The t-pebbling number is eventually linear in t

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Submitted: Apr 4, 2010; Accepted: Jun 22, 2011; Published: Jul 22, 2011
Mathematics Subject Classification: 05C99, 05C35

Abstract

In graph pebbling games, one considers a distribution of pebbles on the vertices of a graph, and a *pebbling move* consists of taking two pebbles off one vertex and placing one on an adjacent vertex. The *t-pebbling number* $\pi_t(G)$ of a graph G is the smallest m such that for every initial distribution of m pebbles on $V(G)$ and every target vertex x there exists a sequence of pebbling moves leading to a distibution with at least t pebbles at x. Answering a question of Sieben, we show that for every graph G, $\pi_t(G)$ is *eventually linear* in t; that is, there are numbers a, b, t_0 such that $\pi_t(G) = at + b$ for all $t \geq t_0$. Our result is also valid for weighted graphs, where every edge $e = \{u, v\}$ has some integer weight $\omega(e) \geq 2$, and a pebbling move from u to v removes $\omega(e)$ pebbles at u and adds one pebble to v.

1 Introduction

Let $G = (V, E)$ be an undirected graph. A *pebbling distribution* on G is a function $p\colon V \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. A *pebbling move* consists of taking two pebbles off a vertex u and adding one pebble on an adjacent vertex v (we can think of this as paying a toll of one pebble for using the edge $\{u, v\}$). We also say that we *move one pebble* from u to v.

More generally, we consider a graph G together with a weight function $\omega\colon E \to \{2, 3, 4, \ldots\}$ on edges. If an edge $e = \{u, v\}$ has weight $\omega(e)$, then we pay a toll of

*Supported by Global COE Program "Computationism as a Foundation for the Sciences" and Grant-in-Aid for Scientific Research from Ministry of Education, Science and Culture, Japan, and Japan Society for the Promotion of Science.

$\omega(e) - 1$ pebbles for moving one pebble along e. (So the unweighted case corresponds to $\omega(v) = 2$ for all $v \in V(G)$.)

More formally, if $e = \{u, v\} \in E$ and p is a pebbling distribution such that $p(u) \geq \omega(e)$, then a pebbling move allows us to replace p with the distriution p' given by

$$p'(w) = \begin{cases} p(u) - \omega(e) & \text{for } w = u, \\ p(v) + 1 & \text{for } w = v, \\ p(w) & \text{otherwise.} \end{cases}$$

For a vertex $x \in V(G)$, let $\pi_t(G, \omega, x)$ be the smallest integer m such for all distributions p of m pebbles there is a distribution q with $q(x) \geq t$ that can be reached from p by a sequence of pebbling moves. The *t-pebbling number* of (G, ω) is defined as $\pi_t(G, \omega) = \max\{\pi_t(G, \omega, x) : x \in V(G)\}$ and we write $\pi_t(G)$ for the unweighted case with $\omega \equiv 2$.

Graph pebbling originated in combinatorial number theory and group theory. The pebbling game (unweighted and with $t = 1$) was suggested by Lagarias and Saks, and in print it first appears in Chung [2]. For more background we refer to two recent surveys by Hurlbert [4, 5].

For some graph classes the (unweighted) t-pebbling number has been determined exactly. We have $\pi_t(K_n) = 2t + n - 2$ for the complete graph, $\pi_t(C_{2n}) = t2^n$ and $\pi_t(C_{2n-1}) = t2^{n-1} + 2\lfloor \frac{2^n}{3} \rfloor - 2^{n-1} + 1$ for the cycle, and $\pi_t(Q_d) = t2^d$ for the cube (see [7]). All of these are linear functions of t. Moreover, one can show that the t-pebbling number of any tree is linear in t by using the methods of [8]. It is shown in [6] that for the complete bipartite graph, we have $\pi_t(K_{m,n}) = \max\{2t+m+n-2, 4t+m-2\}$, which is linear in t but only for t sufficiently large.

Sieben [8] asked whether the t-pebbling number is always linear for $t \geq t_0$ where t_0 is some constant. We answer this question affirmatively. A similar result is known in Ramsey theory: the Ramsey number of t copies of a graph G is eventually linear in t (see [1]).

To formulate our result, let us define, for every two vertices $u, v \in V(G)$, the *multiplicative distance*

$$\operatorname{mdist}(u, v) := \min\left\{ \prod_{e \in E(P)} \omega(e) : \ P \text{ is a } u\text{-}v\text{-path in } G \right\},$$

(in particular, $\operatorname{mdist}(u, u) = 1$ because the empty product equals 1). The function $\log(\operatorname{mdist}(u, v))$ clearly defines a metric on $V(G)$. Further, for $x \in V(G)$ we set

$$r_x := \max\{\operatorname{mdist}(x, v) : v \in V(G)\}.$$

Theorem 1. *For every graph G with edge weight function ω and for every $x \in V(G)$ there exist b and t_0 such that that for all $t \geq t_0$*

$$\pi_t(G, \omega, x) = r_x t + b.$$

Consequently, for a suitable $t_0 = t_0(G, \omega)$, $\pi_t(G, \omega)$ is a linear function of t for all $t \geq t_0$.

As a corollary, we immediately obtain a result from Hersovici et al. [3] about fractional pebbling:

$$\lim_{t\to\infty} \frac{\pi_t(G)}{t} = 2^{\mathrm{diam}(G)},$$

where $\mathrm{diam}(G)$ denotes the diameter of G in the usual shortest-path metric. Indeed, for the weight function $\omega \equiv 2$ we have $\max_{x\in V(G)} r_x = 2^{\mathrm{diam}(G)}$.

Unfortunately, our proof of Theorem 1 is existential, and it yields no upper bound on t_0. It would be interesting to find upper bounds on (the minimum necessary) t_0 in terms of G and ω, or lower bounds showing that a large t_0 may sometimes be needed.

2 Proof of Theorem 1

First we check that

$$\pi_t(G, \omega, x) \geq r_x t \tag{1}$$

for all t. To this end, we consider the distribution p_0 with $r_x t - 1$ pebbles, all placed at a vertex y with $\mathrm{mdist}(x, y) = r_x$. We claim that, starting with p_0, it is impossible to obtain t pebbles at x.

To check this, we define the *potential* of a pebbling distribution p as

$$F(p) := \sum_{v\in V(G)} \frac{p(v)}{\mathrm{mdist}(v, x)}.$$

It is easy to see that this potential is nonincreasing under pebbling moves (using the "multiplicative triangle inequality" $\mathrm{mdist}(u, x) \leq \omega(\{u, v\})\mathrm{mdist}(v, x)$). Now $F(p_0) < t$, while any distribution q with at least t pebbles at x has $F(q) \geq t$, which proves the claim and thus also (1).

Next, we define the function

$$f(t) := \pi_t(G, \omega, x) - r_x t.$$

We have $f(t) \geq 0$ for all t by (1). Let $n := |V(G)|$; we claim that f is nonincreasing for all $t \geq n$. Once we show this, Theorem 1 will be proved, since a nonincreasing nonnegative function with integer values has to be eventually constant.

So we want to prove that, for all $t \geq n$, we have $f(t) \leq f(t-1)$, which we rewrite to

$$\pi_t(G, \omega, x) \leq \pi_{t-1}(G, \omega, x) + r_x. \tag{2}$$

To this end, we consider an arbitrary pebbling distribution p with $m := \pi_{t-1}(G, \omega, x) + r_x$ pebbles. By (1) we obtain $m \geq r_x(t-1) + r_x = r_x t \geq r_x n$. So by the pigeonhole principle, there exists a vertex y with $p(y) \geq r_x$.

Let us temporarily remove r_x pebbles from y. This yields a distribution with at least $\pi_{t-1}(G, \omega, x)$ pebbles, and, by definition, we can convert it by pebbling moves to a distribution with at least $t - 1$ pebbles at x. Now we add the r_x pebbles back to y and move them toward x, and in this way we obtain one additional pebble at x. This verifies (2), and the proof of Theorem 1 is finished.

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