(v)$. Furthermore, the function $f(n)$ for A grows faster than the $f(n)$ for F. This is enough to show that the coefficients of F are negligible compared to those of A because of Comment 2 above. We will use these ideas without explicit mention when considering error bounds.

4 Proof of Theorem 1

The value of $A_g(r)$ in this section is simply the value assumed in the proof of Theorem 3 in Section 2. The formula for $A_g(r)$ will be derived in Section 9.

For $g = 0$ we find it easier to verify that the formulas in Theorem 1 agree with known results. The $g = 0$ case for general maps will follow when we use [4] to evaluate $A_g(r)$ in Section 9. For maps with $i+1$ vertices and $j+1$ faces the number of 2-connected planar maps equals [14]

$$\frac{(2i+j-2)!\,(2j+i-2)!}{i!\,j!\,(2i-1)!\,(2j-1)!}$$

and the number 3-connected planar maps is asymptotic to

$$\frac{1}{3^5 ij}\binom{2i}{j+3}\binom{2j}{i+3}$$

uniformly as $\max(i,j) \to \infty$ [13]. The verification of Theorem 1 now requires only some straightforward estimates of factorials and the fact that $t_0 = \frac{2}{\sqrt{\pi}}$.

We now assume $g > 0$.

We derive the 1-connected case from Theorem 3. Lemma 1 tells us that $xQ_g(x,y)$ counts 1-connected maps by vertices and edges. Now apply Theorem 3 and Lemma 2. With $A_g(r)$ given by Theorem 3, it follows that

$$\hat{M}_g(n,m;1) \;\sim\; \frac{A_g(r)}{\sigma_1(r)\sqrt{2\pi}} n^{5g/2-3} \rho(r)^{-n} \eta_1(r)^{-m}$$

where

$$\begin{aligned} \frac{m}{n} &= \frac{-d\,\log\rho(r)}{d\,\log\eta_1(r)} = \frac{(1+r)(1+r+r^2)}{r^2(2+r)}, \\ \sigma_1^2(r) &= \frac{-d^2\,\log\rho(r)}{(d\,\log\eta_1(r))^2} = \frac{(4+7r+4r^2)(1+2r)(1+r+r^2)}{6r^4(2+r)^2(1+r)}. \end{aligned}$$

This gives Theorem 1(i). (Of course, we could also have cited [4], but we need the derivation from Theorem 3 so that we can evaluate $A_g(r)$ later.)

Our proof for 2- and 3-connected maps uses Lemma 1 in connection with Theorem 3 and Lemma 2. We obtain upper and lower bounds from Lemma 1(c,d). We show that Lemma 2 can be applied to both bounds and that the asymptotics are the same.

Upper bounds are provided by $\hat{Q}$ and $Q^\star$. These can be treated in the same manner as Theorem 1(i) was derived from Q. Let $E(x, y)$ be the errors in these upper bounds. We handle $E(x, y)$ as discussed at the end of Section 3, namely $E(x, y) = \tilde{O}(F(x, y))$ where F is well-behaved. We now turn to $F(x, y)$.

2-Connected maps: We bound the quadrangulations that have non-contractible 2-cycles and are counted by $\hat{Q}_g(x, y)$. The argument is essentially the same as that used in [7]. The only difference is that we keep track of both the number of faces and the number of black vertices.

We first study quadrangulations counted by $\hat{Q}_g(x, y)$ which contain a separating non-contractible cycle C of length 2.

Cutting through C gives two near-quadrangulations. After closing the resulting two 2-cycles, we obtain a rooted quadrangulation $\mathcal{Q}_1$ with a distinguished edge, which has genus $0 < j < g$, and another rooted quadrangulation $\mathcal{Q}_2$ with genus $g - j$. The quadrangulation $\mathcal{Q}_1$ may contain contractible 2-cycles which contain the distinguished edge d in its interior. Hence $\mathcal{Q}_1$ is decomposed into a rooted quadrangulation counted by $y\frac{\partial}{\partial y}\hat{Q}_j(x, y)$ and a sequence of rooted quadrangulations counted by $y\frac{\partial}{\partial y}\hat{Q}_0(x, y)$. Thus the generating function for $\mathcal{Q}_1$ is

$$\tilde{O}\left(x^{-1}\frac{\partial\hat{Q}_j(x, y)}{\partial y}\Big(1 - y\,\partial\hat{Q}_0(x, y)/\partial y\Big)^{-1}\right).$$

For convergence of $\sum(y\,\partial\hat{Q}_0(x, y)/\partial y))^k$ it suffices to show that $y\,\partial\hat{Q}_0(x, y)/\partial y < 1$ for positive x and y since it is a power series with nonnegative coefficients. Since

$$\frac{y\,\partial\hat{Q}_0(x, y)}{\partial y} = \frac{2(r + s)}{(2 + r)(2 + s)},$$

the result is immediate. Also note that this implies that $1 - y\,\partial\hat{Q}_0(x, y)/\partial y$ does not vanish for $|x| \le \rho(r)$.

Similarly the quadrangulation $\mathcal{Q}_2$ may contain contractible 2-cycles containing its root edge in its interior. So $\mathcal{Q}_2$ is decomposed into a rooted quadrangulation counted by $\hat{Q}_{g-j}(x, y)$ and a sequence of rooted quadrangulations counted by $y\frac{\partial}{\partial y}\hat{Q}_0(x, y)$. Hence the generating function of the quadrangulations with a separating non-contractible 2-cycle is bounded above coefficient-wise by

$$\sum_{j=1}^{g-1} x^{-1}\Big(1 - y\,\partial\hat{Q}_0(x, y)/\partial y\Big)^{-2}\,\frac{\partial\hat{Q}_j(x, y)}{\partial y}\hat{Q}_{g-j}(x, y), \tag{16}$$

which is algebraic with nonnegative coefficients.

Since $1 - y\,\partial\hat{Q}_0(x, y)/\partial y \neq 0$, the function given in (16) has only one singularity on the circle of convergence and near that singularity is $O((1 - x/\rho(r))^p)$ where

$$p = \left(\frac{3 - 5j}{2} - 1\right) + \frac{3 - 5(g - j)}{2} = \frac{3 - 5g}{2} + \frac{1}{2}. \tag{17}$$

Thus we can apply Lemma 2 to see that the error is negligible.

Next we consider quadrangulations counted by $\hat{Q}_g(x,y)$ which contain a non-separating non-contractible cycle C of length 2. Cutting through C gives a near-quadrangulation of genus $g-1$ with two 2-cycles. After closing the resulting two 2-cycles, we obtain a rooted quadrangulation $\mathcal{Q}$ with two distinguished edges. The quadrangulation $\mathcal{Q}$ may contain contractible 2-cycles which contain a distinguished edge in its interior. Hence $\mathcal{Q}$ is decomposed into a rooted quadrangulation counted by $y^2\frac{\partial^2\hat{Q}_{g-1}(x,y)}{(\partial y)^2}$ and two sequences of rooted quadrangulations counted by $y\frac{\partial\hat{Q}_0(x,y)}{\partial y}$. Hence the bound in this case is

$$\left(1 - y\partial\hat{Q}_0(x,y)/\partial y\right)^{-2} y^2\frac{\partial^2\hat{Q}_{g-1}(x,y)}{(\partial y)^2}.$$

Reasoning as in the previous paragraph, this gives a negligible contribution to the asymptotics.

Now Theorem 1(ii) follows from Lemma 1 and Theorem 3 using

$$\frac{m}{n} = \frac{-d\,\log\rho(r)}{d\,\log\eta_2(r)} = \frac{1+r}{r} \quad \text{and} \quad \sigma_2^2(r) = \frac{-d^2\,\log\rho(r)}{(d\,\log\eta_2(r))^2} = \frac{(2+r)(1+2r)}{6r^2(1+r)}.$$

Proof of Theorem 1(iii): We prove that almost all quadrangulations counted by $Q^\star(x,y)$ have no non-contractible cycles of length 2 or 4. The argument is similar to the one used above, and is identical to the one used in [7]. We note here

$$\frac{m}{n} = \frac{-d\,\log\rho(r)}{d\,\log\eta_3(r)} = \frac{3(1+r)}{1+2r} \quad \text{and} \quad \sigma_3^2(r) = \frac{-d^2\,\log\rho(r)}{(d\,\log\eta_3(r))^2} = \frac{3r(2+r)}{2(1+r)(1+2r)^2}.$$

5 Face Widths of 3-Connected Maps and Graphs

Robertson and Vitray [25] have shown that, if a 3-connected map $\mathcal{M}$ in a surface Σ_g of genus g has $\mathrm{fw}(M) > 2g+2$, then its underlying graph has a unique embedding in Σ_g and is not embeddable in a surface of lower genus.

Our goal is to prove Theorem 4 below. Then Theorem 2(i) follows from Theorem 1 by counting vertex-labeled, 3-connected, rooted maps. To obtain Theorem 2(iii) for 3-connected graphs, it suffices to use (7) with r chosen so that $y=1$; that is, $\eta_3(r)=1$. In other words, $r=\sqrt{7}/2-1$. This gives

$$x_3 = \rho(r) = \frac{7\sqrt{7}-17}{32} \doteq 0.04751.$$

By Comment 4 after Lemma 2, the number of edges is concentrated around its mean which is asymptotically $\frac{3(1+r)}{1+2r}n$.

Applying the "transfer theorem" [16, Sec. VI.3] to (7) and using Theorem 4, one obtains

$$\left(\frac{3(1+r)}{1+2r}n\right)\frac{G_g(n;3)}{n!} \;\sim\; \frac{C(r)n^{5(g-1)/2}}{4\,\Gamma\left(\frac{5g-3}{2}\right)}.$$

After some algebra we obtain Theorem 2(iii) for 3-connected graphs, with

$$\begin{aligned}
\beta_3 &= \frac{2\sqrt{3}\,(1+2r)^2(1+r)^{3/2}(2+r)^{5/2}}{r^6} \doteq 1.48590 \cdot 10^5, \\
\alpha_3 &= \frac{1}{4\beta_3} \doteq 1.68248 \cdot 10^{-6}.
\end{aligned}$$

Theorem 4 (Large Face Width) *Let $L_g(n,m;c)$ be the number of maps counted by $\hat{M}_g(n,m;3)$ that have face width at least c and let $L_g(x,y) = \sum_{n,m} L_g(n,m;c)x^n y^m$. Then, for fixed $g > 0$,*

$$L_g(x,y) = xQ_g^\star(x,y) + \tilde{O}(B_1(x,y)), \tag{18}$$

$$\frac{\partial G_{g,3}(x,y)}{\partial y} = \frac{x}{4y}Q_g^\star(x,y) + \tilde{O}(B_2(x,y)), \tag{19}$$

where every singularity of B_i is a singularity of $Q_g^\star$ and

$$B_i(x,y) = O\left((1-x/\rho(r))^{5(g-1)/2+1/2}\right) \quad \text{as} \quad x \to \rho(r)$$

for $y = \eta_3(r)$, uniformly for $r \in N(\epsilon)$.

We show that almost all simple quadrangulations have no non-contractible cycles of length less than any constant c. We need only consider cycles of length $2k$ where $c \geq 2k > 4$ since we may limit attention to simple quadrangulations. Let C be a non-contractible cycle of length $2k$ in a simple quadrangulation counted by $Q_g^\star(x,y)$. As in previous arguments, we consider separating and non-separating separately

Case 1. Suppose C is separating. Cutting through C and filling the two holes with disks, we obtain a rooted simple quadrangulation $\mathcal{Q}_1$ with a distinguished face of degree $2k$, which has genus $0 < j < g$, and another rooted simple near-quadrangulation $\mathcal{Q}_2$ with genus $g - j$ and root face degree $2k$. We may quadrangulate the faces of degree $2k$ by inserting a vertex in the interior of the face, but this may create separating quadrangles near the cycle C. We can get around this technical problem by gluing a special near-quadrangulation $\mathcal{M}_0$ to the face bounded by C. For example, the near-quadrangulation $\mathcal{M}_0$ can be constructed using two copies of the $2k$-cycle, one inside the other, adding edges between the two corresponding vertices of the cycles, and inserting a new vertex inside the interior $2k$-cycle and joining this new vertex to every other vertex of the cycle. As a result we obtain a simple quadrangulation of genus j with a distinguished $\mathcal{M}_0$, and another simple quadrangulation of genus $g - j$ rooted at $\mathcal{M}_0$. Thus the generating function of simple quadrangulations in this case is bounded by

$$\tilde{O}\left(x^i y^l \left(\sum_{j=1}^{g-1} Q_{g-j}^\star(x,y) \frac{\partial Q_j^\star(x,y)}{\partial x}\right)\right)$$

for some fixed integers i, l. As in previous arguments, this leads to a negligible contribution.

Case 2. Now suppose C is non-separating. Cutting through C, filling the two holes with disks, and then quadrangulating the resulting two faces as in Case 1, we obtain a rooted simple quadrangulation of genus $g-1$ with two distinguished $\mathcal{M}_0$. Thus the generating function of simple quadrangulations in this case is bounded by

$$\tilde{O}\left(x^i y^l \frac{\partial^2 Q^\star_{g-1}(x,y)}{(\partial x)^2}\right)$$

for some fixed integers i, l. Again, the contribution is negligible. This gives (18). Robertson and Vitray's result [25] implies that $L_g(n,m;2g{+}3)n!/(4m)$ counts 3-connected graphs of genus g with face width at least $2g+3$ and so (19) follows.

6 From 3-connected graphs to 2-connected graphs

Since the results for 2-connected planar graphs follow from [8], we assume $g>0$ in this section.

Definition 7 ((Planar networks) *A* planar network *is a graph $\mathcal{G}$ together with two distinguished vertices v_0 and v_1 (the* poles*) such that the graph obtained by adding the edge $e=\{v_0,v_1\}$ (if it is not already in $\mathcal{G}$) is 2-connected and planar. In contrast to the usual labeled graph, the poles of a labeled network are not labeled.*

As in [8] we use $D(x,y)$ to denote the generating function for planar networks; that is, $[(x^i/i!)y^m]\,D(x,y)$ is the number of planar networks with m edges i vertices *not including* the poles v_0 and v_1.

We will be expanding various functions about singularities. To help us remember which coefficient goes with which function, we introduce some notation. If $F(x)$ has a singularity at $x=r$ and we expand it in powers of $(1-x/r)$, then $F^{[t]}$ denotes the coefficient of $(1-x/r)^t$ in the expansion.

We begin with a review of some results for planar graphs. It is convenient to use essentially the same notation and parametrization as in [8]. That paper has three parameters, u, v and t. The parameters u and v are related to r and s by

$$u=\frac{r(2+s)}{4-rs} \quad\text{and}\quad v=\frac{s(2+r)}{4-rs}; \tag{20}$$

or equivalently,

$$r=\frac{2u}{1+v} \quad\text{and}\quad s=\frac{2v}{1+u}. \tag{21}$$

The parameter t is used on the singular curve $rs=1$ and is given by

$$t=\frac{1}{1+2r}.$$

It also uses the following functions of t. (When our notation differs from [8], we have indicated the [8] notation parenthetically.)

$$\begin{aligned}
\alpha(t) &= 144+592t+664t^2+135t^3+6t^4-5t^5 \\
\beta(t) &= 3t(1+t)(400+1808t+2527t^2+1155t^3+237t^4+17t^5) \\
\gamma(t) &= 1296+10272\,t+30920\,t^2+42526\,t^3+23135\,t^4 \\
&\quad -1482\,t^5-4650\,t^6-1358\,t^7-405\,t^8-30t^9 \\
h(t) &= \frac{t^2(1-t)\,(18+36\,t+5t^2)}{2(3+t)(1+2t)\,(1+3\,t)^2} \\
\rho_2(t) &= \frac{(1+3t)(1-t)^3}{16t^3} && \text{(called } x_0 \text{ in [8])} \\
\lambda_2(t) &= \frac{1+2t}{(1+3t)\,(1-t)}e^{-h(t)}-1 && \text{(called } y_0 \text{ in [8])} \\
\mu(t) &= \frac{(1+t)(3+t)^2(1+2t)^2(1+3t)^2\lambda_2(t)}{t^3(1+\lambda_2(t))\alpha(t)} \\
\sigma^2(t) &= \frac{(3+t)^2(1+2t)^2(1+3t)^2\lambda_2(t)}{3t^6(1+t)(1+\lambda_2(t))^2\alpha(t)^3} \\
&\quad \times\Big(3t^3(1+t)^2\alpha(t)^2-(1-t)(3+t)(1+2t)(1+3t)^2\lambda_2(t)\gamma(t)\Big) \\
D^{[0]}(t) &= \frac{3t^2}{(1-t)(1+3t)} && \text{(called } D_0 \text{ in [8])} \\
D^{[1]}(t) &= -\frac{48t^2(1+t)(1+2t)^2(18+6t+t^2)}{(1+3t)\beta(t)} && \text{(called } D_2 \text{ in [8])} \\
D^{[3/2]}(t) &= 384t^3(1+t)^2(1+2t)^2(3+t)^2\alpha(t)^{3/2}\beta(t)^{-5/2} && \text{(called } D_3 \text{ in [8]).}
\end{aligned}$$

As was pointed out in [20], a factor of t is missing in D_2 of [8]. We note that $\rho_2(t)=\rho(r)$.

Throughout the rest of the paper, we adopt the following notation, with $\epsilon>0$ not necessarily the same at each appearance,

$$T(\epsilon) \;=\; \{te^{i\theta}:\epsilon\le t\le 1-\epsilon,\ |\theta|\le\epsilon\} \quad\text{and}\quad \Delta(\rho,\epsilon) \;=\; \{z:|z|\le\rho+\epsilon\}-[\rho,\rho+\epsilon].$$

It is known [8, 20] that for each $t\in T(\epsilon)$, $D(x,\lambda_2(t))$ and $G_{0,2}(x,\lambda_2(t))$ are all analytic in a $\Delta(\rho_2(t),\epsilon)$ region. Also from [8, 20], we have

$$\begin{aligned}
D(x,y) &= D^{[0]}(t)+D^{[1]}(t)(1-x/\rho_2(t))+D^{[3/2]}(t)(1-x/\rho_2(t))^{3/2} \qquad (22)\\
&\quad +O\left((1-x/\rho_2(t))^2\right), \\
\frac{\partial D}{\partial y} &= \frac{D^{[0]\prime}(t)}{\lambda_2'(t)}+\frac{D^{[1]}(t)\rho_2'(t)}{\rho_2(t)\lambda_2'(t)}+O\left((1-x/\rho_2(t))^{1/2}\right), \qquad (23)
\end{aligned}$$

as $x\to\rho_2(t)$, uniformly for $y=\lambda_2(t)$ and $t\in T(\epsilon)$.

We now turn our attention to $G_{g,2}(x,y)$ and $G_g(n,m;2)$. Since the planar case $g=0$ has already been done [8, 20], we deal with the nonplanar case and prove the following theorem. A logarithm arises for $g=1$ from integrating a function raised to the power $(3-5g)/2$. As a consequence, $g=1$ requires separate treatment in later theorems.

Theorem 5 *Let $B_g(t)$ be as in Theorem 2(ii). There are generating functions $E_{g,2}(x,y)$ which are analytic in a $\Delta(\rho_2(t),\epsilon)$ region for each $t\in T(\epsilon)$ such that*

$$\begin{aligned} G_{1,2}(x,y) &= B_1(t)\ln\left(\frac{1}{1-x/\rho_2(t)}\right)+\tilde{O}(E_{1,2}(x,y)) \\ G_{g,2}(x,y) &= B_g(t)\,\Gamma\left(\frac{5g-5}{2}\right)\Big(1-x/\rho_2(t)\Big)^{-5(g-1)/2}+\tilde{O}(E_{g,2}(x,y)) \quad \text{for } g>1. \end{aligned}$$

The radius of convergence $R(c)$ of $E_{g,2}(x,c)$ satisfies $R(c)>R(|c|)$ for $c\neq|c|$. As $x\to\rho_2(t)$, we have, uniformly for $y=\lambda_2(t)$ and $t\in T(\epsilon)$,

$$E_{g,2}(x,y)=h(y)+O\left((1-x/\rho_2(t))^{-5g/2+3}\right)$$

for some function $h(y)$.

Proof: Since the planar case has been done in [8], we will use induction on g and assume $g>0$ below. Write $G_{g,2}(x,y)=F(x,y)+E(x,y)$ where $F(x,y)$ counts 2-connected graphs containing a unique nonplanar 3-connected component and $E(x,y)$ counts the remaining 2-connected graphs. We will analyze $F(x,y)$ and show that the contribution of $E(x,y)$ is negligible.

The dominant singularity is extracted from the $F(x,y)$ part and the remainder, along with the $E(x,y)$ bound, can be incorporated into $E_{g,2}$.

We begin with F. A 2-connected graph $\mathcal{F}$ counted by F contains a unique 3-connected component of genus g and all other 3-connected components of $\mathcal{F}$ are planar.

Thus we have

$$F(x,y) \;=\; G_{g,3}(x,D(x,y)).$$

It follows from (19) that

$$\frac{\partial}{\partial y}F(x,y) \;=\; \frac{xQ_g^\star(x,D(x,y))}{4D(x,y)}\frac{\partial D(x,y)}{\partial y}+\tilde{O}\left(B_2(x,D(x,y))\frac{\partial D(x,y)}{\partial y}\right),$$

and hence

$$F(x,y)=\int\frac{xQ_g^\star(x,D(x,y))}{4D(x,y)}\frac{\partial D(x,y)}{\partial y}dy+\tilde{O}\left(\int B_2(x,D(x,y))\frac{\partial D(x,y)}{\partial y}dy\right). \qquad (24)$$

Although we do not know $xQ_g^\star(x,y)$ exactly, we can still obtain an asymptotic estimate for the above integral because the coefficients of D are nonnegative and we have (7). We first use Theorem 4 and (7) to obtain the singular expansion for $xQ_g^\star(x,D(x,y))$ at the singularity $x=\rho(r)=\rho_2(t)$, with $y=\lambda_2(t)$ fixed. We have from (7)

$$xQ_g^\star(x,D)=C(r)(1-x/\rho(r))^{(3-5g)/2}+O\left((1-x/\rho(r))^{(4-5g)/2}\right),$$

as $x \to \rho(r)$. As in the proofs of (7) for $\hat{Q}_g(x,y)$ and $Q^\star_g(x,y)$, it is important to note that $D = D(x,y)$ is a function of x for each fixed y, and hence $\rho(r)$ is a function of x through the relation $D = \eta_3(r)$. It follows from (22) that

$$\begin{aligned}
\frac{d}{dx}\left(1-\frac{x}{\rho(r)}\right)\bigg|_{x=\rho(r)} &= \frac{-1}{\rho(r)}\left(1-\frac{\rho'(r)}{\eta_3'(r)}\frac{\partial D}{\partial x}\bigg|_{x=\rho(r)}\right) \\
&= \frac{-1}{\rho(r)}\left(1+\frac{\rho'(r)}{\eta_3'(r)}\frac{D^{[1]}}{\rho(r)}\right) \\
&= \frac{-1}{\rho_2(t)}\frac{3(1+t)(1+3t)\alpha(t)}{\beta(t)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(1-x/\rho(r))^{(3-5g)/2} &= \left(\frac{3(1+t)(1+3t)\alpha(t)}{\beta(t)}\right)^{(3-5g)/2}(1-x/\rho_2(t))^{(3-5g)/2} \\
&\quad +O\left((1-x/\rho(r))^{(4-5g)/2}\right),
\end{aligned}$$

as $x \to \rho_2(t)$ with $y = \lambda_2(t)$ fixed. We remind the reader that t and r are related by $t = \frac{1}{1+2r}$. Thus, temporarily using the notation

$$H(t) = \sqrt{\frac{3}{1+r}}\frac{A_g(r)}{4D(x,y)(1+2r)}(2+r)^{5g-4}\left(\frac{3(1+t)(1+3t)\alpha(t)}{\beta(t)}\right)^{(3-5g)/2}\Gamma\left(\frac{5g-3}{2}\right),$$

we have

$$\begin{aligned}
F(x,y) &= \int H(t)\Big(1-x/\rho_2(t)\Big)^{(3-5g)/2}\frac{\partial D(x,y)}{\partial y}\frac{\lambda_2'(t)}{\rho_2'(t)}d\rho_2 \\
&\quad +\tilde{O}\left(\int B_2(x,D(x,y))\frac{\partial D(x,y)}{\partial y}\frac{\lambda_2'(t)}{\rho_2'(t)}d\rho_2\right).
\end{aligned}$$

Noting that $B_2(x,D(x,y))$ has a singular expansion at $x = \rho_2(t)$ of lower order, we obtain from (23) that

$$\begin{aligned}
F(x,y) &= H(t)\left(\frac{D^{[0]\prime}(t)}{\lambda_2'(t)}+\frac{\rho_2'(t)D^{[1]}(t)}{\lambda_2'(t)\rho_2(t)}\right)\frac{\lambda_2'(t)\rho_2(t)}{\rho_2'(t)}f_g(\rho_2) \\
&\quad +O\left(\Big(1-x/\rho_2\Big)^{(6-5g)/2}\right) \\
&= B_g(t)f_g(\rho_2)+O\left(\Big(1-x/\rho_2\Big)^{(6-5g)/2}\right), \qquad (25)
\end{aligned}$$

where $B_g(t)$ is defined in Theorem 2(ii), $f_1(\rho_2) = -\ln(1-x/\rho_2)$, and

$$f_g(\rho_2) = \frac{(1-x/\rho_2(t))^{-5(g-1)/2}}{5(g-1)/2} \quad \text{when } g > 1.$$

We now show that $E(x, y)$ is negligible compared with $F(x, y)$.

For each graph counted by $E(x, y)$, there are at least two nonplanar 3-connected components. In this case there is a 2-cut $\{a, b\}$ that either splits $\mathcal{G}$ into two nonplanar pieces or gives a single piece with a lower genus. We consider these two cases separately. As in Section 4, there is a non-contractible simple closed curve C intersecting $\mathcal{G}$ only at a and b. As an aside, we note that this means the face width of $\mathcal{G}$ is at most 2 and hence intuitively the graphs in this class should be negligible; however, we have not proved a large face-width result for 2-connected graphs. The following analysis basically proves such a large face-width result and is very similar to the one used above for 3-connected graphs (maps).

Case 1. Cutting through C splits $\mathcal{G}$ into 2-connected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{G}_1$ is embeddable in the orientable surface of genus $j > 0$ and $\mathcal{G}_2$ is embeddable in the orientable surface of genus $g - j > 0$. Also $\mathcal{G}_1$ and $\mathcal{G}_2$ each has a distinguished edge (joining vertices a and b). Hence the generating function of the 2-connected graphs in this case is bounded by (applying Lemma 3)

$$\sum_{j=1}^{g-1} \tilde{O}\left(\frac{\partial G_{g-j,2}(x,y)}{\partial y} \frac{\partial G_{j,2}(x,y)}{\partial y}\right).$$

Case 2. Cutting through C reduces $\mathcal{G}$ into a 2-connected graph $\mathcal{G}_1$ which is embeddable in the orientable surface of genus $g - 1$, and $\mathcal{G}_1$ has two distinguished edges (joining the copies of a and b). Hence the generating function in this case is bounded by

$$\tilde{O}\left(\frac{\partial^2 G_{g-1,2}(x,y)}{(\partial y)^2}\right).$$

By induction, it is easily seen that the contributions in both cases satisfy Lemma 2 with the same parameters as $F(x, y)$, except that the exponent of n obtained in the asymptotics is less than the exponent of n in the asymptotics for F.

We need to establish Lemma 2(a,e). It is important to note that the dominant singularities of $G_{g,3}(x, D(x, y))$ are the same for each genus g because $G_{g,3}(x, y)$ (more precisely $Q_g^\star(x, y)$) have the same dominant singularities.

This completes the proof of Theorem 5.

Now Theorem 2(ii) follows immediately using Lemma 2.

Theorem 2(iii) for 2-connected graphs follows by setting $y = \lambda_2(t) = 1$ in Theorem 5 (i.e., $t = \hat{t} \doteq 0.62637$) and applying the "transfer" theorem. We note that

$$\begin{aligned}
\beta_2 &= \frac{8}{9(1+\hat{t})(1-\hat{t})^6}\left(\frac{\beta(\hat{t})}{\alpha(\hat{t})}\right)^{5/2} \doteq 7.6150 \cdot 10^4, \\
\alpha_2 &= \frac{1}{4\beta_2} \doteq 3.28299 \cdot 10^{-6}.
\end{aligned}$$

7 From 2-connected graphs to 1-connected graphs

Since the composition depends only on the vertices, there is no need to keep track of the number of edges if we only care about the number of graphs with n vertices. This makes the arguments much simpler as we are dealing with univariate functions. From now on, we will focus on $y = 1$, although the results extend to all y near 1 as done in [20] for planar graphs. We note that it would be possible to extend the result to the whole range of y, provided that the condition $R(y) > R(|y|)$ when $y \neq |y|$ for the radius of convergence in Lemma 2 can be verified for $G_{g,1}(x, y)$. However, we have not verified this technical condition.

Since the planar case is dealt with in [20], we assume $g > 0$.

Let x_1 be the smallest positive singularity of $G_{0,1}(x)$. Giménez and Noy [20, p. 320] showed that

$$x_2 = x_1 G'_{0,1}(x_1) \tag{26}$$

and $G_{0,1}(x)$ is analytic in a $\Delta(x_1, \epsilon)$ region.

As in the previous section, let $\hat{t} \doteq 0.62637$ be determined by $\lambda_2(\hat{t}) = 1$. From [20, Lemma 6], we have the following singular expansion at $x_2 = \rho_2(\hat{t}) \doteq 0.03819$,

$$G_{0,2}(x) \;=\; G_{0,2}^{[0]} + G_{0,2}^{[1]}(1 - x/x_2) + G_{0,2}^{[2]}(1 - x/x_2)^2 + G_{0,2}^{[5/2]}(1 - x/x_2)^{5/2} + \ldots, \tag{27}$$

where $G_{0,2}^{[j]} = G_{0,2}^{[j]}(\hat{t})$, and in particular

$$G_{0,2}^{[0]} \doteq 7.397 \cdot 10^{-4}, \quad G_{0,2}^{[1]} \doteq -1.4914 \cdot 10^{-3} \quad \text{and} \quad G_{0,2}^{[2]} \doteq 7.672 \cdot 10^{-4}.$$

Define

$$\begin{aligned} A \;&=\; \frac{(3\hat{t} - 1)(1 + \hat{t})^3 \ln(1 + \hat{t})}{16\hat{t}^3} + \frac{(1 + 3\hat{t})(1 - \hat{t})^3 \ln(1 + 2\hat{t})}{32\hat{t}^3} \\ &\quad + \frac{(1 - \hat{t})(185\hat{t}^4 + 698\hat{t}^3 - 217\hat{t}^2 - 160\hat{t} + 6)}{64\hat{t}(1 + 3\hat{t})^2(3 + \hat{t})}, \\ x_1 \;&=\; \tfrac{1}{16}\sqrt{1 + 3\hat{t}}(1 - \hat{t})^3 \hat{t}^{-3} e^A \;\doteq\; 0.03673. \end{aligned}$$

It was shown in [20] that

$$G_{0,1}(x) \;=\; G_{0,1}^{[0]} + G_{0,1}^{[1]}(1 - x/x_1) + G_{0,1}^{[2]}(1 - x/x_1)^2 + G_{0,1}^{[5/2]}(1 - x/x_1)^{5/2} + \ldots \tag{28}$$

$$P(x) \;:=\; xG'_{0,1}(x) \;=\; P^{[0]} + P^{[1]}(1 - x/x_1) + P^{[3/2]}(1 - x/x_1)^{3/2} + \ldots, \tag{29}$$

where

$$P^{[0]} = -G_{0,1}^{[1]}, \quad P^{[1]} = -2G_{0,1}^{[2]} - G_{0,1}^{[0]} \doteq -0.03979 \quad \text{and} \quad P^{[3/2]} = -5G_{0,1}^{[5/2]}/2.$$

We also note that [20, (4.7)] $G_{0,1}^{[0]} = G_{0,1}(x_1) = x_2 + G_{0,2}^{[0]} + G_{0,2}^{[1]} \doteq 0.03744$. The following theorem summarizes the main results of this section.

Theorem 6 *Fix $g > 0$. We have $G_{g,1}(x) = F(x) + \tilde{O}(E(x))$ where*

(i) $F(x)$ and $E_g(x)$ are analytic in a $\Delta(x_1, \epsilon)$;

(ii) as $x \to x_1$,

$$F(x) \sim \begin{cases} \alpha_2\beta_2 t_1 \ln\left(\dfrac{1}{1-x/x_1}\right) & \text{if } g = 1, \\ \alpha_2\beta_2^g t_g\,\Gamma\left(\dfrac{5g-5}{2}\right)\left(\dfrac{-x_2}{P^{[1]}}\right)^{5(g-1)/2} (1-x/x_1)^{-5(g-1)/2} & \text{if } g > 1; \end{cases}$$

(iii) as $x \to x_1$, $E_1(x) = C + O\left((1-x/x_1)^{1/2}\right)$ for some constant C and $E(x) = O\left((1-x/x_1)^{-5g/2+3}\right)$ when $g > 1$.

Proof: We again apply induction on g. Let $\mathcal{G}$ be a connected graph of genus g rooted at a vertex v. It is well known that $\mathcal{G}$ is (uniquely) decomposed into a set of blocks (2-connected pieces) and the genus of $\mathcal{G}$ is the sum of the genera of all blocks [2]. We divide all connected graphs of genus $g > 0$ into two classes according to whether there is a block of genus g or not and will show that the second class is negligible.

Case 1 (Genus g block). We attach a planar 1-connected graph to each vertex of the genus g to connected block. Thus the generating function for this case is

$$F(x) = G_{g,2}(xG'_{0,1}(x)).$$

Since $G_{g,2}(x)$ is bounded termwise above and below by functions analytic in a $\Delta(x_2, \epsilon)$ region, it follows from (26), the same holds for $F(x)$ in a $\Delta(x_1, \epsilon)$ region. For $g > 1$, it follows from Theorem 5 and (29) that

$$\begin{aligned} G_{g,2}(x) &= \alpha_2\beta_2^g t_g\,\Gamma\left(\frac{5g-5}{2}\right)(1-x/x_2)^{-5g/2+5/2} + O\left((1-x/x_2)^{-5g/2+3}\right) \\ G_{g,2}(xG'_{0,1}(x)) &= \alpha_2\beta_2^g t_g\,\Gamma\left(\frac{5g-5}{2}\right)(-P^{[1]}/x_2)^{-5g/2+5/2}(1-x/x_1)^{-5g/2+5/2} \\ &\quad + O\left((1-x/x_1)^{-5g/2+3}\right). \end{aligned}$$

It follows that

$$\begin{aligned} F(x) &= \alpha_2\beta_2^g t_g\,\Gamma\left(\frac{5g-5}{2}\right)(-P^{[1]}/x_2)^{-5g/2+5/2}(1-x/x_1)^{-5g/2+5/2} \\ &\quad + O\left((1-x/x_1)^{-5g/2+3}\right). \end{aligned} \tag{30}$$

The formula for $g = 1$ is similar except that it involves a logarithm:

$$F(x) = \alpha_2\beta_2 t_1 \ln\left(\frac{1}{1-x/x_1}\right) + O(1). \tag{31}$$

As in previous proofs, we use bounds on the functions to split F into F_g and a contribution to E_g which are analytic in a $\Delta(x_1, \epsilon)$.

Case 2 (No genus g block). In this case, there is at least one vertex v such that $\mathcal{G}$ can be viewed as two nonplanar graphs joined at v. Hence an upper bound for graphs in this class is given by the generating function

$$\sum_{j=1}^{g-1} xG'_{j,1}(x)G'_{g-j,1}(x).$$

It follows by induction on g that each summand is bounded by a function analytic in a $\Delta(x_1, \epsilon)$ region and, as $x \to x_1$ in this region each bound is bounded by

$$O\left((1-x/x_1)^{-5j/2+3/2}(1-x/x_1)^{-5(g-j)/2+3/2}\right) \;=\; O\left((1-x/x_1)^{-5g/2+3}\right). \tag{32}$$

This completes the proof of Theorem 6.

Now Theorem 2(iii) for 1-connected graphs follows immediately using the "transfer theorem". We obtain

$$\beta_1 \;=\; \left(\frac{-x_2}{P^{[1]}}\right)^{5/2}\beta_2 \;\doteq\; 6.87242\cdot 10^4, \quad \text{and} \quad \alpha_1 \;=\; \frac{1}{4\beta_1} \;\doteq\; 3.63773\cdot 10^{-6}.$$

8 From 1-connected graphs to all graphs

The case $g = 0$ is treated in [20]. We treat $g > 1$. The case $g = 1$ is similar to $g > 1$ except that $\ln(1-x/x_1)$ appears. Let $F(x)$ denote the generating function of these graphs containing a connected component of genus g. Then we have

$$\begin{aligned} F(x) &= G_{g,1}(x)\exp(G_{0,1}(x)) \\ &= \alpha_1\beta_1^g \exp(G_{0,1}(x_1))t_g\,\Gamma\left(\frac{5g-5}{2}\right)(1-x/x_1)^{5(1-g)/2} + O\left((1-x/x_1)^{-5g/2+3}\right). \end{aligned}$$

Again the case that there are two components with positive genus is (by induction) bounded by

$$O\left(\sum_{j=1}^{g-1} G_{j,1}(x)G_{g-j,0}(x)\right) \;=\; O\left((1-x/x_1)^{-5g/2+3}\right).$$

Thus

$$\begin{aligned} G_{g,0}(x) &= G_{g,1}(x)\exp(G_{0,1}(x)) + O\left((1-x/x_1)^{-5g/2+3}\right) \\ &= \alpha_1\beta_1^g \exp(G_{0,1}(x_1))t_g\,\Gamma\left(\frac{5g-5}{2}\right)(1-x/x_1)^{5(1-g)/2} + O\left((1-x/x_1)^{-5g/2+3}\right). \end{aligned}$$

This completes the proof of Theorem 2 (using the "transfer" theorem again) with

$$\alpha_0 \;=\; \alpha_1\exp(G_{0,1}(x_1)) \;\doteq\; 3.77651\cdot 10^{-6} \;\text{ and }\; \beta_0 = \beta_1.$$

9 A formula for $A_g(r)$

In this section we obtain a formula for $A_g(r)$ using [4] and recently derived information [17] for $t_g(r)$.

Let $T_g(n,j)$ be the number of rooted maps of genus g with i faces and j vertices.

By duality, we may interchange the role of vertices and faces, and we do so. By Euler's formula, $T_g(n,j)$ is also the number of rooted maps of genus g with i vertices and $m = j+n+2g-2$ edges.

By Theorem 1, we have

$$\begin{aligned} T_g(n,j) &= [x^n y^m]\hat{M}_g(x,y) \\ &\sim \Big(C_1(r)A_g(r)n^{5g/2-3}\Big)\rho(r)^{-n}\eta_1(r)^{-m} \\ &\sim C_1(r)A_g(r)(n/j)^{5g/4-3/2}(nj)^{5g/4-3/2}\rho(r)^{-n}\eta_1(r)^{-n-j+2-2g} \end{aligned} \tag{33}$$

Note that

$$\frac{j}{n} = \frac{m}{n} - 1 + \frac{2-2g}{n} = \frac{1+2r}{r^2(2+r)} + \frac{2-2g}{n}.$$

It follows that the value of r in [4, Theorem 2] differs from our r by $O(1/n)$. Replacing one r with the other inside the large parentheses of (33) does not change the asymptotics. We must show that is also true for $f = \rho(r)^{-n}\eta_1(r)^{-m}$. This can be done by expanding $\log f$ in a power series about r and noting that the linear term vanishes after we set m/n to the value given in Theorem 1(i). It follows that replacing r by $r+O(1/n)$ changes $\log f$ by $nO(1/n^2) = o(1)$. Hence we may freely use either value of r in (33). Thus we obtain

$$T_g(n,j) \sim \left(C_1(r)A_g(r)\eta_1(r)^{2-2g}\left(\frac{r^2(2+r)}{1+2r}\,nj\right)^{5g/4-3/2}\right)(\rho(r)\eta_1(r))^{-n}\,\eta_1(r)^{-j}.$$

Comparing this with [4, Theorem 2] we obtain

$$t_g(r) = C_1(r)A_g(r)\eta_1(r)^{2-2g}\left(\frac{r^2(2+r)}{1+2r}\right)^{5g/4-3/2}$$

and so by Theorem 1(i)

$$\begin{aligned} A_g(r) &= \frac{\eta_1(r)^{2g-2}}{C_1(r)}\left(\frac{1+2r}{r^2(2+r)}\right)^{5g/4-3/2} t_g(r) \\ &= \frac{2^4 r^3(2+r)^{1/2}(1+r+r^2)^{7/2}(4+7r+4r^2)^{1/2}}{(1+2r)^3} \\ &\qquad \times \left(\frac{(1+2r)^{13/2}}{2^8 r^5(1+r+r^2)^8(2+r)^{5/2}}\right)^{g/2} t_g(r). \end{aligned} \tag{34}$$

It was shown by the second author [17] that

$$t_g(r) = c(r)[d(r)]^g \, t_g, \tag{35}$$

where

$$\begin{aligned} c(r) &= \frac{r^3(1+2r)(2+r)}{32\sqrt{\pi}(4+7r+4r^2)^{1/2}(1+r+r^2)^{7/2}}, \\ d(r) &= \frac{32\sqrt{3}(1+r+r^2)^4(1+r)^{3/2}}{r^{7/2}(2+r)^{5/4}(1+2r)^{5/4}}. \end{aligned}$$

Combining (34) and (35) gives

$$A_g(r) \;=\; \frac{r^6(2+r)^{3/2}}{2\sqrt{\pi}(1+2r)^2}\left(\frac{12(1+2r)^4(1+r)^3}{r^{12}(2+r)^5}\right)^{g/2} t_g.$$

10 Remarks on Nonorientable Surfaces

The study of graphs embeddable in nonorientable surfaces can proceed in a manner akin to the approach we have used for the orientable case, giving Theorems 1 and 2 for a nonorientable surface with $2g$ crosscaps provided t_g is replaced by the nonorientable map constant p_g. There are some issues that will have to be dealt with.

(a) The projective plane will require some special care because the singular expansion of the generating function behaves like

$$M_{1/2}(x,y) = g(y) + h(y)(1 - x/\rho(y))^{1/4} + \cdots,$$

whose dominant term has a positive exponent. For (7) it simply involves subtracting the value of $xq(x,y)$ on the singular curve from the left side. For products like $M_{1/2}(x,y)M_{g-1/2}(x,y)$, the dominant term in the singular expansion is not simply the sum $(1/4)+(3/2)-5(g-1/2)/2$ as in (17). Rather, it should be $(3/2)-5(g-1/2)/2$ when $g>1$ and $1/4$ when $g=1$. It can still be checked that the exponent of the singular expansion of a product like (16) is higher than $(3-5g)/2$.

(b) Theorem 3 requires minor adjustment. In (a), $q(x,y)$ is no longer rational in r and s because it involves $\sqrt{1-rs}$, but it is still algebraic. The adjustment to (7) for the projective plane was noted in (a).

(c) Careful attention to the proofs in this paper reveals that relative errors obtained for the asymptotics are typically $O(n^{-1/2})$. In the nonorientable case they will often be $O(n^{-1/4})$ because g increases in half-integer steps rather than integer steps.

(d) Numerical estimation of graphs and maps will be difficult for all but small genus because computing p_g is difficult. The original recursions for t_g and p_g [3] are quite complicated; however a practical recursion was found as noted in the remark following Theorem 2. None is known for p_g, but Garoufalidis and Mariño [19] conjectured that

$$p_g \;=\; \frac{v_{2g-1}}{2^{g-2}\Gamma\left(\frac{5g-3}{2}\right)},$$

where v_g satisfies

$$v_g \;=\; \frac{1}{2\sqrt{3}}\left(-3a_{g/2}+\frac{5g-6}{2}v_{g-1}+\sum_{k=1}^{g-1}v_k v_{g-k}\right),$$

and a_j is defined in (1), with the understanding that $a_j=0$ when j is not an integer.

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