Mr. Paint and Mrs. Correct go fractional

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Submitted: Mar 14, 2011; Accepted: Jun 22, 2011; Published: Jul 1, 2011
Mathematics Subject Classifications: 05C15, 05C57, 68W27

Abstract

We study a fractional counterpart of the on-line list colouring game "Mr. Paint and Mrs. Correct" introduced recently by Schauz. We answer positively a question of Zhu by proving that for any given graph the on-line choice ratio and the (off-line) choice ratio coincide. On the other hand it is known from the paper of Alon et al. that the choice ratio equals the fractional chromatic number. It was also shown that the limits used in the definitions of these last two notions can be realised. We show that this is not the case for the on-line choice ratio. Both our results are obtained by exploring the strong links between the on-line choice ratio, and a new on-line game with probabilistic flavour which we introduce.

1 Introduction

We start with a short introduction of the basic notions. Consider a finite and simple graph $G = (V, E)$. A list assignment L is a function assigning a set $L(v)$ to each vertex $v \in V$. We say that L is a-long if $|L(v)| \geqslant a$ for each vertex $v \in V$.

A function Φ assigning a subset $\Phi(v) \subseteq L(v)$ to each vertex $v \in V$ is an (L, b)-colouring if the following conditions are met:

- $\forall v \in V \;\; |\Phi(v)| \geqslant b$,
- $\forall (u, w) \in E \;\; \Phi(u) \cap \Phi(w) = \emptyset$.

The graph G is (a, b)-colourable if there exists an (L_a, b)-colouring from a list assignment L_a in which $L_a(v) := \{0, \ldots, a-1\}$ for all $v \in V$. If for any given a-long list assignment L there exists an (L, b)-colouring, then G is called (a, b)-choosable.

Following [1] we define the fractional chromatic number $\chi_{\mathrm{F}}(G)$ and the choice ratio $\mathrm{ch}_{\mathrm{F}}(G)$:

- $\chi_{\mathrm{F}}(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-colourable} \right\}$,

- $\mathrm{ch}_{\mathrm{F}}(G) = \inf\left\{\frac{a}{b} : G \text{ is } (a,b)\text{-choosable}\right\}$.

Observe that the standard notions of chromatic number $\chi(G)$ and choice number $\mathrm{ch}(G)$ can be equivalently restated as follows:

- $\chi(G) = \min\{a : G \text{ is } (a,1)\text{-colourable}\}$,
- $\mathrm{ch}(G) = \min\{a : G \text{ is } (a,1)\text{-choosable}\}$.

Fact 1. *The values of the infimum in the definitions of $\chi_{\mathrm{F}}(G)$ and $\mathrm{ch}_{\mathrm{F}}(G)$ are attained.*

See [1] for a proof. As a consequence, the infimum can be replaced by the minimum in the definitions of $\chi_{\mathrm{F}}(G)$ and $\mathrm{ch}_{\mathrm{F}}(G)$.

The following inequalities between the four chromatic parameters follow directly from their definitions:

$$\begin{array}{ccc} \chi_{\mathrm{F}}(G) & \leqslant & \chi(G) \\ \wedge\!\!| & & \wedge\!\!| \\ \mathrm{ch}_{\mathrm{F}}(G) & \leqslant & \mathrm{ch}(G) \end{array}$$

The game "Mr. Paint and Mrs. Correct", or PC-game for short, introduced recently by Shauz [2], is a two player game. The game is played in rounds on a graph $G = (V, E)$ which is known in advance to both players Mr. Paint and Mrs. Correct. Moreover, two parameters a and b are fixed in advance. At the very beginning each vertex v has an empty list $L_0(v) = \emptyset$ assigned to it. These initially empty lists will eventually contain exactly a colours as Mr. Paint is going to fill the lists during the game. The goal of Mrs. Correct is to incrementally construct an (L, b)-colouring φ from the resulting list assignment. Mr. Paint, by smartly filling the lists $L(v)$, tries to prevent the assignment φ, being built by Mrs. Correct, from resulting in the correct (L, b)-colouring. At the very beginning Mrs. Correct is forced to set $\varphi_0(v) = \emptyset$ for all $v \in V$.

In the i-th round Mr. Paint chooses a nonempty subset of vertices $P_i \subseteq V$ and appends a new colour c_i to the lists assigned to the vertices in P_i so that:

$$L_i(v) = \begin{cases} L_{i-1}(v) \cup \{c_i\}, & \text{if } v \in P_i, \\ L_{i-1}(v), & \text{otherwise.} \end{cases}$$

It is forbidden for Mr. Paint to construct any lists having more than a elements. He instantly loses the game if he decides to break this rule.

At this point Mrs. Correct chooses an independent subset $C_i \subseteq P_i$ and assigns colour c_i to the vertices in C_i to get:

$$\varphi_i(v) = \begin{cases} \varphi_{i-1}(v) \cup \{c_i\}, & \text{if } v \in C_i, \\ \varphi_{i-1}(v), & \text{otherwise.} \end{cases}$$

If after l rounds all the lists are full, i.e. $|L_l(v)| = a$ for every $v \in V$, the game ends, after Mrs. Corrects move. Mrs. Correct wins if φ_l is a valid (L_l, b)-colouring of G that is, if $\varphi_l(v) \geqslant b$ for all $v \in V$. Otherwise Mr. Paint wins. Observe that the rule forbidding

Mr. Paint to construct too long lists assures that the game ends after at most $a \cdot |V|$ rounds.

We say that a graph G is on-line (a, b)-choosable if Mrs. Correct has a winning strategy in the PC-game on the graph G with the parameters a and b. The on-line choice number $\mathrm{ch}^{\mathrm{OL}}(G)$ and the on-line choice ratio $\mathrm{ch}_{\mathrm{F}}^{\mathrm{OL}}(G)$ are defined as follows:

- $\mathrm{ch}^{\mathrm{OL}}(G) = \min\{a : G \text{ is on-line } (a, 1)\text{-choosable}\}$,
- $\mathrm{ch}_{\mathrm{F}}^{\mathrm{OL}}(G) = \inf\left\{\frac{a}{b} : G \text{ is on-line } (a, b)\text{-choosable}\right\}$.

The following inequalities are straightforward:

$$\begin{array}{ccc} \mathrm{ch}_{\mathrm{F}}(G) & \leqslant & \mathrm{ch}(G) \\ \wedge\!\backslash & & \wedge\!\backslash \\ \mathrm{ch}_{\mathrm{F}}^{\mathrm{OL}}(G) & \leqslant & \mathrm{ch}^{\mathrm{OL}}(G) \end{array}$$

2 Balanced Distribution Game

To understand better the possible strategies for Mr. Paint and for Mrs. Correct in the PC-games we introduce a new game. It is again a two person game, between Nominator and Distributor, and we call it the Balanced Distribution Game, or the BD-game for short. In this game, informally, Nominator constructs a sequence (N_i) of subsets of a set Π of participants – one subset at a time. Distributor consecutively assigns goods to the participants in the new subset N_i. More precisely, he chooses a commodity δ_i out of a finite set Γ of commodities, and then distribute only goods of this commodity to the participants in N_i, one good for each of the participants. The number of goods of each commodity is unlimited. Distributor's goal is to ensure that at the end of the game each single participant got a well balanced spectrum of goods, that is, about equally many goods of each commodity.

In the preparatory phase of the game players establish some rules of the next phase of the game. Nominator chooses a finite set Π of participants and a finite set Γ of commodities of goods. Then he chooses a real number $0 \leqslant \varepsilon < 1$ – an acceptable deviation from the perfectly balanced distribution. Distributor responds by carefully picking a number $k > 0$. This number determines how many goods each of the participants will receive or, equivalently, how many times each participant will occur in the subsets N_i constructed by Nominator.

After the preparatory phase the game is played in rounds. In the i-th round Nominator nominates a non-empty subset $N_i \subseteq \Pi$. Sets presented by Nominator need not be pairwise different. Distributor responds by choosing any $\delta_i \in \Gamma$ and by giving one good of the commodity δ_i to each participant $\pi \in N_i$. The following two parameters are necessary to determine the winner in the BD-game.

- $\mathrm{occ}_i(\pi) = |\{j : \pi \in N_j \text{ and } j \leqslant i\}|$ – the number of goods given to the participant $\pi \in \Pi$ in the first i rounds,

- $\text{gds}_i(\pi,\gamma) = |\{j : \pi \in N_j \text{ and } \gamma = \delta_j \text{ and } j \leqslant i\}|$ – the number of goods of the commodity $\gamma \in \Gamma$ given to the participant $\pi \in \Pi$ in the first i rounds.

We will omit the subscripts and use the parameters $\text{occ}(\cdot)$ and $\text{gds}(\cdot,\cdot)$ which change their values as the game is played.

Nominator is not allowed to construct sequences with $\text{occ}(\pi) > k$. He instantly loses the game if he decides to break this rule. The game actually ends when each of the participants from Π was nominated exactly k times, i.e. $\text{occ}(\pi) = k$ for all $\pi \in \Pi$. As Nominator presents only non-empty subsets, the game comes to an end after at most $k \cdot |\Pi|$ rounds. Distributor wins if the inequality

$$\text{gds}(\pi,\gamma) \geqslant (1-\varepsilon)\frac{k}{|\Gamma|}$$

holds for each π and γ. Otherwise Nominator wins.

We will also consider a variant of BD-game in which values of Π, Γ and ε are fixed instead of being chosen by Nominator. We will see that the most important parameter is ε and the situation splits dramatically between $\varepsilon > 0$ and $\varepsilon = 0$.

One of the natural strategies for Distributor is to choose k "big enough" and then pick each δ_i "uniformly at random from Γ". We omit the somewhat troublesome details, as they will not be needed, and assure the reader that for $\varepsilon > 0$ this randomised strategy brings victory to Distributor with probability tending to 1 as k tends to infinity. The following lemma gives a straightforward derandomisation of this idea.

Lemma 2. *In BD-games with $\varepsilon > 0$, Distributor has a winning strategy.*

Proof. We describe a strategy that leads to the defeat of Nominator. In the preparatory phase Distributor, knowing Π, Γ and $\varepsilon > 0$, fixes an arbitrary numbering of $\Gamma = \{\gamma_0,\ldots,\gamma_{r-1}\}$ and sets

$$k = \left\lceil \frac{2^{|\Pi|}\,|\Gamma|}{\varepsilon} \right\rceil.$$

In the i-th round, Distributor needs to select δ_i for the participants in N_i nominated by Nominator. If this is the first time the subset N_i appears in the sequence $(N_1,\ldots,N_i)$ then Distributor simply chooses $\delta_i = \gamma_0$. Otherwise, let $j < i$ be the biggest number such that $N_j = N_i$. Distributor finds q such that $\delta_j = \gamma_q$ and chooses δ_i to be the next one in Γ, i.e. $\delta_i = \gamma_{((q+1) \mod r)}$.

Suppose the game end after round l. Consider the possible values of $\text{gds}(\pi,\gamma)$ at the end of the game. For a fixed participant π let γ_{min}, γ_{max} be such that $\text{gds}(\pi,\gamma_{min}) = \min_{\gamma\in\Gamma}\text{gds}(\pi,\gamma)$ and $\text{gds}(\pi,\gamma_{max}) = \max_{\gamma\in\Gamma}\text{gds}(\pi,\gamma)$.

Now suppose that a subset $N \subseteq \Pi$ occurs s times in the sequence $(N_1,\ldots,N_l)$. Observe that Distributor responded to those s occurrences of the subset N in a round robin manner, and γ_j was chosen to be the response exactly $\left\lceil\frac{s-j}{r}\right\rceil$ times. This means that subset N contributes at most 1 to the value of the difference $\text{gds}(\pi,\gamma_{max}) - \text{gds}(\pi,\gamma_{min})$. This, together with the fact that $\operatorname{gds}(\pi, \gamma_{max})$ is not smaller than the average value $\frac{k}{|\Gamma|}$ of $\operatorname{gds}(\pi, \gamma)$, allows us to state

$$\operatorname{gds}(\pi, \gamma) \geqslant \operatorname{gds}(\pi, \gamma_{min}) \geqslant \operatorname{gds}(\pi, \gamma_{max}) - 2^{|\Pi|} \geqslant \frac{k}{|\Gamma|} - 2^{|\Pi|} \geqslant \frac{k}{|\Gamma|}(1 - \varepsilon).$$

Since this holds for all participants $\pi \in \Pi$, Distributor has won. □

From the results of [1] one can infer that in the off-line version of Balanced Distribution Game (i.e. when all sets N_i are given to Distributor in a single batch) Distributor has a winning strategy even for $\varepsilon = 0$. In the BD-games with $|\Pi| = 1$ or $|\Gamma| = 1$ Distributor has an obvious winning strategy. In the BD-games with $\varepsilon = 0$ and $|\Pi| = 2$ a winning strategy for Distributor also exists:

- set $k = |\Gamma|$ (each of the two participants will receive one good of every commodity),
- before the i-th round renumber Π to $\{\pi_1, \pi_2\} = \Pi$ so that $\operatorname{occ}(\pi_1) \geqslant \operatorname{occ}(\pi_2)$,
- if $\pi_1 \in N_i$, set δ_i to be one of the goods not received yet by π_1,
- if $\pi_1 \notin N_i$, set δ_i to be one of the goods received by π_1 but not by π_2.

The next lemma shows that the situation changes dramatically in the remaining cases.

Lemma 3. *In BD-games with fixed $\varepsilon = 0$, $|\Pi| \geqslant 3$ and $|\Gamma| \geqslant 2$, Nominator has a winning strategy.*

Proof. Assume that ε, Π and Γ are given as in the Lemma and that Distributor has already set k. If $k = 1$ Nominator plays $N_1 = \Pi$ and wins instantly. We will assume $k \geqslant 2$ for the rest of the proof.

We call a configuration of the two participants (π_1, π_2) insecure if $\operatorname{occ}(\pi_1) = \operatorname{occ}(\pi_2)$ and $\operatorname{gds}(\pi_1, \gamma_1) \neq \operatorname{gds}(\pi_2, \gamma_1)$ for some γ_1. Observe that if at any point of the game Nominator finds an insecure configuration (π_1, π_2) then he can win the game using the following approach. Nominator simply plays $k - \operatorname{occ}(\pi_1)$ times the subset $\{\pi_1, \pi_2\}$. No matter how Distributor responds, $\operatorname{occ}(\pi_1) = \operatorname{occ}(\pi_2) = k$ and $\operatorname{gds}(\pi_1, \gamma_1) \neq \operatorname{gds}(\pi_2, \gamma_1)$ still holds. Without loss of generality this allows us to assume that $\operatorname{gds}(\pi_1, \gamma_1) \neq \frac{k}{|\Gamma|}$. This means that the goods given to π_1 are not in balance, and therefore $\operatorname{gds}(\pi_1, \gamma) < \frac{k}{|\Gamma|}$ for some $\gamma \in \Gamma$. Nominator still needs to finish the game and he will do so by playing singletons $N_i = \{\pi\}$ as long as there exists a π with $\operatorname{occ}(\pi) < k$. Distributor loses the game, for he failed to balance the distribution of the goods received by π_1.

Now we show how Nominator can win the game. He starts with fixing three different participants $\pi_1, \pi_2, \pi_3 \in \Pi$. In round 1 Nominator plays $N_1 = \{\pi_1, \pi_2\}$ and after Distributor's response δ_1 he plays $N_2 = \{\pi_1, \pi_3\}$. Distributor must respond with $\delta_2 = \delta_1$, for otherwise the configuration (π_2, π_3) becomes insecure. Nominator continues with $N_3 = \{\pi_2\}$

forcing $\delta_3 = \delta_1$. Indeed, for $\delta_3 \neq \delta_2 = \delta_1$ the configuration (π_1, π_2) is insecure. The following table presents the state of the game during the first rounds:

i	N_i	δ_i	$\operatorname{occ}(\pi_1)$	$\operatorname{occ}(\pi_2)$	$\operatorname{occ}(\pi_3)$
1	$\{\pi_1, \pi_2\}$	δ_1	1	1	0
2	$\{\pi_1, \pi_3\}$	δ_1	2	1	1
3	$\{\pi_2\}$	δ_1	2	2	1

Nominator may continue alternating the moves $\{\pi_1, \pi_3\}$ and $\{\pi_2\}$ and Distributor must always respond with the same δ_1 or create an insecure configuration. When Nominator can no longer continue the simple alternations, the situation is as follows:

i	N_i	δ_i	$\operatorname{occ}(\pi_1)$	$\operatorname{occ}(\pi_2)$	$\operatorname{occ}(\pi_3)$
$2k-2$	$\{\pi_1, \pi_3\}$	δ_1	k	$k-1$	$k-1$
$2k-1$	$\{\pi_2\}$	δ_1	k	k	$k-1$

Again Nominator can finish the game by playing singletons as long as needed. He wins, for the distributions of the goods owned by the participants π_1 and π_2 are as far from being balanced as one can get – all of the goods received by them are of the single commodity δ_1. This is unacceptable for $|\Gamma| \geqslant 2$. □

3 On-line choice ratio

Now that we understand the basic mechanics of the BD-game, we will use this knowledge to prove the main results concerning the on-line choice ratio $\operatorname{ch}^{\mathrm{OL}}_{\mathrm{F}}(G)$.

Theorem 4. *For any graph G we have $\operatorname{ch}^{\mathrm{OL}}_{\mathrm{F}}(G) = \chi_{\mathrm{F}}(G)$.*

Proof. Recall that inequality $\operatorname{ch}^{\mathrm{OL}}_{\mathrm{F}}(G) \geqslant \chi_{\mathrm{F}}(G)$ follows immediately from the definitions. We are going to prove the converse inequality by reducing the PC-game to the BD-game. For the graph $G = (V, E)$, let L_a be the list assignment with $L_a(v) = \{0, \dots, a-1\}$ for all $v \in V$. Fact 1 supplies us with a, b such that $\chi_{\mathrm{F}}(G) = \frac{a}{b}$ and an (L_a, b)-colouring Φ. Given any real number $\tau > 0$ we will find a' and b' such that G is on-line (a', b')-choosable and $\frac{a'}{b'} \leqslant \frac{a}{b}(1+\tau)$. This will allow us to conclude that $\operatorname{ch}^{\mathrm{OL}}_{\mathrm{F}}(G) = \frac{a}{b}$.

To find a' and b' and construct a strategy for Mrs. Correct, we are going to use Distributor's winning strategy in the Balanced Distribution Game and the colouring Φ. We may imagine that Mrs. Correct plays simultaneously as Nominator in the BD-game (with $\Pi = V$, $\Gamma = \{0, \dots, a-1\}$ and $\varepsilon = \frac{\tau}{1+\tau}$), against Distributor using a winning strategy. She translates P_i played by Mr. Paint to $N_i := P_i$ for the BD-game, and then takes Distributor's response δ_i to calculate her response in the PC-game. The idea is that, at the end of the game, for each vertex v, Distributor partitioned the occurrences of v in the sets P_i into a different types of about equal size. For each occurrence of v in P_i, Mrs. Correct is going to include vertex v in her response C_i if the type chosen by Distributor coincides with one of the colours assigned to v in the colouring Φ. Mrs. Correct's strategy grants that a fraction of about $\frac{a}{b}$ of the occurrences will be included in her responses. In detail, we proceed as follows.

By Lemma 2 we know that Distributor has a winning strategy in the BD-game with $\Pi = V$, $\Gamma = \{0, \ldots, a-1\}$ and $\varepsilon = \frac{\tau}{1+\tau}$. In particular this strategy tells Distributor the value of k with which to respond initially. Now we are ready to set $a' = k$ and $b' = \left\lceil (1-\varepsilon)\frac{b}{a}k \right\rceil$ and start the PC-game. Note that

$$\frac{a'}{b'} \leqslant \frac{a}{b}\frac{1}{1-\varepsilon} = \frac{a}{b}(1+\tau).$$

In the i-th round, Mr. Paint presents a set P_i. In order to find her response $C_i \subseteq P_i$, Mrs. Correct plays $N_i = P_i$ as Nominator in the BD-game. Distributor responds with $\delta_i \in \{0, \ldots, a-1\}$. Mrs. Correct includes vertex v into her response $C_i \subseteq P_i$ if and only if δ_i is one of the colours assigned to v in the colouring Φ, i.e.

$$C_i = P_i \cap \{v \in V : \delta_i \in \Phi(v)\}.$$

This is a valid response in the PC-game, as $\{v \in V : \delta_i \in \Phi(v)\}$ is an independent set.

The PC-game ends when each list contains $a' = k$ colours. This means that every participant was nominated exactly k times and the BD-game finishes in the very same moment. Distributor wins the BD-game, thus $\mathrm{gds}\,(v,d) \geqslant (1-\varepsilon)\frac{k}{a}$ for each vertex v and $d \in \{0, \ldots, a-1\}$. Since $|\Phi(v)| = b$ we know that each vertex was given a set $\varphi(v)$ of at least b' colours in the colouring constructed by Mrs. Correct, for

$$|\varphi(v)| = \sum_{d \in \Phi(v)} \mathrm{gds}\,(v,d) \geqslant \left\lceil b(1-\varepsilon)\frac{k}{a} \right\rceil = b'.$$

□

The next theorem shows that, in contrast to Fact 1, the infimum in the definition of the on-line choice ratio is not always reached.

Theorem 5. *There is an infinite family $\mathcal{G}$ of graphs, such that for every graph $G \in \mathcal{G}$ we have*

$$\mathrm{ch}^{\mathrm{OL}}_{\mathrm{F}}(G) \notin \left\{ \frac{a}{b} : G \text{ is on-line } (a,b)\text{-choosable} \right\}.$$

Proof. For the finite sets Π, Γ with $|\Pi| \geqslant 3$ and $|\Gamma| \geqslant 2$ let the graph $G_{\Pi,\Gamma} = (V, E)$ be a complete $|\Gamma|$-partite graph with $|\Pi|$ vertices in each partite set. For the further reference we simply put:

- $V = \Pi \times \Gamma$,
- $E = \{\{(\pi_1, \gamma_1), (\pi_2, \gamma_2)\} \subseteq V : \gamma_1 \neq \gamma_2\}$.

The graph $G_{\Pi,\Gamma}$ is obviously $(|\Gamma|, 1)$-colourable – one can assign a different colour to each partite set $\Pi \times \{\gamma\}$. Observe that graph G contains $|\Gamma|$-cliques, for example $\{\pi\} \times \Gamma$ for any $\pi \in \Pi$. Assume that G is (a, b)-colourable. Colours assigned to the vertices of any $|\Gamma|$-clique need to be different, thus $a \geqslant b \cdot |\Gamma|$ holds. It follows that $\chi_{\mathrm{F}}(G) = \chi(G) = |\Gamma|$. Theorem 4 allows us to conclude that $\mathrm{ch}_{\mathrm{F}}^{\mathrm{OL}}(G) = |\Gamma|$.

Put $\mathcal{G} = \{G_{\Pi,\Gamma} : |\Pi| \geqslant 3 \wedge |\Gamma| \geqslant 2\}$. Now suppose, to the contrary, that some $G_{\Pi,\Gamma} \in \mathcal{G}$ is on-line (a, b)-choosable for some a, b with $\frac{a}{b} = |\Gamma|$. This means that Mrs. Correct has a winning strategy $\mathcal{S}$ in the PC-game on the graph $G_{\Pi,\Gamma}$ with the parameters a and b.

Leading to a contradiction with Lemma 3 we will find a winning strategy for Distributor in the BD-game with Π, Γ and $\varepsilon = 0$ using a reduction to the PC-game on the graph $G_{\Pi,\Gamma}$ so that strategy $\mathcal{S}$ can be used.

The construction of Distributor's strategy is as follows. In the preparatory phase of the BD-game Distributor responds with $k = a$. In the i-th round, when Nominator nominates $N_i \subseteq \Pi$, Distributor plays $P_i = N_i \times \Gamma$ in the PC-game as Mr. Paint. Strategy $\mathcal{S}$ tells Mrs. Correct to respond with an independent set $C_i \subseteq P_i$. Set C_i contains vertices from at most one copy $\Pi \times \{\gamma\}$ of Π. Distributor sets his response δ_i in the BD-game to be γ such that $C_i \subseteq N_i \times \{\gamma\}$. Observe that the participant π receives one good of the commodity γ each time the vertex (π, γ) is present in the set C_i.

The BD-game ends when each participant was nominated a times. This means that each vertex in G has a list of length a attached to it. The PC-game is over and Mrs. Correct has won using the strategy $\mathcal{S}$, so each vertex is present in at least b of the sets C_i. This means that $\mathrm{gds}(\pi, \gamma) \geqslant b = \frac{k}{|\Gamma|}$, and that Distributor has managed to construct a perfectly balanced distribution. This contradiction with Lemma 3 ends the proof. $\square$

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