On the number of subsequences with a given sum in a finite abelian group

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Abstract

Suppose G is a finite abelian group and S is a sequence of elements in G. For any element g of G, let $N_g(S)$ denote the number of subsequences of S with sum g. The purpose of this paper is to investigate the lower bound for $N_g(S)$. In particular, we prove that either $N_g(S) = 0$ or $N_g(S) \geq 2^{|S|-D(G)+1}$, where $D(G)$ is the smallest positive integer ℓ such that every sequence over G of length at least ℓ has a nonempty zero-sum subsequence. We also characterize the structures of the extremal sequences for which the equality holds for some groups.

1 Introduction

Suppose G is a finite abelian group and S is a sequence over G. The enumeration of subsequences with certain prescribed properties is a classical topic in Combinatorial Number

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Theory going back to Erdős, Ginzburg and Ziv [6, 14, 15] who proved that $2n-1$ is the smallest integer such that every sequence S over a cyclic group C_n has a subsequence of length n with zero-sum. This raises the problem of determining the smallest positive integer ℓ such that every sequence S of length at least ℓ has a nonempty zero-sum subsequence. Such an integer ℓ is called the *Davenport constant* [4] of G, denoted by $D(G)$, which is still unknown in general.

For any g of G, let $N_g(S)$ denote the number of subsequences of S with sum g. In 1969, J. E. Olson [24] proved that $N_0(S) \geq 2^{|S|-D(G)+1}$ for every sequence S over G of length $|S| \geq D(G)$. Subsequently, several authors [1, 2, 3, 5, 8, 9, 11, 13, 16, 17, 18, 20] obtained a huge variety of results on the number of subsequences with prescribed properties. However, for any arbitrary g of G, the lower bound of $N_g(S)$ remains undetermined.

In this paper, we determine the best possible lower bound of $N_g(S)$ for an arbitrary g of G. We also characterize the structures of the extremal sequences which attain the lower bound for some groups.

2 Notation and lower bound

Our notation and terminology are consistent with [10]. We briefly gather some notions and fix the notation concerning sequences over abelian group. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and non-negative integers, respectively. For integers $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 : a \leq x \leq b\}$. Throughout, all abelian groups are written additively. For a positive integer n, let C_n denote a cyclic group with n elements.

For a sequence $S = g_1 \cdot \ldots \cdot g_m$ of elements in G, we use $\sigma(S) = \sum_{i=1}^{m} g_i$ denote the sum of S. By λ we denote the empty sequence and adopt the convention that $\sigma(\lambda) = 0$. A subsequence $T|S$ means $T = g_{i_1} \cdot \ldots \cdot g_{i_k}$ with $\{i_1, \ldots, i_k\} \subseteq [1, m]$; we denote by I_T the *index set* $\{i_1, \ldots, i_k\}$ of T, and identify two subsequences S_1 and S_2 if $I_{S_1} = I_{S_2}$. We denote $-T = (-g_{i_1}) \cdot \ldots \cdot (-g_{i_k})$. Let $S_1, \ldots, S_n$ be n subsequences of S, denote by $\gcd(S_1, \ldots, S_n)$ the subsequence of S with index set $I_{S_1} \bigcap \cdots \bigcap I_{S_n}$. We say two subsequences S_1 and S_2 are *disjoint* if $\gcd(S_1, S_2) = \lambda$. If S_1 and S_2 are disjoint, then we denote by S_1S_2 the subsequence with index set $I_{S_1} \bigcup I_{S_2}$; if $S_1|S_2$, we denote by $S_2S_1^{-1}$ the subsequence with index set $I_{S_2} \setminus I_{S_1}$. Define $\sum(S) = \{\sum_{i \in I} g_i : \phi \neq I \subseteq [1, m]\}$, and $\sum^{\bullet}(S) = \sum(S) \cup \{0\}$.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *zero-sum free sequence* if $0 \notin \sum(S)$,
- a *minimal zero-sum sequence* if $S \neq \lambda$, $\sigma(S) = 0$, and every $T|S$ with $1 \leq |T| < |S|$ is zero-sum free,
- a *unique factorial sequence* if $0 \nmid S$ and if $S = T_1 \cdot \ldots \cdot T_k S'$, where $T_1, \ldots, T_k$ are all the minimal zero-sum subsequences of S.

Define

$$\mathscr{N}_1(G) = \max\{|S| : S \text{ is a unique factorial sequence over } G\}$$

where the maximum is taken when S runs over all unique factorial sequences over G.

Remark 1. The concept of unique factorial sequence was first introduced by Narkiewicz in [21] for zero-sum sequence. For recent progress on unique factorial sequences we refer to [12].

For an element g of G, let

$$N_g(S) = |\{I_T : T|S \text{ and } \sigma(T) = g\}|$$

denote the number of subsequences T of S with sum $\sigma(T) = g$. Notice that we always have $N_0(S) \geq 1$.

Theorem 2. If S is a sequence over a finite abelian group G and $g \in \sum^\bullet(S)$, then $N_g(S) \geq 2^{|S|-D(G)+1}$.

Proof. We shall prove the theorem by induction on $m = |S|$. The case of $m \leq D(G) - 1$ is clear. We now consider the case of $m \geq D(G)$. Choose a subsequence $T|S$ of minimum length with $\sigma(T) = g$, and a nonempty zero-sum subsequence $W|T(-(ST^{-1}))$. By the minimality of $|T|$, W is not a subsequence of T, for otherwise TW^{-1} is a shorter subsequence of S with $\sigma(TW^{-1}) = g$. Choose a term $a|W$ with $a \nmid T$, and let $X = \gcd(W, T)$. Then, $-a|ST^{-1}$ such that $g = \sigma(T) \in \sum^\bullet(S(-a)^{-1})$ and $(g - \sigma(X)) - (0 - \sigma(X) - a) = g + a = \sigma(TX^{-1}(-(W(Xa)^{-1}))) \in \sum^\bullet(S(-a)^{-1})$. By the induction hypothesis, $N_g(S) = N_g(S(-a)^{-1}) + N_{g+a}(S(-a)^{-1}) \geq 2^{m-D(G)} + 2^{m-D(G)} = 2^{m-D(G)+1}$. This completes the proof of the theorem. □

Notice that the result in [24] that $N_0(S) \geq 2^{|S|-D(G)+1}$ for any sequence S over G, together with the following lemma, also gives Theorem 2.

Lemma 3. If S is a sequence over a finite abelian group G, then for any $T|S$ with $\sigma(T) = g \in \sum^\bullet(S)$,

$$N_g(S) = N_0(T(-(ST^{-1}))).$$

Proof. Let $\mathcal{A} = \{X|S : \sigma(X) = g\}$ and $\mathcal{B} = \{Y|T(-(ST^{-1})) : \sigma(Y) = 0\}$. It is clear that $|\mathcal{A}| = N_g(S)$ and $|\mathcal{B}| = N_0(T(-(ST^{-1})))$. Define the map $\varphi : \mathcal{A} \to \mathcal{B}$ by $\varphi(X) = TX_1^{-1}(-X_2)$ for any $X \in \mathcal{A}$, where $X_1 = \gcd(X, T)$ and $X_2 = \gcd(X, ST^{-1})$. It is straightforward to check that φ is a bijection, which implies $N_g(S) = N_0(T(-(ST^{-1})))$. □

We remark that the lower bound in Theorem 2 is best possible. For any $g \in G$ and any $m \geq D(G)-1$, we construct the extremal sequence S over G of length m with respect to g as follows: Take a zero-sum free sequence U over G with $|U| = D(G) - 1$. Clearly, U contains a subsequence T with $\sigma(T) = g$. For $S = T(-(UT^{-1}))0^{m-D(G)+1}$, by Lemma 3, $N_g(S) = N_0(U0^{m-D(G)+1}) = 2^{m-D(G)+1}$.

Proposition 4. If S is a sequence over a finite abelian group G such that $N_h(S) = 2^{|S|-D(G)+1}$ for some $h \in G$, then $N_g(S) \geq 2^{|S|-D(G)+1}$ for all $g \in G$.

Proof. If there exists g such that $N_g(S) < 2^{|S|-D(G)+1}$, then

$$N_h(S(h-g)) = N_h(S) + N_g(S) < 2^{|S|+1-D(G)+1}$$

is a contradiction to Theorem 2 since $h \in \sum^\bullet(S) \subseteq \sum^\bullet(S(h-g))$. □

3 The structures of extremal sequences

In this section, we study sequence S for which $N_g(S) = 2^{|S|-D(G)+1}$. By Lemma 3, we need only pay attention to the case $g = 0$. Also, as $N_g(0S) = 2N_g(S)$, it suffices to consider the case $0 \nmid S$. For $|S| \geq D(G) - 1$, define

$$E(S) = \{g \in G : N_g(S) = 2^{|S|-D(G)+1}\}.$$

Lemma 5. Suppose S is a sequence over a finite abelian group G with $0 \nmid S$, $|S| \geq D(G)$ and $0 \in E(S)$. If a is a term of a zero-sum subsequence T of S, then

$$E(S) + \{0, -a\} \subseteq E(Sa^{-1}).$$

Proof. Since $0, -a \in \sum^\bullet(Sa^{-1})$, by Theorem 2, $N_0(Sa^{-1}) \geq 2^{|S|-D(G)}$ and $N_{-a}(Sa^{-1}) \geq 2^{|S|-D(G)}$. On the other hand, $N_0(Sa^{-1}) + N_{-a}(Sa^{-1}) = N_0(S) = 2^{|S|-D(G)+1}$ and so $N_0(Sa^{-1}) = N_{-a}(Sa^{-1}) = 2^{|S|-D(G)}$. Hence, by Proposition 4, $N_g(Sa^{-1}) \geq 2^{|S|-D(G)}$ for all $g \in G$. Now, for every $h \in E(S)$, $N_h(Sa^{-1}) + N_{h-a}(Sa^{-1}) = N_h(S) = 2^{|S|-D(G)+1}$ and so $N_h(Sa^{-1}) = N_{h-a}(Sa^{-1}) = 2^{|S|-D(G)}$, i.e., $\{h, h-a\} \subseteq E(Sa^{-1})$. This proves $E(S) + \{0, -a\} \subseteq E(Sa^{-1})$. □

Lemma 6 ([14], Lemma 6.1.3, Lemma 6.1.4)**.** Let $G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with $n_1|n_2|\cdots|n_r$, and H be a subgroup of G, then $D(G) \geq D(H) + D(G/H) - 1$ and $D(G) \geq \sum_{i=1}^{r}(n_i - 1) + 1$.

Lemma 7. If S is a sequence over a finite abelian group G such that $E(S)$ contains a non-trivial subgroup H of G, then $H \cong \bigoplus_{i=1}^{r} C_2$ and $D(G) = D(G/H) + r$.

Proof. Suppose $H \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$, where $n_1|n_2|\cdots|n_r$, and assume that $S = g_1 \cdot \ldots \cdot g_m$. Consider the canonical map $\varphi : G \to G/H$ and let $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_m)$ be a sequence over G/H. Then

$$|H| \cdot 2^{|S|-D(G)+1} = \sum_{h \in H} N_h(S) = N_0(\varphi(S)) \geq 2^{|\varphi(S)|-D(G/H)+1}.$$

It follows from Lemma 6 that $|H| \geq 2^{D(G)-D(G/H)} \geq 2^{D(H)-1}$, and so

$$\prod_{i=1}^{r} n_i \geq 2^{\sum_{i=1}^{r}(n_i-1)} = \prod_{i=1}^{r} 2^{n_i-1}.$$

Hence, $n_i = 2$ for all i, which gives $H \cong \bigoplus_{i=1}^{r} C_2$ and $D(G) = D(G/H) + r$. □

Lemma 8. ([22], Proposition 9; [12], Lemma 3.9) Let G be a finite abelian group, and let $S = S_1 \cdot \ldots \cdot S_r$ be a unique factorial zero-sum sequence over G, where $S_1, \ldots, S_r$ are all the minimal zero-sum subsequences of S. Then, $|S_1| \cdots |S_r| \leq |G|$.

Lemma 9. Let G be a finite abelian group, and let $S = S_1 \cdot \ldots \cdot S_r S'$ be a unique factorial sequence over G, where $S_1, \ldots, S_r$ are all the minimal zero-sum subsequences of S and S' is empty or zero-sum free. Then, $|S_1| \cdots |S_r| \max\{1, |S'|\} \leq |G|$.

Proof. If $|S'| \leq 1$ then $|S_1| \cdots |S_r| \max\{1, |S'|\} = |S_1| \cdots |S_r| \leq |G|$ follows from Lemma 8. Now assume that $|S'| \geq 2$. In a similar way to the proof of Proposition 9 in [22] (or Lemma 3.9 in [12]) one can prove that $|S_1| \cdots |S_r||S'| \leq |G|$. □

Lemma 10. If G is a finite abelian group then $\mathscr{N}_1(G) \leq \log_2 |G| + D(G) - 1$.

Proof. Let S be a unique factorial sequence over G with $|S| = \mathscr{N}_1(G)$. Then, $S = S_1 \cdot \ldots \cdot S_r S'$ with $S_1, \ldots, S_r$ are all the minimal zero-sum subsequences of S. By Lemma 9, $|S_1| \cdots |S_r| \leq |G|$. It follows from $|S_i| \geq 2$ for every $i \in [1, r]$ that $r \leq \log_2 |G|$. Take an element $x_i \in S_i$ for every $i \in [1, r]$. Since $S_1, \ldots, S_r$ are all the minimal zero-sum subsequences of S, we have that $S_1 \cdot \ldots \cdot S_r S'(x_1 \cdot \ldots \cdot x_r)^{-1}$ is zero-sum free. It follows that $|S| - r = |S_1 \cdot \ldots \cdot S_r S'| - r \leq D(G) - 1$. Therefore, $\mathscr{N}_1(G) = |S| \leq \log_2 |G| + D(G) - 1$. □

Now, we consider the case $G = C_n$. Notice that $D(C_n) = n$.

Theorem 11. For $n \geq 3$, if S is a sequence over the cyclic group C_n with $0 \nmid S$ and $N_0(S) = 2^{|S|-n+1}$, then $n - 1 \leq |S| \leq n$ and $S = a^{|S|}$, where a generates C_n.

Proof. Suppose S is a sequence over the cyclic group C_n with $0 \nmid S$ and $N_0(S) = 2^{|S|-n+1}$. We first show by induction that

$$S = a^{|S|} \tag{1}$$

where $\langle a \rangle = C_n$. For $|S| = n - 1$, we have $N_0(S) = 1$, i.e., S is a zero-sum free sequence, and (1) follows readily.

For $|S| \geq n$, since $N_0(S) = 2^{|S|-n+1} \geq 2$, S contains at least one nonempty zero-sum subsequence T. Take an arbitrary term c from T. By Lemma 5, $0 \in E(Sc^{-1})$. It follows from the induction hypothesis that $Sc^{-1} = a^{|S|-1}$ for some a generating C_n. By the arbitrariness of c, we conclude that (1) holds.

To prove $|S| \leq n$, we suppose to the contrary that $|S| \geq n + 1$. By (1) and Lemma 5,

$$0 \in E(a^{n+1}). \tag{2}$$

We see that $N_0(a^{n+1}) \geq 1 + \binom{n+1}{n} > 4$, a contraction with (2). □

Notice that Theorem 11 is not true for $n = 2$, since for any sequence S over C_2 with $0 \nmid S$, we always have $N_0(S) = 2^{|S|-2+1}$.

While the structure of a sequence S over a general finite abelian group G with $0 \nmid S$ and $N_0(S) = 2^{|S|-D(G)+1}$ is still not known, we have the following result for the case when $|G|$ is odd.

Theorem 12. If S is a sequence over a finite abelian group G of odd order with $0 \nmid S$ and $N_0(S) = 2^{|S|-D(G)+1}$, then S is unique factorial and the number of minimal zero-sum subsequences of S is $|S| - D(G) + 1$, and therefore $|S| \leq \mathscr{N}_1(G) \leq D(G) - 1 + \log_2 |G|$.

Proof. We first note that if S is a unique factorial sequence, i.e., $S = S_1 \cdot \ldots \cdot S_\ell S'$ where $S_1, \ldots, S_\ell$ are all the minimal zero-sum subsequences of S, then $2^\ell = N_0(S) = 2^{|S|-D(G)+1}$, which implies that $\ell = |S| - D(G) + 1$, and that $|S| \leq \mathscr{N}_1(G) \leq \log_2 |G| + D(G) - 1$ follows from Lemma 10. Therefore, it suffices to show that S is a unique factorial sequence.

We proceed by induction on $|S|$. If $|S| = D(G)$, then $N_0(S) = 2$ and so S contains exactly one nonempty zero-sum subsequence, and we are done. Now assume

$$|S| \geq D(G) + 1.$$

If all the minimal zero-sum subsequences of S are pairwise disjoint, then the conclusion follows readily. So we may assume that there exist two distinct minimal zero-sum subsequences T_1 and T_2 with $\gcd(T_1, T_2) \neq \lambda$. Take a term $a | \gcd(T_1, T_2)$. By Lemma 5, $0 \in E(Sa^{-1})$ and so Sa^{-1} contains $r = |S| - D(G) \geq 1$ pairwise disjoint minimal zero-sum subsequences $T_3, T_4, \ldots, T_{r+2}$ by the induction hypothesis. Now we need the following claim.

Claim A. There is no term which is contained in exactly one T_i, where $i \in [1, r + 2]$.

Proof of Claim A. Assume to the contrary that, there is a term b such that $b | T_t$ for some $t \in [1, r + 2]$, and such that $b \nmid T_i$ for every $i \in [1, r + 2] \setminus \{t\}$. By Lemma 5, we have $0 \in E(Sb^{-1})$. It follows from the induction hypothesis that Sb^{-1} contains exactly r minimal zero-sum subsequences, which is a contradiction. This proves Claim A. □

Choose a term c in T_1 but not in T_2. By Claim A, we have that c is in another T_i, say T_{r+2} and so not in any of $T_3, T_4, \ldots, T_{r+1}$. Again Sc^{-1} contains exactly r disjoint minimal zero-sum subsequences, which are just $T_2, T_3, \ldots, T_{r+1}$. If $r \geq 2$, noticing that $\gcd(T_{r+1}, T_i) = \lambda$ for every $i \in [2, r + 2] \setminus \{r + 1\}$, it follows from Claim A that $T_{r+1} | T_1$, which is a contradiction to the minimality of T_1. Therefore,

$$r = 1.$$

Then $N_0(S)$=4 and T_1, T_2, T_3 are all the minimal zero-sum subsequences of S. If there is some $d | \gcd(T_1, T_2, T_3)$, then Sd^{-1} contains no minimal zero-sum subsequence, which is impossible. Thus $\gcd(T_1, T_2, T_3) = \lambda$. Let $X = \gcd(T_2, T_3), Y = \gcd(T_1, T_3)$ and $Z = \gcd(T_1, T_2)$. It follows from Claim A that $T_1 = YZ, T_2 = XZ$ and $T_3 = XY$. Therefore, $\sigma(Y) + \sigma(Z) = \sigma(X) + \sigma(Z) = \sigma(X) + \sigma(Y) = 0$. This gives that $2\sigma(X) = 2\sigma(Y) = 2\sigma(Z) = 0$. Since $|G|$ is odd, it follows that $\sigma(X) = 0$, which is a contradiction. This completes the proof of the theorem. □

If we further assume that $E(S) = \{0\}$ in Theorem 12, the structure of S can be further restricted.

Corollary 13. If S is a sequence over a finite abelian group G of odd order with $0 \nmid S$ and $E(S) = \{0\}$, then S is a unique factorial zero-sum sequence and the number of minimal zero-sum subsequences of S is $|S| - D(G) + 1$. Therefore, $|S| \leq \mathscr{N}_1(G) \leq \log_2 |G| + D(G) - 1$.

Proof. By Theorem 12, S is unique factorial and contains exactly $r = |S| - D(G) + 1$ minimal zero-sum subsequences $T_1, \ldots, T_r$ (say). Therefore, $S = T_1 \cdot \ldots \cdot T_r W$. For any subsequence X of S with $\sigma(X) = \sigma(W)$, if $W \nmid X$, then SX^{-1} is a zero-sum subsequence containing terms in W, which is impossible. So $W | X$, and then $\sigma(XW^{-1}) = 0$. This gives $X = T_{i_1} \cdot \ldots \cdot T_{i_s} W$ with $1 \leq i_1 < \cdots < i_s \leq r$. Hence, $N_{\sigma(W)}(S) = 2^r$ and then $\sigma(W) \in E(S) = \{0\}$ implying $W = \lambda$. Now $|S| \leq \mathscr{N}_1(G) \leq \log_2 |G| + D(G) - 1$ follows from Lemma 10. □

Remark 14. The following example shows that Theorem 12 does not hold for all finite abelian groups. Let $G = C_2 \oplus C_{2n_1} \oplus \cdots \oplus C_{2n_r} = \langle e\rangle \oplus \langle e_1\rangle \oplus \cdots \oplus \langle e_r\rangle$ with $1 \le n_1|\cdots|n_r$ and $D(G) = d^*(G)+1$. For any $m \ge D(G)+1$, take $S = e^{m-D(G)+2} \cdot \prod_{i=1}^{r} e_i^{2n_i-1}$. It is easy to check that $N_0(S) = \binom{k}{0} + \binom{k}{2} + \cdots + \binom{k}{2\lfloor \frac{k}{2}\rfloor} = 2^{k-1}$ where $k = m - D(G) + 2$, and that S is not a unique factorial sequence.

The property that S contains exactly $|S|-D(G)+1$ minimal zero-sum subsequences, all of which are pairwise disjoint, implies that $|S|$ is bounded as in the case of Theorem 11 for cyclic groups. In general, we have the following theorem.

Theorem 15. For any finite abelian group $G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with $n_1|n_2|\cdots|n_r$, (i) implies the three equivalent statements (ii), (iii) and (iv).

(i) Any sequence S over G with $0 \nmid S$ and $N_0(S) = 2^{|S|-D(G)+1}$, contains exactly $|S| - D(G) + 1$ minimal zero-sum subsequences, all of which are pairwise disjoint.

(ii) There is a natural number $t = t(G)$ such that $|S| \le t$ for every sequence S over G with $0 \nmid S$ and $N_0(S) = 2^{|S|-D(G)+1}$.

(iii) For any subgroup H of G isomorphic to C_2, $D(G) \ge D(G/H) + 2$.

(iv) For any sequence S over G, $E(S)$ contains no non-trivial subgroup of G.

Proof. (i) $\Rightarrow$ (ii). Since S contains exactly $|S|-D(G)+1$ minimal zero-sum subsequences, all of which are pairwise disjoint, we have that $|S| \ge 2(|S| - D(G) + 1)$ which gives $|S| \le 2D(G) - 2$.

(ii) $\Rightarrow$ (iii). Assume to the contrary that $D(G) = D(G/H) + 1$ for some subgroup $H = \{0, h\}$ of G. Let $\varphi : G \to G/H$ be the canonical map, and let $m = D(G/H)$. We choose a sequence $S = g_1 \cdot \ldots \cdot g_m$ over G such that $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_m)$ is a minimal zero-sum sequence over G/H, and $\sigma(S) = h$ in G. Since

$$N_0(S) + N_h(S) = N_0(\varphi(S)) = 2 = 2 \cdot 2^{|S|-D(G)+1}$$

and $N_0(S)$ and $N_h(S)$ are not zero, by theorem 2, $N_0(S) = N_h(S) = 2^{|S|-D(G)+1}$. Since $N_0(Sh^k) = N_0(Sh^{k-1}) + N_h(Sh^{k-1}) = N_h(Sh^k)$, by induction we have $N_0(Sh^k) = N_h(Sh^k) = 2^{|Sh^k|-D(G)+1}$ for all k, a contradiction to the assumption in (ii).

(iii) $\Rightarrow$ (iv). Suppose to the contrary that there exists a sequence S over G such that $E(S)$ contains a non-trivial subgroup H of G. By Lemma 7, $H \cong \bigoplus_{i=1}^{s} C_2$ and $D(G) = D(G/H)+s$. Hence, $E(S)$ contains a subgroup $H' \cong C_2$. If $D(G) \ge D(G/H')+2$, then by Lemma 6, $D(G) \ge D(G/H') + 2 \ge D(H/H') + D((G/H')/(H/H')) + 1 = s + 1 + D(G/H) > D(G)$, a contradiction.

(iv) $\Rightarrow$ (ii). For $|S| \ge D(G)$, that is, $N_0(S) = 2^{|S|-D(G)+1} > 1$, there exists a nonempty zero-sum subsequence T_1 of S and a term $a_1|T_1$. By Lemma 5, $0 \in E(S) \subseteq E(Sa_1^{-1})$. By (iv), $\langle -a_1\rangle \nsubseteq E(Sa_1^{-1})$. Let k be the minimum index such that $k(-a_1) \notin$

$E(Sa_1^{-1})$, that is, $\{0, -a_1, \ldots, (k-1)(-a_1)\} \subseteq E(Sa_1^{-1})$ but $k(-a_1) \notin E(Sa_1^{-1})$. Then, $N_{(k-1)(-a_1)}(Sa_1^{-1}) = 2^{|Sa_1^{-1}|-D(G)+1}$ but $N_{k(-a_1)}(Sa_1^{-1}) \neq 2^{|Sa_1^{-1}|-D(G)+1}$. Thus,

$$N_{(k-1)(-a_1)}(S) = N_{(k-1)(-a_1)}(Sa_1^{-1}) + N_{k(-a_1)}(Sa_1^{-1}) \neq 2^{|S|-D(G)+1}$$

and so $(k-1)(-a_1) \notin E(S)$. This means

$$E(S) \subsetneq E(Sa_1^{-1}).$$

If $|Sa_1^{-1}| \geq D(G)$, a similar argument shows that there exists a nonempty zero-sum subsequence T_2 of Sa_1^{-1} and a term $a_2|T_2$, thus, $E(Sa_1^{-1}) \subsetneq E(Sa_1^{-1}a_2^{-1})$. We continue this process to get $a_1, a_2, \ldots, a_{|S|-D(G)+1}$ of S such that

$$E(S) \subsetneq E(Sa_1^{-1}) \subsetneq \cdots \subsetneq E(Sa_1^{-1}a_2^{-1} \cdot \ldots \cdot a_{|S|-D(G)+1}^{-1}).$$

Since $|E(Sa_1^{-1}a_2^{-1} \cdot \ldots \cdot a_{|S|-D(G)+1}^{-1})| \leq |G|$, we conclude $|S| \leq D(G) + |G| - 1 := t$. □

4 Concluding remarks

We are interested in the structure of a sequence S over a finite abelian group G such that $N_0(S) = 2^{|S|-D(G)+1}$. Based on the experiences in Section 3, we have the following two conjectures.

Conjecture 16. Suppose G is a finite abelian group in which $D(G) \geq D(G/H) + 2$ for every subgroup H of G isomorphic to C_2. If S is a sequence over G with $0 \nmid S$ and $N_0(S) = 2^{|S|-D(G)+1}$, then S contains exactly $|S| - D(G) + 1$ minimal zero-sum subsequences, all of which are pairwise disjoint.

Notice that this conjecture holds when G is cyclic or $|G|$ is odd. The second conjecture concerns the length of S.

Conjecture 17. Suppose $G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ where $1 < n_1|n_2| \cdots |n_r$ and $D(G) = d^*(G) + 1 = \sum_{i=1}^r (n_i - 1) + 1$. Let S be a sequence over G such that $0 \nmid S$ and $E(S) \neq \emptyset$ contains no non-trivial subgroup of G, then $|S| \leq d^*(G) + r$.

The following example shows that if Conjecture 17 holds, then the upper bound $d^*(G)+r = \sum_{i=1}^r n_i$ is best possible. Let $G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r} = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_r \rangle$ with $1 < n_1|n_2| \cdots |n_r$. Clearly, $S = \prod_{i=1}^r e_i^{n_i}$ is an extremal sequence with respect to 0 and of length $d^*(G) + r$.

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