Note on highly connected monochromatic subgraphs in 2-colored complete graphs

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Abstract

In this note, we improve upon some recent results concerning the existence of large monochromatic, highly connected subgraphs in a 2-coloring of a complete graph. In particular, we show that if $n \geq 6.5(k-1)$, then in any 2-coloring of the edges of K_n, there exists a monochromatic k-connected subgraph of order at least $n-2(k-1)$. Our result improves upon several recent results by a variety of authors.

1 Introduction

It is easy to see that for any graph G, either G or its complement is connected. This is equivalent to saying there exists a connected color in any 2-coloring of K_n. However, when we try to find a subgraph with higher connectivity, we cannot hope to find such a spanning subgraph. In order to see this, consider the following example. All standard notation comes from [3].

Consider the following example from [1]. Let $G_n = H_1 \cup \cdots \cup H_5$ where H_i is a red complete graph K_{k-1} for $i \leq 4$ and H_5 is a red $K_{n-4(k-1)}$ where $n > 4(k-1)$. To this structure, we add all possible red edges between H_5, H_1 and H_2 and from H_1 to H_3 and from H_2 to H_4. All edges not already colored in red are colored in blue. In either color, there is no k-connected subgraph of order larger than $n-2(k-1)$.

Since a spanning monochromatic subgraph is more than we could hope for, we consider finding a highly connected subgraph that is as large as possible. Along this line, Bollobás and Gyárfás proposed the following conjecture.

Conjecture 1 ([1]) *For $n > 4(k-1)$, every 2-coloring of K_n contains a k-connected monochromatic subgraph with at least $n-2(k-1)$ vertices.*

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In order to see that the bound on n is the best possible, consider the example G_n above with $n = 4(k-1)$ (so $H_5 = \emptyset$). In [1], the authors showed that this conjecture is true for $k \leq 2$. Also, in [4], Liu, Morris and Prince showed the conjecture holds for $k = 3$, but for other cases, it remains open. As a weaker result, in [6] the authors proved the following.

Theorem 1 ([6]) *If $n \geq 13k-15$ then every 2-coloring of K_n contains a monochromatic k-connected subgraph of order at least $n-2(k-1)$.*

In a related result, Bollobás and Gyárfás also proved the following.

Theorem 2 ([1]) *If Conjecture 1 holds for $4(k-1) < n < 7(k-1)$ then Conjecture 1 is true.*

In this note, we improve both of these results as follows:

Theorem 3 *If $n > 6.5(k-1)$ then any 2-coloring of K_n contains a monochromatic k-connected subgraph of order at least $n-2(k-1)$.*

By improving the constant from 13 to 6.5, we also slightly improve other results from [5] in some cases. As these improvements are very minor, we omit details. Since any k-connected graph has the minimum degree at least k, we immediately obtain the following corollary.

Corollary 4 *If $n > 6.5(k-1)$, then any 2-coloring of K_n contains a monochromatic subgraph of order at least $n-2(k-1)$ with the minimum degree at least k.*

This corollary slightly improves a result in [2], which deals with the monochromatic large subgraph with a specified minimum degree in general graphs. When we focus on complete graphs, their work shows that the conclusion holds if $n \geq 7k+4$.

2 Proof of Theorem 3

Consider a 2-coloring G of K_n with the colors red and blue. The proof proceeds by induction on k. The cases for $k \leq 2$ follow from [1] and the case $k = 3$ follows from [4] but we will not need this assuption so we simply suppose $k \geq 3$. By induction, there exists a $(k-1)$-connected subgraph in one color (suppose red) of order at least $n-2(k-2)$. If this subgraph is k-connected, this is a desired subgraph so we may assume the connectivity is exactly $k-1$.

Let G_r be the largest $(k-1)$-connected red subgraph and consider a minimum cutset C (of order $k-1$) of G_r. Let A^C and B^C be a bipartition of the vertices of $G_r \setminus C$ such that A^C (and likewise B^C) is the union of vertices in components of $G_r \setminus C$ and we choose such unions with $|A^C| \geq |B^C|$ and $|B^C|$ maximum. Choose such a cutset C so that $|B^C|$

is maximized and define $A' = A^C$ and $B' = B^C$. By definition, all edges between A' and B' are blue. This forms a complete bipartite graph in blue. Define $D' = G \setminus G_r$.

First suppose $|B'| \geq k$, which implies that this blue complete bipartite graph is k-connected. Note that $|D'| \leq 2(k-2)$ and, since $|G_r|$ is maximum, every vertex in D' has at most $k-2$ red edges to G_r. This means that each vertex of D' must have at least $|G_r| - (k-2)$ blue edges to G_r. More specifically, each vertex must have at least $|A' \cup B'| - (k-2) > k$ blue edges to $A' \cup B'$ (since $n \geq 5k-7$). This means that $A' \cup B' \cup D'$ induces a blue k-connected graph of order exactly $n-(k-1)$, thus proving the theorem in this case.

Hence, we assume $|B'| < k$. Let B be the set of vertices satisfying the following conditions:

1. $|B|$ is maximum subject to $|B| < 3(k-1)$.

2. Each vertex of B has at most $k-1$ red edges to $G \setminus B$.

Certainly such a set B exists since both D' and $D' \cup B'$ satisfy Property 2 and we know that $|D'| \leq 2(k-1)$ and $|B'| \geq 1$.

Claim 1 $|B| \geq 2(k-1)$.

Proof of Claim 1: Suppose $|B| < 2(k-1)$ and consider the graph G_r induced on the red edges in $G \setminus B$. If this graph is k-connected, it would be a desired subgraph so we know $\kappa(G_r) \leq k-1$. As above, if there exists a cutset C' of G_r and a partition of the components of $G_r \setminus C'$ so that each part has order at least k, then we could find a k-connected blue subgraph of order at least $n - |C'| \geq n-(k-1)$ which would again be a desired subgraph (note that each vertex of B has at least k blue edges to G_r). Hence, there exists a cutset C' of order $|C'| \leq k-1$ and a set of vertices B^* (think of a component of $G_b \setminus C'$) of order $|B^*| \leq k-1$ which have red edges only to C' in G_r. The set $B \cup B^*$ forms a set larger than B satisfying Properties 1 and 2, a contradiction. $\square_{Claim\ 1}$

Let $A = G \setminus B$ and consider the blue bipartite graph G_b induced on $A \cup B$. Since we have assumed $n > 6.5(k-1)$, we see that $|A| \geq 3.5(k-1)+1$. At this point, it is worth while to note that, by Lemma 10 in [5], Theorem 3 holds for $n > 8(k-1)$. Part of what remains of our proof is a strengthening of the ideas presented in [5].

We now claim that there exists a large k-connected subgraph of G_b which serves as a desired structure. Hence, we restrict our attention to G_b. Assume G_b is not k-connected. Consider a minimum cutset C with $|C| \leq k-1$.

Claim 2 $C \subseteq B$.

Proof of Claim 2: In order to prove this claim, it suffices to show that a cutset of order at most $k-1$ cannot separate two vertices of B. This would imply that any cutset including vertices of A is not minimal and hence, complete the proof.

Each vertex of B has at least $|A|-(k-1)$ edges to A which means that each pair of vertices in B shares at least $|A|-2(k-1)\geq k$ common neighbors (note that this requires only $n>6(k-1)$). Hence, no pair of vertices in B can be separated by a cutset of order at most $k-1$, thereby proving the claim. $\square_{Claim\ 2}$

Since G_b is bipartite, every component of $G_b\backslash C$ which does not contain a vertex of B is a single vertex (in A). Hence, each of these vertices has degree at most $|C|\leq k-1$. Let A^* be the vertices $v\in A$ with $d_b(v)\leq k-1$. Our first goal is to show that $|A^*|=t\leq 2(k-1)$. From the definitions, there are at most $t(k-1)+|B|(|A|-t)$ blue edges between A and B. Conversely, recall that there are at least $|B|(|A|-(k-1))$ edges between A and B since each vertex of B has many blue edges to A. This means

$$t(k-1)+|B|(|A|-t)\geq |B|(|A|-(k-1))$$

which, using the fact that $|B|\geq 2(k-1)$, implies

$$t\leq \frac{|B|(k-1)}{|B|-(k-1)}\leq 2(k-1), \tag{1}$$

as required.

Let $A''=A\setminus A^*$ and let $G''_b=B\cup A''=G_b\setminus A^*$ (the graph remaining after the removal of the above singleton vertices). We would now like to show that G''_b, which has order $n-t\geq n-2(k-1)$, is k-connected. Let C'' be a minimum cutset of G''_b and suppose $|C''|\leq k-1$. Let t' be the maximum red degree from vertices in $B\setminus C''$ into A^*. From this we get the following inequalities

$$t(|B|-(k-1))\leq e_r(B,A^*)\leq t'|B\setminus C''|+t|B\cap C''|$$

which implies

$$t'\geq t-\frac{t(k-1)}{|B\setminus C''|}. \tag{2}$$

We would now like to show that $|A''\setminus C''|\leq 2(k-1)-t'$. In order to accomplish this task, let X and Y be the two components (or collections of components) of $G''_b\setminus C''$ and choose a vertex $v\in B\setminus C''$ such that $e_r(v,A^*)=t'$. Notice that, by the definition of A^* and $G''_b=G_b\setminus A^*$, we know that $B\cap X$ and $B\cap Y$ are both nonempty. Without loss of generality, suppose $v\in B\cap X$. Since all edges from v to $A''\cap Y$ are red, we know that $|A''\cap Y|\leq k-1-t'$. Now let v' be a vertex in $B\cap Y$. Since all edges from v' to $A''\cap X$ are red, we get $|A''\cap X|\leq k-1$. These two bounds show that $|A''\setminus C''|\leq 2(k-1)-t'$.

Using (1) and (2), this implies

$$\begin{aligned}
n &= |C''|+|A''\setminus C''|+|B\setminus C''|+t\\
&\leq |C''|+2(k-1)-t'+|B\setminus C''|+t\\
&\leq (k-1)+2(k-1)-\left(t-\frac{t(k-1)}{|B\setminus C''|}\right)+|B\setminus C''|+t\\
&\leq 3(k-1)+|B\setminus C''|+\frac{|B|(k-1)^2}{[|B|-(k-1)]|B\setminus C''|}.
\end{aligned}$$

Hence, we need only show that

Fact 1

$$|B \setminus C''| + \frac{|B|(k-1)^2}{[|B| - (k-1)]|B \setminus C''|} \leq 3.5(k-1).$$

Proof: In order to prove this fact, we maximize the left hand side (LHS) over the values $2(k-1) \leq |B| \leq 3(k-1)$ and $(k-1) \leq |B \setminus C''| \leq 3(k-1)$ (also certainly $|B| \geq |B \setminus C''|$). It is easy to see this maximum occurs at one of the boundary points of our allowed values so we need only check these points. The largest value occurs when $|B| = |B \setminus C''| = 3(k-1)$ which yeilds $LHS \leq 3.5(k-1)$. $\square_{Fact\ 1}$

Hence $n \leq 3(k-1) + 3.5(k-1) = 6.5(k-1)$ which is a contradiction, completing the proof of Theorem 3. $\square$

Since we actually know $|B| < 3(k-1)$, the result in Fact 1 (and hence Theorem 3) may be improved slightly. For the sake of simplicity, this computation is omitted.

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References

[1] B. Bollobás and A. Gyárfás. Highly connected monochromatic subgraphs. *Discrete Mathematics*, 308(9):1722–1725, 2008.

[2] Y. Caro and R. Yuster. The order of monochromatic subgraphs with a given minimum degree. *Electron. J. Combin.*, 10:R32, 8 pp., 2003.

[3] G. Chartrand and L. Lesniak. *Graphs & Digraphs*. Chapman & Hall/CRC, Boca Raton, FL, fourth edition, 2005.

[4] H. Liu, R. D. Morris, and N. Prince. Highly connected monochromatic subgraphs: Addendum. *Manuscript*.

[5] H. Liu, R. D. Morris, and N. Prince. Highly connected multicoloured subgraphs of multicoloured graphs. *Discrete Math.*, 308(22):5096–5121, 2008.

[6] H. Liu, R. D. Morris, and N. Prince. Highly connected monochromatic subgraphs of multicolored graphs. *J. Graph Theory*, 61(1):22–44, 2009.