Enumeration of Varlet and Comer hypergroups

H. Aghabozorgi

Department of Mathematics
Yazd University, Yazd, Iran

h_aghabozorgi1@yahoo.com

M. Jafarpour

Department of Mathematics
Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran
m.j@mail.vru.ac.ir

B. Davvaz

Department of Mathematics
Yazd University, Yazd, Iran

davvaz@yazduni.ac.ir

Submitted: Feb 12, 2011; Accepted: May 29, 2011; Published: Jun 14, 2011
Mathematics Subject Classifications: 20N20, 05E15.

Abstract

In this paper, we study hypergroups determined by lattices introduced by Varlet and Comer, especially we enumerate Varlet and Comer hypergroups of orders less than 50 and 13, respectively.

1 Basic definitions and results

An algebraic hyperstructure is a natural generalization of a classical algebraic structure. More precisely, an algebraic hyperstructure is a non-empty set H endowed with one or more *hyperoperations* that associate with two elements of H not an element, as in a classical structure, but a subset of H. One of the interests of the researchers in the field of hyperstructures is to construct new hyperoperations using graphs [18], binary relations [2, 5, 7, 8, 9, 11, 15, 21, 23], n-ary relations [10], lattices [16], classical structures [13], tolerance space [12] and so on. Connections between lattices and hypergroupoids have been considered since at least three decades, starting with [24] and followed by [3, 14, 17]. This paper deals with hypergroups derived from lattices, in particular we study some properties of the hypergroups defined by J.C. Varlet [24] and S. Comer [3] that called here Varlet hypergroups and Comer hypergroups, respectively. Using the results of [1, 22] we enumerate the number of non isomorphic Varlet and Comer hypergroups of orders less than 50 and 13, respectively.

Let us briefly recall some basic notions and results about hypergroups; for a comprehensive overview of this subject, the reader is referred to [4, 6, 25]. For a non-empty set H, we denote by $\mathcal{P}^*(H)$ the set of all non-empty subsets of H. A non-empty set H, endowed with a mapping, called *hyperoperation*, $\circ : H^2 \longrightarrow \mathcal{P}^*(H)$ is named *hypergroupoid*. A hypergroupoid which satisfies the following conditions: (1) $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$ (the *associativity*), (2) $x \circ H = H = H \circ x$, for all $x \in H$ (the *reproduction axiom*) is called a *hypergroup*. In particular, an associative hypergroupoid is called a *semihypergroup* and a hypergroupoid that satisfies the reproduction axiom is called a *quasihypergroup*. If A and B are non-empty subsets of H, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Let $(H, \circ)$ and $(H', \circ')$ be two hypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if it satisfies the condition: for any $x, y \in H$, $f(x \circ y) \subseteq f(x) \circ' f(y)$. f is a *good homomorphism* if, for any $x, y \in H$, $f(x \circ y) = f(x) \circ' f(y)$. We say that the two hypergroups are *isomorphic* if there is a good homomorphism between them which is also a bijection.

Join spaces were introduced by W. Prenowitz and then applied by him and J. Jantosciak both in Euclidian and in non Euclidian geometry [19, 20]. Using this notion, several branches of non Euclidian geometry were rebuilt: descriptive geometry, projective geometry and spherical geometry. Then, several important examples of join spaces have been constructed in connection with binary relations, graphs and lattices. In order to define a join space, we need the following notation: If a, b are elements of a hypergroupoid $(H, \circ)$, then we denote $a/b = \{x \in H \mid a \in x \circ b\}$. Moreover, by A/B we intend the set $\bigcup_{a \in A, b \in B} a/b$.

A commutative hypergroup $(H, \circ)$ is called a *join space* if the following condition holds for all elements a, b, c, d of H:

$$a/b \cap c/d \neq \emptyset \implies a \circ d \cap b \circ c \neq \emptyset.$$

Definition 1.1. [24] Let $\mathfrak{L}_{\leq} = (L, \wedge, \vee)$ be a lattice with join $\vee$, meet $\wedge$ and order relation $\leq$ and let:

$$\forall (a, b) \in L^2, \quad a \circ b = \{x \in L \mid a \wedge b \leq x \leq a \vee b\}.$$

Theorem 1.2. *[24] For a lattice $\mathfrak{L}_{\leq}$ the following are equivalent:*

(1) $\mathfrak{L}_{\leq}$ is distributive;

(2) $\mathbb{L}_{\leq} = (L, \circ)$ is a join space.

The class of intervals of elements of $\mathfrak{L}_{\leq} = (L, \wedge, \vee)$ is denoted by $I(\mathfrak{L}_{\leq})$, that is:

$$I(\mathfrak{L}_{\leq}) = \{[a, b] \mid (a, b) \in L^2, a \leq b\},$$

where $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

Theorem 1.3. *For the join space $\mathbb{L}_{\leq}$ given in Theorem 1.2, the following equality holds:*

$$Sub(\mathbb{L}_{\leq}) = I(\mathfrak{L}_{\leq}) = \{x \circ y | (x, y) \in L^2\},$$

where $Sub(\mathbb{L}_{\leq})$ is the class of subhypergroups of $\mathbb{L}_{\leq}$.

Proof. Let $[a, b] \in I(\mathfrak{L}_{\leq})$. Then, for any $x, y \in [a, b]$ we have $a \leq x \leq b$ and $a \leq y \leq b$. These lead to $a \leq x \wedge y \leq x \vee y \leq b$ and so $x \circ y = [x \wedge y, x \vee y] \subseteq [a, b]$. Moreover, $[a, b] \circ x = x \circ [a, b] = \bigcup_{t \in [a,b]} x \circ t = [a, x] \cup [x, b] = [a, b]$. Conversely, let $H \in Sub(\mathbb{L}_{\leq})$, $a \circ b \subseteq H$, for all $a, b \in H$. Hence, $[a \wedge b, a \vee b] \subseteq H$. In particular, one obtains that H is closed under the operations $\wedge$ and $\vee$. Let $A = \{a_i\}_{i \in I}$ and $B = \{b_i\}_{i \in J}$ are the sets of minimal and maximal elements of H, respectively with respect to the order on L. If $|I| \geq 2$, then we can choose two distinct elements of A, say a, a', it follows that $a \wedge a' \in H$ a contradiction. In this way, A contains a unique element, say a_0. Similarly, B contains a unique element, say b_0. It is clear that $H = [a_0, b_0]$. We can easily see that the equality $I(\mathfrak{L}_{\leq}) = \{x \circ y | (x, y) \in L^2\}$ holds. □

Theorem 1.4. *Let $\mathfrak{L}_{\leq} = (L, \wedge, \vee)$ be a distributive lattice. Define on the set $I(\mathfrak{L}_{\leq})$, the following hyperoperation*

$$[x, y] \odot_R [z, w] = Sub([x \wedge z, y \vee w]).$$

Then, $(I(\mathfrak{L}_{\leq}), \odot_R)$ is a hypergroup.

Proof. Using previous theorem, it is clear that $\odot_R$ is a well defined hyperoperation. We prove $\odot_R$ is associative. To this end we have:

$$\begin{aligned}
([x_1, y_1] \odot_R [x_2, y_2]) \odot_R [x_3, y_3] &= Sub([x_1 \wedge x_2, y_1 \vee y_2]) \odot_R [x_3, y_3] \\
&= Sub([(x_1 \wedge x_2) \wedge x_3, (y_1 \vee y_2) \vee y_3]) \\
&= Sub([x_1 \wedge (x_2 \wedge x_3), y_1 \vee (y_2 \vee y_3)]) \\
&= [x_1, y_1] \odot_R Sub([x_2 \wedge x_3, y_2 \vee y_3]) \\
&= [x_1, y_1] \odot_R ([x_2, y_2] \odot_R [x_3, y_3]).
\end{aligned}$$

□

2 Enumeration of finite Varlet hypergroups

It is well known that every binary relation ρ on a finite set L, with $cardL = n$, may be represented by a Boolean matrix $M(\rho)$ and conversely every Boolean matrix of order n defines on L a binary relation. Indeed, let $L = \{a_1, ..., a_n\}$; a Boolean matrix of order n is constructed in the following way: the element in the position (i, j) of the matrix is 1, if $(a_i, a_j) \in \rho$ and it is 0 if $(a_i, a_j) \notin \rho$ and vice versa. Hence, on every set with n elements, 2^{n^2} partial hypergroupoids can be defined. Recall that in a Boolean algebra the following properties hold: $0 + 1 = 1 + 0 = 1 + 1 = 1$, while $0 + 0 = 0$, and $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$. Moreover, if ρ is a binary relation on L, then $M(\rho^2) = M^2(\rho)$.

Proposition 2.1. *Let $\mathfrak{L}_{\leq}$ and $\mathfrak{L}_{\leq'}$ be two finite lattices and (t_{ij}), (t'_{ij}) be their associated matrices, respectively. Then, $\mathfrak{L}_{\leq}$ and $\mathfrak{L}_{\leq'}$ are isomorphic if and only if $t_{ij} = t'_{\sigma(i)\sigma(j)}$, for a permutation σ of the set $\{1, 2, \ldots, n\}$.*

Definition 2.2. Let $\mathfrak{L}_{\leq}$ be a finite lattice. The matrix $M(\leq)$ is called *very good* if and only if $\mathbb{L}_{\leq}$ is a Varlet hypergroup.

Proposition 2.3. *If $M = (t_{ij})_{n\times n}$ is a very good matrix and $M^2 = (s_{ij})$, then the following assertions hold:*

(1) $t_{ii} = 1$, for all $1 \leq i \leq n$;

(2) $t_{ij} = 1 \Rightarrow t_{ji} = 0$, for all $i \neq j$ and $1 \leq i, j \leq n$;

(3) $M^2 \leq M$, (i.e., $s_{ij} = 1 \Rightarrow t_{ij} = 1$, for all $1 \leq i, j \leq n$);

(4) there exists i, with $1 \leq i \leq n$, such that $t_{ij} = 1$, for all $1 \leq j \leq n$;

(5) there exists j, with $1 \leq j \leq n$, such that $t_{ij} = 1$, for all $1 \leq i \leq n$.

The matrix $T = (t_{ij})_{n\times n}$, with

$$t_{ij} = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise,} \end{cases}$$

for any $i, j \in \{1, 2, \ldots, n\}$, is a very good matrix that we call it n-triangular and the corresponding hypergroup is a Varlet hypergroup.

In the following, we give in terms of matrices a necessary and sufficient condition such that two Varlet hypergroups associated with two lattices on the same set L, are isomorphic.

Proposition 2.4. *Let $L = \{a_1, \ldots, a_n\}$ be a finite set, $\leq$ and $\leq'$ be two order relations on L and $M(\leq) = (t_{ij})$, $M(\leq') = (t'_{ij})$ be their associated matrices. If $t_{ij} = t'_{\sigma(i)\sigma(j)}$, for a permutation σ of the set $\{1, 2, \ldots, n\}$, then the following assertions hold:*

(1) $a_i \leq a_j \Leftrightarrow a_{\sigma(i)} \leq' a_{\sigma(j)}$;

(2) $a_i \wedge a_j = a_k \Leftrightarrow a_{\sigma(i)} \wedge a_{\sigma(j)} = a_{\sigma(k)}$;

(3) $a_i \vee a_j = a_k \Leftrightarrow a_{\sigma(i)} \vee a_{\sigma(j)} = a_{\sigma(k)}$.

Theorem 2.5. *Let $\mathfrak{L}_{\leq}$ and $\mathfrak{L}_{\leq}$ be two finite distributive lattices and let $M(\leq) = (t_{ij})$ and $M(\leq') = (t'_{ij})$ be their associated matrices. The hypergroups $\mathbb{L}_{\leq}$ and $\mathbb{L}_{\leq'}$ are isomorphic if and only if $t_{ij} = t'_{\sigma(i)\sigma(j)}$, for a permutation σ of the set $\{1, 2, \ldots, n\}$.*

Proof. Let $L = \{a_1, \ldots, a_n\}$ and $\theta : \mathbb{L}_{\leq} \longrightarrow \mathbb{L}_{\leq'}$ be an isomorphism. Then, $\theta(a_i \circ a_j) = \theta(a_i) \circ' \theta(a_j)$ and so

$$\{\theta(a_k) \mid a_i \wedge a_j \leq a_k \leq a_i \vee a_j\} = \{a_s | \theta(a_i) \wedge \theta(a_j) \leq' a_s \leq' \theta(a_i) \vee \theta(a_j)\}.$$

Thus, we have $a_i \wedge a_j \leq a_k \leq a_i \vee a_j$ if and only if $\theta(a_i) \wedge \theta(a_j) \leq' \theta(a_k) \leq' \theta(a_i) \vee \theta(a_j)$. Suppose that $\theta(a_j) = a_{\sigma(j)}$, for a permutation σ of the set $\{1, 2, \ldots, n\}$. We show that $t_{ij} = t'_{\sigma(i)\sigma(j)}$. If $t_{ij} = 0$, then we can easily see that $t'_{\sigma(i)\sigma(j)} = 0$. Now, suppose that $t_{ij} = 1$. Then, we have $t'_{\sigma(i)\sigma(j)} = 1$ or $t'_{\sigma(j)\sigma(i)} = 1$. Since $t_{ji} = 0$ the case $t'_{\sigma(j)\sigma(i)} = 1$ would not occur. Thus, we have $t'_{\sigma(i)\sigma(j)} = 1$. Conversely, note that, for a permutation σ of the set $\{1, 2, \ldots, n\}$, we have $a_i \leq a_j \Leftrightarrow a_{\sigma(i)} \leq' a_{\sigma(j)}$. Consider the map $\varphi : \mathbb{L}_{\leq} \to \mathbb{L}_{\leq'}$ with $\varphi(a_i) = a_{\sigma(i)}$. Clearly, φ is a bijection and by using previous proposition we have:

$$\{\varphi(a_k) \mid a_i \wedge a_j \leq a_k \leq a_i \vee a_j\} = \{a_{\sigma(k)} \mid a_{\sigma(i)} \wedge a_{\sigma(j)}) \leq' a_{\sigma(k)} \leq' a_{\sigma(i)} \vee a_{\sigma(j)}\}.$$

Therefore, $\varphi(a_i \circ a_j) = \varphi(a_i) \circ' \varphi(a_j)$ and the proof is completed. $\square$

We say that a Boolean matrix is *reflexive*, *antisymmetric* or *transitive* if the associated binary relation is reflexive, antisymmetric or transitive, respectively.

We say that two very good matrices are *isomorphic* if the Varlet hypergroups obtained by them are isomorphic.

Theorem 2.6. *Let $M = (t_{ij})_{n\times n}$ and $M' = (t'_{ij})_{m\times m}$ be two very good matrices. Then, $M \oplus M' = (m_{ij})_{k\times k}$, where $k = n + m$, and*

$$m_{ij} = \begin{cases} t_{ij} & \text{if } i \leq n,\ j \leq n \\ t'_{ij} & \text{if } n < i,\ n < j \\ 1 & \text{if } i \leq n,\ j > n \\ 0 & \text{if } n < i,\ j \leq n \end{cases}$$

is a very good matrix.

Proof. Since $M \oplus M' = \begin{pmatrix} M & O' \\ O & M' \end{pmatrix}_{k\times k}$, where O is an $m \times n$ matrix which all entries are zero (i.e., $O = (0)_{m\times n}$), and O' is an $n \times m$ matrix which all entries are one. We have $(M \oplus M')^2 = M^2 \oplus M'^2 \leq M \oplus M'$ and so $M \oplus M'$ is a transitive matrix. Obviously, $M \oplus M'$ is reflexive and antisymmetric. Now, suppose that $L = \{a_1, \ldots, a_{n+m}\}$ and $\leq$ is the associated binary relation of $M \oplus M'$. Then

$$a_i \leq a_j \Leftrightarrow [t_{ij} = 1 \text{ or } t'_{ij} = 1, \text{ and } (i \leq n, \quad j > n)].$$

Hence, $a_i \wedge (a_j \vee a_k) \leq (a_i \wedge a_j) \vee (a_i \wedge a_k)$, for every $(a_i, a_j, a_k) \in L^3$. So, $(L, \leq)$ is a distributive lattice and $M \oplus M'$ is very good. $\square$

Corollary 2.7. Let V_n be the number of non isomorphic Varlet hypergroups of order n. Then, $V_{n+m} \geq V_n V_m$, for all $n, m \in \mathbb{N}$.

Using the results of [22] we can enumerate the number of Varlet hypergroups (up to isomorphism) with cardinality less than 50 which we summarize at the following table.

n=	Number of Varlet hypergroups	n=	Number of Varlet hypergroups
1	1	26	711811
2	1	27	1309475
3	1	28	2413144
4	2	29	4442221
5	3	30	8186962
6	5	31	15077454
7	8	32	27789108
8	15	33	51193086
9	26	34	94357143
10	47	35	173859936
11	82	36	320462062
12	151	37	590555664
13	269	38	1088548290
14	494	39	2006193418
15	891	40	3697997558
16	1639	41	6815841849
17	2978	42	12563729268
18	5483	43	23157428823
19	10006	44	42686759863
20	18428	45	78682454720
21	33749	46	145038561665
22	62162	47	267348052028
23	114083	48	492815778109
24	210189	49	908414736485
25	386292		

3 On Comer hypergroups

Proposition 3.1. *[3] Let $\mathfrak{L}_{\leq} = (L, \wedge, \vee)$ be a Modular lattice. If for all $a, b \in L$ we define*

$$a \bullet b = \{z \in L \ \mid \ z \vee a = a \vee b = b \vee z\},$$

then $\mathbf{L}_{\leq} = (L, \bullet)$ is a hypergroup that we call it "Comer hypergroup".

Definition 3.2. Let $\mathfrak{L}_{\leq}$ be a finite lattice. The matrix $M(\leq)$ is called *good* if and only if $\mathbf{L}_{\leq}$ is a Comer hypergroup.

Theorem 3.3. *Let $\mathfrak{L}_{\leq}$ and $\mathfrak{L}_{\leq}$ be two finite modular lattices and $M(\leq) = (t_{ij}), M(\leq') = (t'_{ij})$ be their associated matrices. The hypergroups $\mathbf{L}_{\leq}$ and $\mathbf{L}_{\leq'}$ are isomorphic if and only if $t_{ij} = t'_{\sigma(i)\sigma(j)}$, for a permutation σ of the set $\{1, 2, \ldots, n\}$.*

Theorem 3.4. *Let $M = (t_{ij})_{n\times n}$, $M' = (t'_{ij})_{m\times m}$ be two good matrices. Then, $M \boxplus M' = (m_{ij})_{k\times k}$, where $k = n + m$ and*

$$m_{ij} = \begin{cases} t_{ij} & \text{if } \ i \le n, \ j \le n \\ t'_{ij} & \text{if } n < i, \ n < j \\ 1 & \text{if } \ (\prod_{s=1}^{n} t_{is} = 1, j > n) \text{ or } (i \le n, \prod_{l=1}^{m} t'_{lj} = 1) \\ 0 & \text{others} \end{cases}$$

is a good matrix.

Proof. We have $M \boxplus M' = \begin{pmatrix} M & O' \\ O & M' \end{pmatrix}_{k\times k}$, where $O = (0)_{m\times n}$ and $O' = (b_{ij})_{n\times m}$, where

$$b_{ij} = 1 \Leftrightarrow [\prod_{s=1}^{n} t_{is} = 1 \ \text{ or } \ \prod_{l=1}^{m} t'_{lj} = 1].$$

So, $(M \boxplus M')^2 = M^2 \boxplus M'^2 \le M \boxplus M'$ and so $M \boxplus M'$ is a transitive matrix. Notice that in M just exists one row and one column which all entries are 1. Now, suppose that $L = L_1 = \{a_1, \dots a_n\} \cup \{a_{n+1}, \dots a_{n+m}\} = L_2$ and $\le$, $\le_1$ and $\le_2$ are the associated binary relations of $M \boxplus M'$, M and M' on L, L_1 and L_2, respectively. Then, we have

$$a_i \le a_j \Leftrightarrow [a_i \le_1 a_j, \text{or} \ \ a_i \le_2 a_j, \ \text{ or } \ a_i = \bigwedge_{s=1}^{n} a_s \ \text{ and or } \ a_j = \bigvee_{s=n+1}^{n+m} a_s].$$

Hence, $(a_i \wedge a_j) \vee (a_i \wedge a_k) = a_i \wedge (a_j \vee (a_i \wedge a_k))$, for every $(a_i, a_j, a_k) \in L^3$, so $(L, \le)$ is a modular lattice and $M \boxplus M'$ is good. $\square$

Corollary 3.5. If C_n is the number of non isomorphic Comer hypergroups of order n, then $C_{n+m} \ge C_n C_m$, for all $n, m \in \mathbb{N}$.

Proposition 3.6. *For every $n \in \mathbb{N}$, $V_n \le C_n$.*

Example 1. Let T and T' be 2-triangular and 3-triangular matrixes. Then, $T \boxplus T'$ is a good matrix which is not very good.

By using the results of [1] we can count the number of Comer hypergroups (up to isomorphism) with the cardinality less than 13 which we summarize at the following table.

n=	1	2	3	4	5	6	7	8	9	10	11	12
Comer hypergroups	1	1	1	2	4	8	16	34	72	157	343	766

References

[1] R. Belohlavek and V. Vychodil, *Residuated Lattices of Size $\leq$ 12*, Mathematics and Statistics, 27 (2010), 147–161.

[2] J. Chvalina, *Commutative hypergroups in the sense of Marty and ordered sets*, Proceedings of the Summer School on General Algebra and Ordered Sets, 1994, Olomouc, Czech Republic, pp. 19-30.

[3] S. Comer, *Multi-valued algebras and their graphical representations*, Math. Comp. Sci. Dep. the Citadel. Charleston, South Carolina, 29409, July 1986.

[4] P. Corsini, *Prolegomena of Hypergroups Theory*, Aviani Editore, 1993.

[5] P. Corsini and V. Leoreanu, *Hypergroups and binary relations*, Algebra Universalis, 43 (2000), 321–330.

[6] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, in: Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[7] I. Cristea and M. Stefanescu, *Binary relations and reduced hypergroups*, Discrete Math., 308 (2008), 3537-3544.

[8] I. Cristea, M. Stefanescu and C. Angheluta, *About the fundamental relations defined on the hypergroupoids associated with binary relations*, European J. Combin., 32 (2011), 72-81.

[9] B. Davvaz and V. Leoreanu-Fotea, *Binary relations on ternary hypergroups*, Communications in Algebra, 38(10) (2010), 3621-3636.

[10] B. Davvaz and T. Vougiouklis, *n-Ary hypergroups*, Iranian Journal of Science and Technology, Transaction A, 30 (A2) (2006), 165-174.

[11] B. Davvaz, W.A. Dudek and S. Mirvakili, *Neutral elements, fundamental relations and n-ary hypersemigroups*, International Journal of Algebra and Computation, 19(4) (2009), 567-583.

[12] S. Hoskova and J. Chvalina, *Discrete transformation hypergroups and transformation hypergroups with phase tolerance space*, Discrete Math., 308 (2008), 4133-4143.

[13] M. Jafarpour and S.Sh. Mousavi, *A construction of hypersemigroups from a family of preordered semigroups*, Scientiae Mathematicae Japonicae On-line, e-2009, 587–594.

[14] M. Konstantinidou and K. Serafimidis, *Sur les P-supertreillis*, in: T. Vougiouklis, (Ed.), New Frontiers in Hyperstructures and Rel. Algebras, Hadronic Press, Palm Harbor, USA, 1996, pp. 139–148.

[15] V. Leoreanu-Fotea and B. Davvaz, *n-hypergroups and binary relations*, European J. Combin., 29 (2008), 1207–1218.

[16] V. Leoreanu-Fotea and B. Davvaz, *Join n-spaces and lattices*, Journal of Multiple-Valued Logic and Soft Computing, 15 (2009), 421–432.

[17] V. Leoreanu-Fotea and I.G. Rosenberg, *Hypergroupoids determined by lattices*, European J. Combin., 31 (2010), 925–931.

[18] J. Nieminen, *Join space graphs*, J. Geom., 33 (1988), 99–103.

[19] W. Prenowitz and J. Jantosciak, *Geometries and join spaces*, J. Reine Angew. Math., 257 (1972), 100–128.

[20] W. Prenowitz and J. Jantosciak, *Join geometries*, Springer-Verlag, UTM, 1979.

[21] I.G. Rosenberg, *Hypergroups and join spaces determined by relations*, Italian J. Pure Appl. Math., 4 (1998), 93–101.

[22] M. Erne, J. Heitzig and J. Reinhold, *On the Number of Distributive Lattices*, Electron. J. Comb., 9(1) (2002), #R24, 23 pp.

[23] S. Spartalis, *The hyperoperation relation and the Corsini's partial or not-partial hypergroupoids (A classification)*, Ital. J. Pure Appl. Math., 24 (2008), 97–112.

[24] J.C. Varlet, *Remarks on distributive lattices*, Bull. de l'Acad. Polonnaise des Sciences, Serie des Sciences Math., Astr. et Phys., 23 (1975), 1143–1147.

[25] T. Vougiouklis, *Hyperstructures and their Representations*, Hadronic Press, Palm Harbor, USA, 1994.