On Han's Hook Length Formulas for Trees

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Abstract

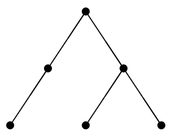
Recently, Han obtained two hook length formulas for binary trees and asked for combinatorial proofs. One of Han's formulas has been generalized to k-ary trees by Yang. Sagan has found a probabilistic proof of Yang's extension. We give combinatorial proofs of Yang's formula for k-ary trees and the other formula of Han for binary trees. Our bijections are based on the structure of k-ary trees associated with staircase labelings.

Keywords: hook length formula, k-ary tree, bijection, staircase labeling.

1 Introduction

Motivated by the hook length formula of Postnikov [6], Han [4] discovered two hook length formulas for binary trees. Han's proofs are based on recurrence relations. He raised the question of finding combinatorial proofs of these two formulas [3, 4]. Yang [9] generalized one of Han's formulas to k-ary trees by using generating functions. A probabilistic proof of Yang's formula has been found by Sagan [7]. By extending Han's expansion technique to k-ary trees, Chen, Gao and Guo [1] gave another proof for Yang's formula. The objective of this paper is to give combinatorial proofs of Yang's formula for k-ary trees and the other formula of Han for binary trees.

Recall that a k-ary tree is a rooted unlabeled tree where each vertex has exactly k subtrees in linear order, where we allow a subtree to be empty. When $k = 2$ (resp., $k = 3$), a k-ary tree is called a binary (resp., ternary) tree. A complete k-ary tree is a k-ary tree for which each internal vertex has exactly k nonempty subtrees. The hook length of a vertex u in a k-ary tree T, denoted by h_u, is the number of vertices of the subtree rooted at u. The hook length multi-set $\mathcal{H}(T)$ of T is defined to be the multi-set of hook lengths of all vertices of T. For example, Figure 1 gives an illustration of the hook length multi-set of a binary tree.



T $\quad \mathcal{H}(T) = \{1, 1, 1, 2, 3, 6\}$

Figure 1: The multi-set of hook lengths of a binary tree.

Let B_n be the set of all binary trees with n vertices. Han [4] discovered the following formulas. He also gave derivations of these formulas in [3] by using the expansion technique.

Theorem 1.1 (Han [4]) *For each positive integer n, we have*

$$\sum_{T \in B_n} \frac{1}{\prod_{h \in \mathcal{H}(T)} h 2^{h-1}} = \frac{1}{n!} \tag{1.1}$$

and

$$\sum_{T \in B_n} \frac{1}{\prod_{h \in \mathcal{H}(T)} (2h+1) 2^{2h-1}} = \frac{1}{(2n+1)!}. \tag{1.2}$$

As pointed out by Han [4], the above two formulas have a special feature that the hook lengths appear as exponents. Yang [9] extended the above formula (1.1) to k-ary trees.

Theorem 1.2 (Yang [9]) *For any positive integers n and k, we have*

$$\sum_{T} \prod_{h \in \mathcal{H}(T)} \frac{1}{h k^{h-1}} = \frac{1}{n!}, \tag{1.3}$$

where the sum ranges over k-ary trees with n vertices.

To give a combinatorial proof of (1.3), we shall define a set $S(n, k)$ of *staircase arrays* on $[k] = \{1, 2, \ldots, k\}$. More precisely, we shall represent an array in $S(n, k)$ in the form $(C_0, C_1, \ldots, C_{n-1})$, where $C_0 = \emptyset$ and for $1 \leqslant i \leqslant n-1$, C_i is a vector of length i with each entry in $[k]$.

We introduce the notion of staircase labelings of a k-ary tree, and we show that the sequences in $S(n, k)$ are in one-to-one correspondence with k-ary trees with n vertices associated with staircase labelings. This leads to a bijective proof of formula (1.3). Based on this bijection, we also obtain a combinatorial interpretation of formula (1.2).

2 A combinatorial proof of (1.3)

Our combinatorial proof of Yang's formula (1.3) is based on the following reformulation

$$\sum_T \frac{n!k^{1+2+\cdots+n}}{\prod_{h\in\mathcal{H}(T)} hk^h} = k^{1+2+\cdots+(n-1)}. \tag{2.1}$$

It is clear that the right-hand side of (2.1) equals the number of sequences in $S(n,k)$. As will be seen, the left hand-side of (2.1) equals the number of k-ary trees with n vertices associated with staircase labelings. We shall give a bijection between $S(n,k)$ and the set of k-ary trees with n vertices associated with staircase labelings.

More precisely, a staircase labeling of a k-ary tree is defined as follows. For a k-ary tree T with n vertices, we use a set $\{C_0, C_1, \ldots, C_{n-1}\}$ of vectors on $[k]$ to label the vertices of T, where C_i contains i elements in $[k]$. Moreover, we impose the following restrictions: for any vertex u with label C_i and a descent (not necessarily a child) v with label C_j, we have $i < j$, that is, the labels on any path from the root to a leaf have increasing lengths; and the $(i+1)$-st entry of C_j is determined by the relative position of the child of u on the path from u to v among its siblings. To be more specific, if the r-th child of u is on the path from u to v, then the $(i+1)$-st entry of C_j is set to be r.

For example, Figure 2 gives a staircase labeling of a ternary tree. For the label of any vertex, the entries that are determined by the labels of its ancestors are written in boldface.

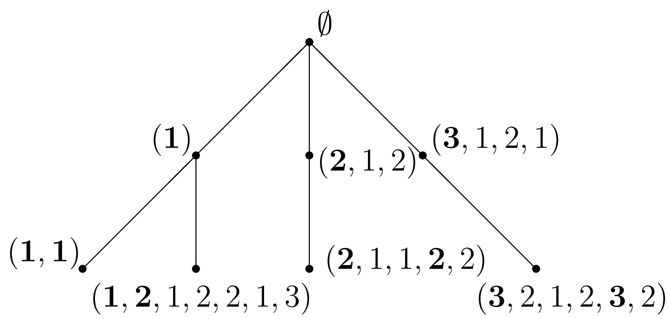


Figure 2: A staircase labeling of a ternary tree

Let $I(n,k)$ denote the set of k-ary trees with n vertices associated with staircase labelings. The following lemma shows that $|I(n,k)|$ is equal to the left-hand side of (2.1).

Lemma 2.1 *For* $n \geqslant 1$,

$$|I(n,k)| = \sum_T \frac{n!k^{1+2+\cdots+n}}{\prod_{h\in\mathcal{H}(T)} hk^h}, \tag{2.2}$$

where the sum ranges over k*-ary trees with* n *vertices.*

Proof. Let $P \in I(n,k)$ be a k-ary tree with a staircase labeling. Suppose that the labels of P are $C_0, C_1, \ldots, C_{n-1}$, where C_i is a vector of length i. Define Q to be the k-ary tree obtained from P by replacing a label C_i with i. Clearly, Q is an increasing k-ary tree in the sense that the label of any internal vertex is smaller than the labels of its children.

We shall consider the question of determining the number of k-ary trees P with n vertices associated with staircase labelings that correspond to a given increasing k-ary tree Q. Clearly, P and Q have the same underlying k-ary tree, denoted by T. In other words, we shall compute the number of staircase labelings of a k-ary tree T with given label length for each vertex. For any vertex u of T, let f_u denote the number of vertices on the path from the root to u. We claim that there are

$$k^{1+\cdots+n-\sum_{u\in T} f_u} \tag{2.3}$$

staircase labelings of T such that a vertex with label i in Q is associated with a vector C_i of length i. To prove (2.3), let u_i be the vertex of Q with label i. Recalling the definition of a staircase labeling, we need to determine how many entries in C_i that are determined by the ancestors of u_i. It can be seen that there are $f_{u_i} - 1$ entries of C_i that are determined by the ancestors of u_i. The other entries can be any element in $[k]$. Hence there are $k^{i+1-f_{u_i}}$ choices for C_i. This implies (2.3).

Note that the number in (2.3) does not depend on the specific increasing labeling of the k-ary tree T. To compute the number of staircase labelings of a k-ary tree T, it suffices to determine the number of increasing labelings of T. It is known that the number of increasing labelings of T equals

$$\frac{n!}{\prod_{h\in\mathcal{H}(T)} h},$$

see Knuth [5] or Gessel and Seo [2]. So we deduce that

$$|I(n,k)| = \sum_T \frac{n!}{\prod_{h\in\mathcal{H}(T)} h} k^{1+\cdots+n-\sum_{u\in T} f_u}, \tag{2.4}$$

where T ranges over k-ary trees with n vertices.

To obtain formula (2.2), we need to establish the following relation

$$\sum_{u\in T} h_u = \sum_{u\in T} f_u. \tag{2.5}$$

This can be justified by observing that both sides of (2.5) count the number of ordered pairs (u,v), where v is a descendant of u in T under the assumption that u is a descendant of itself. Substituting (2.5) into (2.4), we arrive at (2.2). This completes the proof. ∎

We have the following correspondence.

Theorem 2.1 *There is a bijection between $S(n,k)$ and $I(n,k)$.*

Proof. The map φ from $I(n,k)$ to $S(n,k)$ is straightforward, that is, for $P \in I(n,k)$ with a labeling set $\{C_0, C_1, \ldots, C_{n-1}\}$, define

$$\varphi(P) = (C_0, C_1, \ldots, C_{n-1}).$$

We proceed to give the inverse map ϕ from $S(n,k)$ to $I(n,k)$. Given a sequence $(C_0, C_1, \ldots, C_{n-1})$ in $S(n,k)$, we aim to construct a k-ary tree with n vertices associated with a staircase labeling by using the labels $C_0, C_1, \ldots, C_{n-1}$.

The map ϕ can be described as a recursive procedure. Let v_0 be a vertex with label $C_0 = \emptyset$. Clearly, v_0 and its label C_0 form a k-ary tree with a staircase labeling. Let $C_1 = (c_1)$. Adding a vertex v_1 as the c_1-th child of v_0 and assigning the label C_1 to v_1, we get a k-ary tree labeled by C_0 and C_1, denoted by P_1. It can be easily checked that P_1 is a k-ary tree with a staircase labeling. Assume that P_{m-1} ($m \geqslant 2$) is a k-ary tree with a staircase labeling with vertices $v_0, v_1, \ldots, v_{m-1}$ such that for $0 \leqslant i \leqslant m-1$, the vertex v_i has label C_i. Now we construct a k-ary tree with a staircase labeling, denoted by P_m, by adding the vertex v_m to P_{m-1} and assigning the label C_m to v_m.

To determine the position of v_m, we start with the root v_0. Let $C_m = (c_1, c_2, \ldots, c_m)$. If the c_1-th subtree of v_0 is empty, then we add the vertex v_m to P_{m-1} as the c_1-th child of v_0. Otherwise, we arrive at the c_1-th child of v_0, denoted by v_j. Note that the label of v_j is C_j. If the c_{j+1}-th subtree of v_j is empty, then we add the vertex v_m to P_{m-1} as the c_{j+1}-th child of v_j. If the c_{j+1}-th subtree of v_j is not empty, then we arrive at the c_{j+1}-th child of v_j. Repeating this process, we get a k-ary tree P_m labeled by $C_0, C_1, \ldots, C_m$. It is clear that P_m is a k-ary tree with a staircase labeling.

Thus, we obtain a k-ary tree $\phi(C_0, C_1, \ldots, C_{n-1}) = P_{n-1}$, labeled by $C_0, C_1, \ldots, C_{n-1}$. It can be checked that the maps φ and ϕ are inverses of each other. This completes the proof. ∎

In particular, for $k = 2$, the proof of Theorem 2.1 reduces to a combinatorial proof of Han's formula (1.1) for binary trees. Figure 3 gives an illustration of the bijection ϕ for $n = 6$, $k = 2$ and

$$(C_0, C_1, \ldots, C_5) = (\emptyset, (2), (2,1), (1,2,2), (1,2,2,1), (2,2,1,1,2)) \in S(6,2).$$

3 A combinatorial interpretation of (1.2)

In this section, we apply the bijection ϕ constructed in the previous section to give a combinatorial interpretation of formula (1.2). To this end, we reformulate (1.2) in terms of complete binary trees.

Clearly, one can add $n+1$ leaves to a binary tree with n vertices to form a complete binary tree with $2n+1$ vertices. Moreover, a vertex u with hook length h_u in a binary tree becomes an internal vertex with hook length $2h_u + 1$ in the corresponding complete binary tree. Denote by B^c_{2n+1} the set of complete binary trees with $2n+1$ vertices. Then

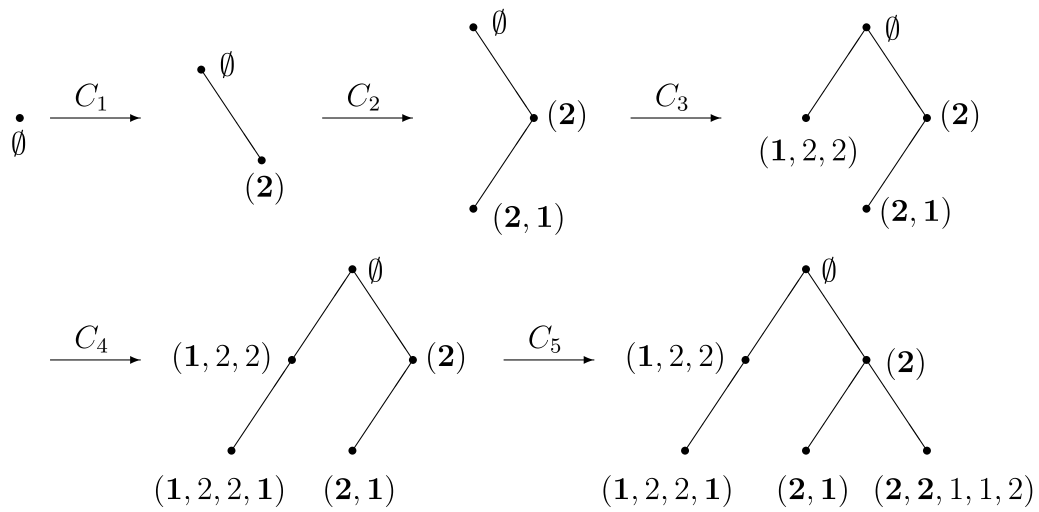


Figure 3: An illustration of the bijection ϕ.

(1.2) is equivalent to the following formula

$$\sum_{T\in B_{2n+1}^c} \frac{1}{\prod_{h\in\mathcal{H}(T)} h2^{h-1}} = \frac{1}{2^n(2n+1)!}. \tag{3.1}$$

In fact, our combinatorial interpretation of (3.1) is based on the following form

$$\sum_{T\in B_{2n+1}^c} \frac{(2n+1)!2^{1+2+\cdots+(2n+1)}}{\prod_{u\in\mathcal{H}(T)} h2^h} = \frac{2^{1+2+\cdots+2n}}{2^n}. \tag{3.2}$$

Combinatorial proof of (3.2). By the argument in the proof of Lemma 2.1, we see that the left-hand side of (3.2) is equal to the number of complete binary trees with $2n+1$ vertices associated with staircase labelings. Let $S'(2n+1,2)$ be the set of sequences in $S(2n+1,2)$ corresponding to complete binary trees with staircase labelings under the bijection ϕ. By the construction of ϕ, we shall give an explanation of the following relation

$$|S'(2n+1,2)| = \frac{1}{2^n}|S(2n+1,2)|. \tag{3.3}$$

Since $|S(2n+1,2)| = 2^{1+2+\cdots+2n}$, we are led to a combinatorial proof of (3.2).

It remains to prove (3.3). To this end, we shall construct a sequence of subsets $M_0, M_1, \ldots, M_n$ of $S(2n+1,2)$ such that

$$S(2n+1,2) = M_0 \supset M_1 \supset \cdots \supset M_n = S'(2n+1,2),$$

and for $1 \leqslant i \leqslant n$,

$$|M_i| = \frac{1}{2}|M_{i-1}|.$$

Let us begin with the definition of the subset M_1 of M_0. Let $(C_0, C_1, \ldots, C_{2n})$ be a sequence in M_0, and let T be the corresponding binary tree with a staircase labeling under the bijection ϕ. If both subtrees of the root of T have an odd number of vertices, then we choose this sequence $(C_0, C_1, \ldots, C_{2n})$ to be in M_1.

We proceed to prove the following relation

$$|M_1| = \frac{1}{2}|M_0|. \tag{3.4}$$

Let $(C_0, C_1, \ldots, C_{2n})$ be a sequence in M_1. Denote by T the corresponding binary tree with a staircase labeling under the bijection ϕ. Assume that for $1 \leqslant i \leqslant 2n$, s_i is the first entry of the vector C_i. By the construction of ϕ, if $s_i = 1$ (resp., $s_i = 2$), then there is a vertex with label C_i in the left (resp., right) subtree of the root of T. Since both subtrees of the root of T have an odd number of vertices, there is an odd number of 1's (or, equivalently, 2's) among $s_1, s_2, \ldots, s_{2n}$. Consider the set $\{1,2\}^{2n}$ of vectors of length $2n$ with entries in $\{1,2\}$. It is clear that there are as many vectors in $\{1,2\}^{2n}$ with an odd number of 1's as vectors in $\{1,2\}^{2n}$ with an even number of 1's. This implies that $|M_1| = |M_0 \backslash M_1|$, and hence we obtain (3.4).

In general, we can define the subset M_{j+1} of M_j for $j \geqslant 1$. Let $(C_0, C_1, \ldots, C_{2n})$ be a sequence in M_j, and let T be the corresponding binary tree with a staircase labeling under the bijection ϕ. Suppose that the vertices of T are $v_0, v_1, \ldots, v_{2n}$ such that the vertex v_i is labeled by C_i. Let $v_{t_0}, v_{t_1}, v_{t_2}, \ldots$ be the internal vertices of T such that the indices are arranged in increasing order, that is, $t_0 < t_1 < t_2 < \cdots$. If both subtrees of v_{t_j} have an odd number of vertices, then this sequence $(C_0, C_1, \ldots, C_{2n})$ is put in M_{j+1}. Using the same argument as that for (3.4), we deduce that

$$|M_{j+1}| = \frac{1}{2}|M_j|.$$

Let $(C_0, C_1, \ldots, C_{2n})$ be a sequence in M_n, and let T be the corresponding binary tree associated with a staircase labeling under the bijection ϕ. It can be seen that T is a binary tree with a staircase labeling such that both subtrees of any internal vertex have an odd number of vertices. It follows that T is a complete binary tree with a staircase labeling, which implies that $M_n = S'(2n+1, 2)$. This completes the proof. ∎

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