# On some Ramsey numbers for quadrilaterals

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**Abstract**

We will prove that $R(C_4, C_4, K_4 - e) = 16$. This fills one of the gaps in the tables presented in a 1996 paper by Arste et al. Moreover by using computer methods we improve lower and upper bounds for some other multicolor Ramsey numbers involving quadrilateral $C_4$. We consider 3 and 4-color numbers, our results improve known bounds.

## 1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let $G$ be such a graph. The vertex set of $G$ is denoted by $V(G)$, the edge set of $G$ by $E(G)$, and the number of edges in $G$ by $e(G)$. $C_m$ denotes the cycle of length $m$, $P_m$ the path on $m$ vertices and $K_m - e$ – the complete graph on $m$ vertices without one edge. For given graphs $G_1, G_2, ..., G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, ..., G_k)$ is the smallest integer $n$ such that if we arbitrarily color the edges of the complete graph of order $n$ with $k$ colors, then it always contains a monochromatic copy of $G_i$ colored with $i$, for some $1 \leq i \leq k$. A coloring of the edges of $n$-vertex complete graph with $m$ colors is called a $(G_1, G_2, ..., G_m; n)$- coloring, if it does not contain a subgraph isomorphic to $G_i$ colored with $i$, for each $i$.

*The Turán number* $T(n, G)$ is the maximum number of edges in any $n$-vertex graph which does not contain a subgraph isomorphic to $G$. A graph on $n$ vertices is said to be *extremal with respect to* $G$ if it does not contain a subgraph isomorphic to $G$ and has exactly $T(n, G)$ edges.

$G_1 \cup G_2$ denotes the graph which consists of two disconnected subgraphs $G_1$ and $G_2$. $kG$ stands for the graph consisting of $k$ disconnected subgraphs $G$.

We will use the following

**Theorem 1 (Woodall, [4])** *Let $G$ be a graph on $n$ ($n \geq 3$) vertices with more than $\frac{n^2}{4}$ edges. Then $G$ contains a cycle of length $k$ for each $k$ ($3 \leq k \leq \lfloor \frac{1}{2}(n+3) \rfloor$).*

## 2 Value of $R(C_4, P_4, K_4 - e)$

In [1] the value of this number is 10. It is incorrect. We prove the following

**Theorem 2**

$$R(C_4, P_4, K_4 - e) = 11$$

**Proof.** First we show a lower bound $R(C_4, P_4, K_4 - e) > 10$ by describing a suitable $(C_4, P_4, K_4 - e; 10)$-coloring. Let us take a graph $2K_3 \cup K_{1,3}$. This graph does not contain a $P_4$, so consider the edges of this graph to be of color 2 in coloring of $K_{10}$. We colored the complement of $2K_3 \cup K_{1,3}$ using colors 1 and 3 as shown in Figure 1, so that there is no $C_4$ in color 1 and no $K_4 - e$ in color 3.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 3 | 3 | 3 | 3 | 2 | 1 | 1 | 1 |
| 3 | 0 | 3 | 2 | 2 | 1 | 1 | 3 | 3 | 1 |
| 3 | 3 | 0 | 1 | 1 | 2 | 3 | 2 | 1 | 3 |
| 3 | 2 | 1 | 0 | 2 | 3 | 1 | 3 | 1 | 3 |
| 3 | 2 | 1 | 2 | 0 | 1 | 3 | 1 | 3 | 3 |
| 3 | 1 | 2 | 3 | 1 | 0 | 3 | 2 | 3 | 1 |
| 2 | 1 | 3 | 1 | 3 | 3 | 0 | 1 | 2 | 2 |
| 1 | 3 | 2 | 3 | 1 | 2 | 1 | 0 | 3 | 3 |
| 1 | 3 | 1 | 1 | 3 | 3 | 2 | 3 | 0 | 1 |
| 1 | 1 | 3 | 3 | 3 | 1 | 2 | 3 | 1 | 0 |

Figure 1: Matrix of $(C_4, P_4, K_4 - e; 10)$-coloring.

Now, we give a proof for the upper bound. This proof can be deduced from Turán numbers for $C_4$, $P_4$ and the Woodall's Theorem given above.

Suppose that for graph $G = K_{11}$ there is a $(C_4, P_4, K_4 - e; 11)$-coloring and let us consider such coloring. Since $R(C_4, P_4, P_3) = 7$ [1] then for each $v \in V(G)$, the number of edges of color 3 incident to $v$ is at most 6, and their total number is at most $6 \cdot 11/2 = 33$. Since $R(C_4, P_4, K_3) = 9$ [1] then there is a triangle $xyz$ with color 3. To avoid a $K_4 - e$ in color 3 there are at most 8 edges in color 3 between a triangle and the remaining vertices of $G$, so total number of edges in color 3 is at most $33 - 4 = 29$. One can count that $V(G) - \{x, y, z\}$ induces in $G$ subgraph in color 3 on 8 vertices and at least 22 edges, and by Woodall's Theorem we have a next triangle in color 3. In fact, we obtain that the total number of edges in color 3 is at most 25. Since $T(11, C_4) = 18$ [2] and $T(11, P_4) = 10$ [3], the number of colored edges in $G$ is at most $18 + 10 + 25 = 53$, which is less than $e(G) = 55$, a contradiction.

□

## 3 Value of $R(C_4, C_4, K_4 - e)$

We prove the following

**Theorem 3**

$$R(C_4, C_4, K_4 - e) = 16$$

**Proof.** First we give a critical $(C_4, C_4, K_4 - e; 15)$-coloring which gives us the lower bound. Let us consider the following graph $H$ which has 45 edges. $H$ consists of 6 triangles $xyz$, $x'y'z'$, $x''y''z''$, $xx'x''$, $yy'y''$, $zz'z''$ and one graph $K_6 - 2K_3$. The edges of subgraph $K_6 - 2K_3$ are partition into 3 groups. Each group has exactly 3 disjoint edges and each edge is join with one vertex from one triangle in such a way that triangles $xyz$, $x'y'z'$, $x''y''z''$ have edges to only one group. The graph $H$ does not contain a $K_4 - e$ and consider the edges of $H$ to be of color 3 in $(C_4, C_4, K_4 - e; 15)$-coloring.

Two graphs of order 15 containing no $C_4$ with the maximum of 30 edges were found in [2]. Let us consider the second extremal graph from [2] which consists of 3 disjoint graphs $\overline{K_{2,2} \cup K_1}$ denoted by $H_1$, $H_2$, $H_3$ and 4 disjoint triangles such that each triangle consists of vertices which belong to each graphs among $H_1$, $H_2$ and $H_3$. Let us consider all 30 edges to be of color 1 in $(C_4, C_4, K_4 - e; 15)$-coloring.

Finally, consider again the edges of the second extremal graph for $T(15, C_4)$ from [2] to be of color 2. The resulting $(C_4, C_4, K_4 - e; 15)$-coloring, proving that $R(C_4, C_4, K_4 - e) > 15$, is shown in Figure 2.

$$\begin{matrix}
0&3&3&3&3&3&3&2&2&2&2&1&1&1&1\\
3&0&3&2&2&2&2&1&1&1&1&3&3&3&3\\
3&3&0&1&1&1&1&3&3&3&3&2&2&2&2\\
3&2&1&0&3&2&1&3&3&2&1&3&3&2&1\\
3&2&1&3&0&1&2&2&1&3&3&2&1&3&3\\
3&2&1&2&1&0&3&1&2&3&3&3&3&1&2\\
3&2&1&1&2&3&0&3&3&1&2&1&2&3&3\\
2&1&3&3&2&1&3&0&2&1&3&2&3&1&3\\
2&1&3&3&1&2&3&2&0&3&1&3&1&3&2\\
2&1&3&2&3&3&1&1&3&0&2&1&3&2&3\\
2&1&3&1&3&3&2&3&1&2&0&3&2&3&1\\
1&3&2&3&2&3&1&2&3&1&3&0&1&2&3\\
1&3&2&3&1&3&2&3&1&3&2&1&0&3&2\\
1&3&2&2&3&1&3&1&3&2&3&2&3&0&1\\
1&3&2&1&3&2&3&3&2&3&1&3&2&1&0
\end{matrix}$$

Figure 2: Matrix of $(C_4, C_4, K_4 - e; 15)$-coloring.

Now, we give a proof that $R(C_4, C_4, K_4 - e) \leq 16$. Assume that the complete graph $G = K_{16}$ is 3-colored, so we have $(C_4, C_4, K_4 - e; 16)$-coloring. Since $R(C_4, C_4, K_3) = 12$ [1] then there exist two triangles with color 3 in graph $G$. Since $R(C_4, C_4, P_3) = 8$ [1] then for each $v \in V(G)$, the number of edges of color 3 incident to $v$ is at most 7, and their total number is at most $7 \cdot 16/2 = 56$. To avoid a $K_4 - e$ in color 3 there are at most 13 edges in color 3 between a first triangle and the 13 remaining vertices of $G$ and there are only 10 edges in this color between a second triangle and the 10 last vertices of $G$. In fact, the total number of edges in color 3 in $G$ is at most $56 - 4 = 52$. Since $T(16, C_4) = 33$ [2], the number of colored edges in $G$ is at most $33 + 33 + 52 = 118$, which is less than $e(G) = 120$, a contradiction.

□

First, we present the following

**Theorem 4**

$$19 \leq R(C_4, K_4 - e, K_4 - e) \leq 22$$

**Proof.** The $(C_4, K_4 - e, K_4 - e; 18)$-coloring, proving lower bound $R(C_4, K_4 - e, K_4 - e) > 18$ is shown in Figure 3.

032233133123122221
303212211312332233
230313222232313112
223031222313231132
311303222132332122
323130322111222333
122223013321323123
312222101321323233
312222310233331231
132311332033222321
213131223302212333
322321113320223213
133232333222011322
231332223212103312
223122331223130332
221113122332333022
231323233231213201
132223331133222210

Figure 3: Matrix of $(C_4, K_4 - e, K_4 - e; 18)$-coloring.

Suppose that the complete graph $G = K_{22}$ is 3-colored, so we have $(C_4, K_4 - e, K_4 - e; 22)$-coloring. Since $R(C_4, K_4 - e, P_3) = 9$ [1] then for each $v \in V(G)$, the number of edges of color 2 and 3 incident to $v$ is at most 8 in each color, and their total number is at most $8 \cdot 22 = 176$. Since $T(22, C_4) = 52$ [8], the number of colored edges in $G$ is at most $176 + 52 = 228$, which is less than $e(G) = 231$, a contradiction.

□

In [6] and [7] we can find the following bounds

**Theorem 5 ([6], [7])**

$$19 \le R(C_4, C_4, K_4) \le 22$$
$$25 \le R(C_4, C_3, K_4) \le 32$$
$$21 \le R(C_4, C_4, C_4, C_3) \le 27$$
$$31 \le R(C_4, C_4, C_4, K_4) \le 50$$
$$28 \le R(C_4, C_4, C_3, C_3) \le 36$$
$$42 \le R(C_4, C_4, C_3, K_4) \le 76$$

We improve all lower bounds for these numbers obtaining such results:

**Theorem 6**

$$20 \le R(C_4, C_4, K_4) \le 22$$
$$27 \le R(C_4, C_3, K_4) \le 32$$
$$24 \le R(C_4, C_4, C_4, C_3) \le 27$$
$$34 \le R(C_4, C_4, C_4, K_4) \le 50$$
$$30 \le R(C_4, C_4, C_3, C_3) \le 36$$
$$43 \le R(C_4, C_4, C_3, K_4) \le 76$$

**Proof.** We only present appropriate colorings proving lower bounds. One can find these colorings in enclosed Appendix. □

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# 4 Appendix

0123323313233133233
1033223323131331332
2302331333233121113
3320232133313132331
3232012313313331332
2233103213123332133
3312230333323231231
3331323031312322313
1233113303232233331
3333333130322313122
2123313323033233133
3331122132303231333
3133333222330313311
1311332323223033133
3323333231331303121
3112121233313330333
2313312331133113023
3313333132331323202
3231231312331313320

Figure 4: Matrix of $(C_4, C_4, K_4; 19)$-coloring.

03131133331313332113212332
30133313313131211333321122
11012331223333323332122311
33103212113311333132333323
13230331132333121323131333
13323032323123311231331112
31313303321332132321113311
33121230231133313311233133
33211332031133313311223113
31213223301332133331313311
13332311110331231323231333
31333131133012223133113231
13313233333103211231331113
31313323321230233311313311
32331313312222013113233123
31232131133213103333113231
21331123331313330113312231
13313233333123131021331213
13332321132331131203131333
33223111113311333130332323
23131312232133213313033133
12233313213131311333301132
21231133331313332112310331
31333131133213122233113031
32123113113311233132333303
22133213313131311333321130

Figure 5: Matrix of $(C_3, C_4, K_4; 26)$-coloring.

```
02222223333444444441111
20234112311444444443341
22044411134233444414123
23401341412444233141344
24410344443323322113414
21433041431444311242244
21144403214312444414332
32114130123444444442342
33144421034221144434231
31314312302444444441242
31423143420444223143244
44243434244013212134421
44342414244101321124433
44343424144310131224422
44423344142231012321434
44432144442123103211434
44432144443211230332414
44411244441112323032424
44141414344322213304413
13413242413444112240144
13134233222444444441043
14241434344232331214401
11344422124132444434310
```

Figure 6: Matrix of $(C_4, C_4, C_4, C_3; 23)$-coloring.

044444444444444444444433332222111 1

404444444444433332211144214442444 4

440333322211444444441441444244444 4

443032133121443142442434444244444 4

443301211231444434442442443444421

443210321322444444413441434444144

443123012112444444444244314444444 4

442312103212441243443444444144344

442311230123424444441443442444444

442123121033443441143442444444444

441232112303434424444341444444442

441112223330414414444443244424444

434444444444403112132144324441444 4

434444442431301213214222444444413

434344414344110321212444342432444

434144424444123032221344441443431

424434444421212303134412244414444

424244434144131230122444344424444

414444444144322211033143444444431

414441444444211232303441144434424

411223231344142142330244444444414

344444444434424344142034411143212

344344444444424414444304231212123

321421343213324424314440324414444

314444144442243423414423013231244

244443444444444444444132101231332

244434442444442144444114310322233

242224441444444444444124223013313

224444444442143412434411332101344

144444444444442344444324112310321

144441434444444444444214232333012

144424444444414344321124433142102

144414444424434144144234423341220

Figure 7: Matrix of $(C_4, C_4, C_4, K_4; 33)$-coloring.

04444444444433333333333222111
40333332221144444333333441444
43022113333233333444441442444
43201213333233331444442444441
43210123333133333444424442414
43121023333133333444441444442
43112203313333221444424441434
42333330122144444332133444441
42333331011242414333333444241
42333312101344444333332442414
41333332110244444331333444422
41221131232012133444443441444
34333334444101122444442333323
34333334244210212444441333313
34333324444112021444214213333
34333324144321201444441123333
34313314444322110442414333323
33444443333444444012212123333
33444443333444444101221333323
33444442331444442210121213333
33444441333444244221012333313
33442423333444141122104333313
33124143323321414211240334343
24444444444433213132333011132
24444444444433123231333102231
21242414424133333333334120124
14444444244433333333333121012
14441434412421332323114332102
14414241142433333333333214220

Figure 8: Matrix of $(C_4, C_4, C_3, C_3; 29)$-coloring.

044444444444444444444444443333333322221111
404444444444443333333333211144443322421144321
440333332111444444333244414332434214434444413
443022213331333314441244434144433443443444
443202121333333333441244441424343342144244 4
44322011321333333342444444344444431443143444
44321101333333332244444414144444433443243444
44312110313333333344444444434444443344342244 4
442313330212441413333444444331434444434 2433
441332312022442443333442443314314444314433
441331331201444443333444444322434444234442 3
441133332210444442313444444333434424434444 32
434333334444012214442333343334444421434441 3
434333334444102124441323343344444442434421 3
434333331244220114444333143342421444344133
434333234444211024414333343344244444342133
434133231444121201424333343344444442434443 3
433444443332444410112313324434234444444224
433442443333444441021333344424244331234444
433414443331444121202313344434144434421444
432124443333214442120133344414144434444444
424244444444433333333102213441332441444444 4
414444444444432333131320213143434344324324 4
414444144244333333333322023441334344444414 4
411444444444433133333311203242434321344344 4
344313134444444444424443333024424123324323 41
343444443333333333444441422011144433423431 2
343124443323333333444444444102244433413431 2
342444441123444443231131241204244344434344
334434444444442444444343421240412113323133
333444443333444242211333344424044334144244
324333334144442443444244414441401332431344
322331334444441444444443332444241032143134 4
241444444442244444344444233331333023323133
224424444444412442433314413334133220331343 3
214313334424444444414443432444342133043234 4
213441243333333334244424442143144334014322
144444424144444444432444443333243321310323 3
144323322444444424441443432444341133243034 4
134444444444442114244442143333123314332303 3
121444444332311333244444444411434443342343 01
113444444333233333444444444122434443342343 10

Figure 9: Matrix of $(C_4, C_4, C_3, K_4; 42)$-coloring.