A strengthening of Brooks' Theorem for line graphs

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Abstract

We prove that if G is the line graph of a multigraph, then the chromatic number $\chi(G)$ of G is at most $\max\left\{\omega(G), \frac{7\Delta(G)+10}{8}\right\}$ where $\omega(G)$ and $\Delta(G)$ are the clique number and the maximum degree of G, respectively. Thus Brooks' Theorem holds for line graphs of multigraphs in much stronger form. Using similar methods we then prove that if G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 9$, then G contains a clique on $\Delta(G)$ vertices. Thus the Borodin-Kostochka Conjecture holds for line graphs of multigraphs.

1 Introduction

We define nonstandard notation when it is first used. For standard notation and terminology see [2]. The clique number of a graph is a trivial lower bound on the chromatic number. Brooks' Theorem gives a sufficient condition for this lower bound to be achieved.

Theorem 1 (Brooks [4])**.** *If G is a graph with $\Delta(G) \geq 3$ and $\chi(G) \geq \Delta(G) + 1$, then $\omega(G) = \chi(G)$.*

We give a much weaker condition for the lower bound to be achieved when G is the line graph of a multigraph.

Theorem 2. *If G is the line graph of a multigraph with $\chi(G) > \frac{7\Delta(G)+10}{8}$, then $\omega(G) = \chi(G)$.*

Combining this with an upper bound of Molloy and Reed [16] on the fractional chromatic number and partial results on the Goldberg Conjecture [8] yields yet another proof of the following result (see [14] for the original proof and [17] for further remarks and a different proof).

Theorem 3 (King, Reed and Vetta [14])**.** *If G is the line graph of a multigraph, then $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$.*

Reed [18] conjectures that the bound $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$ holds for all graphs G. For further information about Reed's conjecture, see King's thesis [11] and King and Reed's proof of the conjecture for quasi-line graphs [13]. Back in the 1970's Borodin and Kostochka [3] conjectured the following.

Conjecture 4 (Borodin and Kostochka [3])**.** *If G is a graph with $\chi(G) \geq \Delta(G) \geq 9$, then G contains a $K_{\Delta(G)}$.*

In [19] Reed proved this conjecture for $\Delta(G) \geq 10^{14}$. The only known connected counterexample for the $\Delta(G) = 8$ case is the line graph of a 5-cycle where each edge has multiplicity 3 (that is, $G = L(3 \cdot C_5)$). We prove that there are no counterexamples that are the line graph of a multigraph for $\Delta(G) \geq 9$. This is tight since the above counterexample for $\Delta(G) = 8$ is a line graph of a multigraph.

Theorem 5. *If G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 9$, then G contains a $K_{\Delta(G)}$.*

In [7], Dhurandhar proved the Borodin-Kostochka Conjecture for a superset of line graphs of *simple* graphs defined by excluding the claw, $K_5 - e$ and another graph D as induced subgraphs. Kierstead and Schmerl [10] improved this by removing the need to exclude D. We note that there is no containment relation between the line graphs of multigraphs and the class of graphs containing no induced claw and no induced $K_5 - e$.

2 The proofs

Lemma 6. *Fix $k \geq 0$. Let H be a multigraph and put $G = L(H)$. Suppose $\chi(G) = \Delta(G) + 1 - k$. If $xy \in E(H)$ is critical and $\mu(xy) \geq 2k + 2$, then xy is contained in a $\chi(G)$-clique in G.*

Proof. Let $xy \in E(H)$ be a critical edge with $\mu(xy) \geq 2k + 2$. Let A be the set of all edges incident with both x and y. Let B be the set of edges incident with either x or y but not both. Then, in G, A is a clique joined to B and B is the complement of a bipartite graph. Put $F = G[A \cup B]$. Since xy is critical, we have a $\chi(G) - 1$ coloring of $G - F$. Viewed as a partial $\chi(G) - 1$ coloring of G this leaves a list assignment L on F with $|L(v)| = \chi(G) - 1 - (d_G(v) - d_F(v)) = d_F(v) - k + \Delta(G) - d_G(v)$ for each $v \in V(F)$. Put $j = k + d_G(xy) - \Delta(G)$.

Let M be a maximum matching in the complement of B. First suppose $|M| \leq j$. Then, since B is perfect, $\omega(B) = \chi(B)$ and we have

$$\begin{aligned} \omega(F) &= \omega(A) + \omega(B) = |A| + \chi(B) \\ &\geq |A| + |B| - j = d_G(xy) + 1 - j \\ &= \Delta(G) + 1 - k = \chi(G). \end{aligned}$$

Thus xy is contained in a $\chi(G)$-clique in G.

Hence we may assume that $|M| \geq j+1$. Let $\{\{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\}\}$ be a matching in the complement of B. Then, for each $1 \leq i \leq j+1$ we have

$$\begin{aligned}|L(x_i)| + |L(y_i)| &\geq d_F(x_i) + d_F(y_i) - 2k \\ &\geq |B| - 2 + 2|A| - 2k \\ &= d_G(xy) + |A| - 2k - 1 \\ &\geq d_G(xy) + 1.\end{aligned}$$

Here the second inequality follows since $\alpha(B) \leq 2$ and the last since $|A| = \mu(xy) \geq 2k+2$. Since the lists together contain at most $\chi(G) - 1 = \Delta(G) - k$ colors we see that for each i,

$$\begin{aligned}|L(x_i) \cap L(y_i)| &\geq |L(x_i)| + |L(y_i)| - (\Delta(G) - k) \\ &\geq d_G(xy) + 1 - \Delta(G) + k \\ &= j + 1.\end{aligned}$$

Thus we may color the vertices in the pairs $\{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\}$ from L using one color for each pair. Since $|A| \geq k+1$ we can extend this to a coloring of B from L by coloring greedily. But each vertex in A has $j+1$ colors used twice on its neighborhood, thus each vertex in A is left with a list of size at least $d_A(v) - k + \Delta(G) - d_G(v) + j + 1 = d_A(v) + 1$. Hence we can complete the $(\chi(G)-1)$-coloring to all of F by coloring greedily. This contradiction completes the proof. □

Theorem 7. *If G is the line graph of a multigraph H and G is vertex critical, then*

$$\chi(G) \leq \max\left\{\omega(G), \Delta(G) + 1 - \frac{\mu(H) - 1}{2}\right\}.$$

Proof. Let G be the line graph of a multigraph H such that G is vertex critical. Say $\chi(G) = \Delta(G) + 1 - k$. Suppose $\chi(G) > \omega(G)$. Since G is vertex critical, every edge in H is critical. Hence, by Lemma 6, $\mu(H) \leq 2k+1$. That is, $\mu(H) \leq 2(\Delta(G) + 1 - \chi(G)) + 1$. The theorem follows. □

This upper bound is tight. To see this, let $H_t = t \cdot C_5$ (i.e. C_5 where each edge has multiplicity t) and put $G_t = L(H_t)$. As Catlin [6] showed, for odd t we have $\chi(G_t) = \frac{5t+1}{2}$, $\Delta(G_t) = 3t - 1$, and $\omega(G_t) = 2t$. Since $\mu(H_t) = t$, the upper bound is achieved.

We need the following lemma which is a consequence of the fan equation (see [1, 5, 8, 9]).

Lemma 8. *Let G be the line graph of a multigraph H. Suppose G is vertex critical with $\chi(G) > \Delta(H)$. Then, for any $x \in V(H)$ there exist $z_1, z_2 \in N_H(x)$ such that $z_1 \neq z_2$ and*

- *$\chi(G) \leq d_H(z_1) + \mu(xz_1)$,*
- *$2\chi(G) \leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2)$.*

Lemma 9. *Let G be the line graph of a multigraph H. If G is vertex critical with $\chi(G) > \Delta(H)$, then*

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$

Proof. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. By Lemma 8 we have $z \in N_H(x)$ such that $\chi(G) \leq d_H(z) + \mu(xz)$. Hence

$$\Delta(G) + 1 \geq d_H(x) + d_H(z) - \mu(xz) \geq d_H(x) + \chi(G) - 2\mu(xz).$$

Which gives

$$\chi(G) \leq \Delta(G) + 1 - \Delta(H) + 2\mu(H).$$

Adding Vizing's inequality $\chi(G) \leq \Delta(H) + \mu(H)$ gives the desired result. $\square$

Combining this with Theorem 7 we get the following upper bound.

Theorem 10. *If G is the line graph of a multigraph, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{7\Delta(G) + 10}{8}\right\}.$$

Proof. Suppose not and choose a counterexample G with the minimum number of vertices. Say $G = L(H)$. Plainly, G is vertex critical. Suppose $\chi(G) > \omega(G)$. By Theorem 7 we have

$$\chi(G) \leq \Delta(G) + 1 - \frac{\mu(H) - 1}{2}.$$

By Lemma 9 we have

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$

Adding three times the first inequality to the second gives

$$4\chi(G) \leq \frac{7}{2}(\Delta(G) + 1) + \frac{3}{2}.$$

The theorem follows. $\square$

Corollary 11. *If G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 11$, then G contains a $K_{\Delta(G)}$.*

With a little more care we can get the 11 down to 9. Our analysis will be simpler if we can inductively reduce to the $\Delta(G) = 9$ case. This reduction is easy using the following lemma from [17] (it also follows from a lemma of Kostochka in [15]). Recently, King [12] improved the $\omega(G) \geq \frac{3}{4}(\Delta(G) + 1)$ condition to the weakest possible condition $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$.

Lemma 12. *If G is a graph with $\omega(G) \geq \frac{3}{4}(\Delta(G) + 1)$, then G has an independent set I such that $\omega(G - I) < \omega(G)$.*

Proof of Theorem 5. Suppose the theorem is false and choose a counterexample F minimizing $\Delta(F)$. By Brooks' Theorem we must have $\chi(F) = \Delta(F)$. Suppose $\Delta(F) \geq 10$. By Lemma 12, we have an independent set I in F such that $\omega(F - I) < \omega(F)$. Expand I to a maximal independent set M and put $T = F - M$. Then $\chi(T) \geq \Delta(F) - 1$ and $\Delta(T) \leq \Delta(F) - 1$. Hence, by minimality of $\Delta(F)$ and Brooks' Theorem, $\omega(F) \geq \omega(T) + 1 \geq \Delta(F)$. This is a contradiction, hence $\chi(F) = \Delta(F) = 9$.

Let G be a 9-critical subgraph of F. Then G is a line graph of a multigraph. If $\Delta(G) \leq 8$, then G is K_9 by Brooks' Theorem giving a contradiction. Hence $\Delta(G) \geq 9$. Since G is critical, it is also connected.

Let H be such that $G = L(H)$. Then by Lemma 6 and Lemma 9 we know that $\mu(H) = 3$. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. Then we have $z_1, z_2 \in N_H(x)$ as in Lemma 8. This gives

$$\begin{aligned} 9 &\leq d_H(z_1) + \mu(xz_1), && (1) \\ 18 &\leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2). && (2) \end{aligned}$$

In addition, we have for $i = 1, 2$,

$$9 \geq d_H(x) + d_H(z_i) - \mu(xz_i) - 1 = \Delta(H) + d_H(z_i) - \mu(xz_i) - 1.$$

Thus,

$$\begin{aligned} \Delta(H) &\leq 2\mu(xz_1) + 1 \leq 7, && (3) \\ \Delta(H) &\leq \mu(xz_1) + \mu(xz_2) + 1. && (4) \end{aligned}$$

Now, let $ab \in E(H)$ with $\mu(ab) = 3$. Then, since G is vertex critical, we have $8 = \Delta(G) - 1 \leq d_H(a) + d_H(b) - \mu(ab) - 1 \leq 2\Delta(H) - 4$. Thus $\Delta(H) \geq 6$. Hence we have $6 \leq \Delta(H) \leq 7$. Thus, by (3), we must have $\mu(xz_1) = 3$.

First, suppose $\Delta(H) = 7$. Then, by (4) we have $\mu(xz_2) = 3$. Let y be the other neighbor of x. Then $\mu(xy) = 1$ and thus $d_H(x) + d_H(y) - 2 \leq 9$. That gives $d_H(y) \leq 4$. Then we have vertices $w_1, w_2 \in N_H(y)$ guaranteed by Lemma 8. Note that $x \notin \{w_1, w_2\}$. Now $4 \geq d_H(y) \geq 1 + \mu(yw_1) + \mu(yw_2)$. Thus $\mu(yw_1) + \mu(yw_2) \leq 3$. This gives $d_H(w_1) + d_H(w_2) \geq 2\Delta(G) - 3 = 15$ contradicting $\Delta(H) \leq 7$.

Thus we must have $\Delta(H) = 6$. By (1) we have $d_H(z_1) = 6$. Then, applying (2) gives $\mu(xz_2) = 3$ and $d_H(z_2) = 6$. Since x was an arbitrary vertex of maximum degree and H is connected we conclude that $G = L(3 \cdot C_n)$ for some $n \geq 4$. But no such graph is 9-chromatic by Brooks' Theorem. □

3 Some conjectures

The graphs $G_t = L(t \cdot C_5)$ discussed above show that the following upper bounds would be tight. Creating a counterexample would require some new construction technique that might lead to more counterexamples to Borodin-Kostochka for $\Delta = 8$.

Conjecture 13. *If G is the line graph of a multigraph, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{5\Delta(G)+8}{6}\right\}.$$

This would follow if the $3\mu(H)$ in Lemma 9 could be improved to $2\mu(H)+1$. The following weaker statement would imply Conjecture 13 in a similar fashion.

Conjecture 14. *If G is the line graph of a multigraph H, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{\Delta(G)+2}{2} + \mu(H)\right\}.$$

Since we always have $\Delta(H) \geq \frac{\Delta(G)+2}{2}$, this can be seen as an improvement of Vizing's Theorem for graphs with $\omega(G) < \chi(G)$.

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