Characteristic polynomials of skew-adjacency matrices of oriented graphs

Yaoping Hou
Department of Mathematics
Hunan Normal University
Changsha, Hunan 410081, China
yphou@hunnu.edu.cn

Tiangang Lei
Department of Mathematical and Physical Sciences
National Nature Science Foundation of China
Beijing 100875, China
leitg@mail.nsfc.gov.cn

Submitted: Jan 18, 2011; Accepted: Jul 4, 2011; Published: Aug 5, 2011
Mathematics Subject Classification: 05C20, 05C50

Abstract

An oriented graph G^σ is a simple undirected graph G with an orientation, which assigns to each edge a direction so that G^σ becomes a directed graph. G is called the underlying graph of G^σ and we denote by $S(G^\sigma)$ the skew-adjacency matrix of G^σ and its spectrum $Sp(G^\sigma)$ is called the skew-spectrum of G^σ. In this paper, the coefficients of the characteristic polynomial of the skew-adjacency matrix $S(G^\sigma)$ are given in terms of G^σ and as its applications, new combinatorial proofs of known results are obtained and new families of oriented bipartite graphs G^σ with $Sp(G^\sigma) = \mathbf{i}Sp(G)$ are given.

1 Introduction

All undirected graphs in this paper are simple and finite. Let G be a graph with n vertices and $A(G) = (a_{i,j})$ the adjacency matrix of G, where $a_{i,j} = a_{j,i} = 1$ if there is an edge ij between vertices i and j in G (denoted by $i \sim j$), otherwise $a_{i,j} = a_{j,i} = 0$. We call G *nonsingular* if the matrix $A(G)$ is nonsingular. The characteristic polynomial $P(G;x) = det(xI - A(G))$ of $A(G)$, where I stands for the identity matrix of order n, is said to be the characteristic polynomial of the graph G. The n roots of the polynomial $P(G;x)$ are said to be the eigenvalues of the graph G. Since $A(G)$ is symmetric, all eigenvalues of $A(G)$ are real and we denote by $Sp(G)$ the adjacency spectrum of G.

Let G^σ (or $\overrightarrow{G}$) be a graph with an orientation, which assigns to each edge of G a direction so that G^σ becomes a directed graph. *The skew-adjacency matrix* $S(G^\sigma) = (s_{i,j})$ is real skew symmetric matrix, where $s_{i,j} = 1$ and $s_{j,i} = -1$ if $i \to j$ is an arc of G^σ, otherwise $s_{i,j} = s_{j,i} = 0$. The *skew-spectrum* $Sp(G^\sigma)$ of G^σ is defined as the spectrum of $S(G^\sigma)$. Note that $Sp(G^\sigma)$ consists of only purely imaginary eigenvalues because $S(G^\sigma)$ is real skew symmetric.

Unlike the adjacency matrix of a graph, there is little research on the skew-adjacency matrix $S(G^\sigma)$, except that in enumeration of perfect matchings of a graph, see [9] and references therein, where the square of the number of perfect matchings of a graph G with a Pfaffian orientation is the determinant of the skew-adjacency matrix $S(G^\sigma)$.

Recently, the skew-energy of G^σ was defined as the energy of matrix $S(G^\sigma)$, that is,

$$\mathcal{E}(G^\sigma) = \sum_{\lambda \in Sp(G^\sigma)} |\lambda|.$$

The concept of the energy of an undirected graph was introduced by Gutman and there has been a constant streams of papers devoted to this topic. The concept of the skew-energy of a simple directed graph (that is, oriented graph) was introduced by Adiga, Balakrishnan and So, and some basic facts are discussed and some open problems are proposed [1], such as,

- *Problem 1: Interpret all the coefficients of the characteristic polynomial of* $S(G^\sigma)$.
- *Problem 2: Find new families of oriented graphs* G^σ *with* $\mathcal{E}(G^\sigma) = \mathcal{E}(G)$.

The motivation of this paper is to address the above two open problems. In section 2 we derive the coefficients of the characteristic polynomial of $S(G^\sigma)$ in terms of G^σ, which is similar to the result of the coefficients of the characteristic polynomial of the adjacency matrix $A(G)$. In section 3 we give some applications of the coefficients theorem: the new combinatorial proofs of known results in [10] are obtained (that is, $Sp(G^\sigma) = \mathbf{i}Sp(G)$ for *some* orientation σ if and only if G is bipartite and $Sp(G^\sigma) = \mathbf{i}Sp(G)$ for *any* orientation G^σ of G if and only if G is acyclic) and some new families of oriented bipartite graphs G^σ with $\mathcal{E}(G^\sigma) = \mathcal{E}(G)$ are given.

2 The skew-characteristic polynomial of G^σ

Let G be a graph. A *linear subgraph* L of G is a disjoint union of some edges and some cycles in G. A *k-matching* $\mathcal{M}$ in G is a disjoint union of k-edges. If $2k$ is the order of G, then a $k-$ matching of G is called a *perfect matching* of G.

Let G be a graph and $A(G)$ be its adjacency matrix and characteristic polynomial of G be

$$P(G;x) = \det(xI - A) = \sum_{i=0}^{n} a_i x^{n-i}. \tag{2.1}$$

Then $a_0(G) = 1, a_1(G) = 0$, and $-a_2(G)$ is the number of edges in G. In general, we have (see [7])

$$a_i = \sum_{L \in \mathcal{L}_i} (-1)^{p_1(L)} (-2)^{p_2(L)}, \tag{2.2}$$

where $\mathcal{L}_i$ denotes the set of all linear subgraphs L of G with i vertices, $p_1(L)$ is the number of components of size 2 in L and $p_2(L)$ is the number of cycles in L. If G is bipartite, then $a_i = 0$ for all odd i, and

$$P(G;x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i b_{2i}(G) x^{n-2i}, \tag{2.3}$$

where all $b_{2i} = (-1)^i a_{2i}$ are nonnegative [4, p. 147].

Let G be a graph and G^σ be an orientation of G and $S(G^\sigma)$ be the skew-adjacency matrix of G^σ. Denote the characteristic polynomial of $S(G^\sigma)$ by

$$P(G^\sigma;x) = \det(xI - S) = \sum_{i=0}^{n} c_i x^{n-i}. \tag{2.4}$$

Then (i) $c_0 = 1$, (ii) c_2 is the number of edges of G, (iii) $c_i \geq 0$ for all i and (iv) all $c_i = 0$ for all odd i since the determinant of any skew symmetric matrix is nonnegative and is 0 if its order is odd. In this section we give c_i in term of G^σ in general. It is based on the combinatorial definition of the determinant of a matrix [6, Section 9.1].

Recall the definition of the determinant of a matrix $M = (m_{i,j})$ is

$$\det M = \sum_{\tau \in Sym(n)} sign(\tau) m_{1,\tau(1)} m_{2,\tau(2)} \cdots m_{n,\tau(n)}, \tag{2.5}$$

where the summation extends over the set $Sym(n)$ of all permutations τ of $\{1, 2, ..., n\}$. Suppose that the permutation τ consists of k permutation cycles of sizes $\ell_1, \ell_2, ..., \ell_k$, respectively, where $\ell_1 + \ell_2 + \cdots + \ell_k = n$. Then $sign(\tau)$ can be computed by

$$sign(\tau) = (-1)^{\ell_1 - 1 + \ell_2 - 1 + \cdots + \ell_k - 1} = (-1)^n (-1)^k. \tag{2.6}$$

Let D_n be the *complete digraph* with vertex set $\{1, 2, ..., n\}$ in which each ordered pair (i, j) of vertices forms an arc of D_n. We assign to each arc (i, j) of D_n the weight $m_{i,j}$ and thereby obtain a weighted digraph. The *weight* of a directed cycle $\gamma : i_1 \to i_2 \to \cdots \to i_t \to i_1$ is defined to be

$$-m_{i_1,i_2} \cdots m_{i_{t-1},i_t} m_{i_t,i_1},$$

the negative of all the product of the weights of arcs.

Let τ be a permutation of $\{1, 2, ..., n\}$ as above. The *permutation digraph* $D(\tau)$ is the digraph with vertices $\{1, 2, ..., n\}$ and with the n arcs $\{(i, \tau(i)) : i = 1, 2, ..., n\}$. The digraph $D(\tau)$ is a spanning sub-digraph of the complete digraph D_n. The directed cycles of $D(\tau)$ are in one-to-one correspondence with the permutation cycles of τ and the arc sets of these directed cycles partition the set of arcs of $D(\tau)$. The *weight* $wt(D\tau)$ *of the permutation digraph* $D(\tau)$ is defined to be the product of the weights of its direct cycles, $wt(D(\tau)) = (-1)^k m_{1,\tau(1)} m_{2,\tau(2)} m_{n,\tau(n)}$. Using (2.5) and (2.6), we obtain

$$\det(M) = (-1)^n \sum_{\tau \in Sym(n)} wt(D(\tau)). \tag{2.7}$$

Let $\mathcal{E}(n)$ denote the set of all permutations τ of $\{1, 2, ..., n\}$ such that the size of all permutation cycles of τ are even.

Lemma 2.1 *[8, Lemma 2.1] If $M = (m_{i,j})$ is an $n \times n$ skew symmetric matrix then*

$$\det M = \sum_{\tau \in \mathcal{E}(n)} sign(\tau) m_{1,\tau(1)} \cdots m_{n,\tau(n)}.$$

If $M = (m_{i,j})$ is an $n \times n$ skew symmetric matrix then

$$\det(M) = (-1)^n \sum_{\tau \in \mathcal{E}(n)} wt(D(\tau)). \tag{2.8}$$

We need also following concepts from [9] in order to interpret all coefficients c_{2i} in term of G^σ.

Let C be an undirected even cycle of G^σ. Now regardless of which of the possible routing around C is chosen, if C contains an even number of oriented edge whose orientation agrees with the routing, then C also contains an even number of edges whose orientation is opposite to the routing. Hence the following definition is independent of the routing chosen.

If C be any undirected even cycle of G^σ, we say C is *evenly oriented relative to* G^σ if it has an even number of edges oriented in the direction of the routing. Otherwise C is *oddly oriented.*

Let $S = (s_{ij})$ be skew-adjacency matrix of an oriented graph G^σ. Note that each undirected cycle C of G^σ correspondences two permutation cycles, and the weights of these two permutation digraphs are -1 if C is evenly oriented relative to G^σ and $+1$ if C is oddly oriented.

We call a linear subgraph L of G *evenly linear* if L contains no cycle with odd length and denote by $\mathcal{EL}_i(G)$ (or $\mathcal{EL}_i$ for short) the set of all evenly linear subgraphs of G with i vertices. For a linear subgraph $L \in \mathcal{EL}_i$ denote by $p_e(L)$ (resp., $p_o(L)$) the number of evenly (resp., oddly) oriented cycles in L relative to G^σ. For a linear subgraph $L \in \mathcal{EL}_n$, L contributes $(-2)^{p_e(L)}2^{p_o(L)}$ to the determinant of $S(G^\sigma)$.

Summarizing the above we have

Lemma 2.2 *If $S(G^\sigma) = (s_{i,j})$ is an $n \times n$ skew-adjacency matrix of the orientation G^σ of a graph G. Then*

$$\det S(G^\sigma) = \sum_{L \in \mathcal{EL}_n} (-2)^{p_e(L)} 2^{p_o(L)},$$

where $p_e(L)$ is the number of evenly oriented cycles of L relative to G^σ and $p_o(S)$ is the number of oddly oriented cycles of L relative to G^σ, respectively.

Note that if n is odd then $\mathcal{EL}_n$ is empty and hence $\det S(G^\sigma) = 0$.

As $(-1)^i c_i$ is the summation of determinants of all principal $i \times i$ submatrices $S(G^\sigma)$, using Lemma 2.2, we have

Theorem 2.3 *Let G be a graph and G^σ be an orientation of G. Then*

$$c_i = \sum_{L \in \mathcal{EL}_i} (-2)^{p_e(L)} 2^{p_o(L)}, \tag{2.9}$$

where $p_e(L)$ is the number of evenly oriented cycles of L relative to G^σ and $p_o(S)$ is the number of oddly oriented cycles of L relative to G^σ, respectively. In particular, $c_i = 0$ if i is odd.

As applications of the above theorem, we can obtain the following result which can be used to find recursions for the characteristic polynomial of some skew-adjacency matrices.

Theorem 2.4 *Let $e = uv$ be an edge of G, then*

$$P(G^\sigma; x) = P(G^\sigma - e; x) + P(G^\sigma - u - v; x) + 2 \sum_{e \in C \in Od(G^\sigma)} P(G^\sigma - C; x) - 2 \sum_{e \in C \in Ev(G^\sigma)} P(G^\sigma - C; x).$$

Proof. Every evenly linear subgraph L of G with i vertices must belong to one of the following four kinds:

(1). $\mathcal{E}_1 : L$ does not contain the edge e;

(2). $\mathcal{E}_2 : L$ contains the edge e but e is not in any cycle component of L;

(3). $\mathcal{E}_3 : L$ contains the edge e and e is contained in some oddly oriented cycle component C of L;

(4). $\mathcal{E}_4 : L$ contains the edge e and e is contained in some evenly oriented cycle component C in L.

Note that any evenly linear subgraph L with i vertices which does not use e is an evenly linear subgraph with i vertices of $G - e$. If an evenly linear subgraph L belongs $\mathcal{E}_2$, then the edge e is a component and L determines an evenly linear subgraph L' of $G - u - v$ with $i - 2$ vertices such that $L = e \cup L'$. For any evenly linear subgraph L belongs to $\mathcal{E}_3$ (or $\mathcal{E}_4$), L determines an evenly linear subgraph L' of $G - C$ with $i - |C|$ vertices for some oddly (resp., evenly) oriented cycle C in G^σ such that $L = C \cup L'$. Hence,

$$\begin{aligned}
c_i(G^\sigma) &= \sum_{L \in \mathcal{EL}_i(G)} (-2)^{p_e(L)} 2^{p_o(L)} \\
&= \sum_{L \in \mathcal{E}_1} (-2)^{p_e(L)} 2^{p_o(L)} + \sum_{L \in \mathcal{E}_2} (-2)^{p_e(L)} 2^{p_o(L)} \\
&\quad + \sum_{L \in \mathcal{E}_3} (-2)^{p_e(L)} 2^{p_o(L)} + \sum_{L \in \mathcal{E}_4} (-2)^{p_e(L)} 2^{p_o(L)} \\
&= \sum_{L' \in \mathcal{EL}_i(G-e)} (-2)^{p_e(L')} 2^{p_o(L')} + \sum_{L' \in \mathcal{EL}_{i-2}(G-u-v)} (-2)^{p_e(L')} 2^{p_o(L')} \\
&\quad + 2 \sum_{e \in C \in Od(G^\sigma)} \sum_{L' \in \mathcal{EL}_{i-|C|}(G-C)} (-2)^{p_e(L')} 2^{p_o(L')} \\
&\quad - 2 \sum_{e \in C \in Ev(G^\sigma)} \sum_{L' \in \mathcal{EL}_{i-|C|}(G-C)} (-2)^{p_e(L')} 2^{p_o(L')} \\
&= c_i(G^\sigma - e) + c_{i-2}(G^\sigma - u - v) + 2 \sum_{e \in C \in Od(G^\sigma)} c_{i-|C|}(G^\sigma - C) \\
&\quad - 2 \sum_{e \in C \in Ev(G^\sigma)} c_{i-|C|}(G^\sigma - C),
\end{aligned}$$

where $Od(G^\sigma)$ (resp., $Ev(G^\sigma)$) is the set of all oddly (resp., evenly) oriented (even) cycles of G^σ. Therefore, the result follows. □

Corollary 2.5 *Let $e = uv$ be an edge of G that is on no even cycle in G. Then*

$$P(G^\sigma; x) = P(G^\sigma - e; x) + P(G^\sigma - u - v; x).$$

Example 2.6 Let $S_{n,3}$ be the unicyclic graph obtained from the star of n vertices by adding an edge and $S^\sigma_{n,3}$ be any orientation of $S_{n,3}$. Then by (2.9),

$$P(S^\sigma_{n,3}; x) = x^n + nx^{n-2} + (n-3)x^{n-4}.$$

Let $S_{n,4}$ be the unicycle graph obtained from the cycle C_4 by adding $n-4$ pendent vertices to a vertex of C_4 and let $S^o_{n,4}$ (resp., $S^e_{n,4}$) be an orientation of graph $S_{n,4}$ such that the unique cycle C_4 in $S_{n,4}$ is oddly (resp., evenly) oriented relative to $S^o_{n,4}$. Then

$$P(S^o_{n,4}; x) = x^n + nx^{n-2} + (2n-4)x^{n-4},$$
$$P(S^e_{n,4}; x) = x^n + nx^{n-2} + (2n-8)x^{n-4}.$$

Let C_n and P_n be the cycle graph and the path graph with n vertices, respectively. In what follows we compute the characteristic polynomial of the skew-adjacency matrix of any orientation of C_n and P_n. Letting $\mathbf{i} = \sqrt{-1}$ and $x = 2\mathbf{i}\sin\tau$, we have $P(P^\sigma_1, x) = 2\mathbf{i}\sin\tau, P(P^\sigma_2, x) = 2\cos 2\tau - 1 = \frac{\cos 3\tau}{\cos\tau}$, and $P({P_n}^\sigma; x) = xP(P^\sigma_{n-1}; x) + P(P^\sigma_{n-2}; x)$ for $n \geq 3$. Using the identities $\sin(\theta+\varphi)-\sin(\theta-\varphi) = 2\cos\theta\sin\varphi$ and $\cos(\theta+\varphi)-\cos(\theta-\varphi) = -2\sin\theta\sin\varphi$ with $\theta = n\tau$ and $\varphi = \tau$, it follows that the solution of the recursion is

Example 2.7

$$P({P_n}^\sigma; x) = \begin{cases} \frac{\cos(n+1)\tau}{\cos\tau}, & n \text{ is even;} \\ \frac{\mathbf{i}\sin(n+1)\tau}{\cos\tau}, & n \text{ is odd.} \end{cases}$$

When $\pi/2 < \tau < -\pi/2$, then values of $x = 2\mathbf{i}\sin\tau$ are distinct and balanced. To obtain the skew-spectrum of P^σ_n, if $j = 1, 2, ..., n$, we may take $\tau = (n+1-2j)\frac{\pi}{2(n+1)}$ when n is even and also when n is odd. Since $\sin\tau = \cos(\frac{\pi}{2} - \tau)$, the skew-spectrum of ${P_n}^\sigma$ is $\{2\mathbf{i}\cos\frac{j\pi}{n+1} | j = 1, 2, ..., n\}$.

Using Corollary 2.4, for any orientation C^σ_n of the cycle C_n, we have

$$P({C_n}^\sigma; x) = \begin{cases} P(P^\sigma_n; x) + P(P^\sigma_{n-2}; x) + 2, & n \text{ is even and the cycle is oddly oriented;} \\ P(P^\sigma_n; x) + P(P^\sigma_{n-2}; x) - 2, & n \text{ is even and the cycle is evenly oriented;} \\ P(P^\sigma_n; x) + P(P^\sigma_{n-2}; x), & n \text{ is odd.} \end{cases}$$

Hence, by the Example 2.7, we have

Example 2.8

$$P(C_n{}^{\sigma};x)=\begin{cases}2\cos n\tau+2, & n \text{ is even and the cycle is oddly oriented;}\\ 2\cos n\tau-2, & n \text{ is even and the cycle is evenly oriented;}\\ 2\mathbf{i}\sin n\tau, & n \text{ is odd.}\end{cases}$$

Hence the skew-spectrum of $C_n{}^{\sigma}$ is $\{2\mathbf{i}\sin\frac{2j\pi}{n}|j=1,2,...,n\}$ if n is odd, and $\{2\mathbf{i}\sin\frac{(2j-1)\pi}{n}|j=1,2,...,n\}$ if n is even and the cycle is oddly oriented, and $\{2\mathbf{i}\sin\frac{2j\pi}{n}|j=1,2,...,n\}$ if n is even and the cycle is evenly oriented.

3 Oriented graphs G^{σ} with $Sp(G^{\sigma})=\mathbf{i}Sp(G)$

Let G be a graph and G^{σ} be an orientation of G. The characteristic polynomials of G and G^{σ} are expressed as (2.1) and (2.4), respectively. Because the roots of $P(G^{\sigma};x)$ are pure imaginary and occur in complex conjugate pairs, while the roots of $P(G;x)$ are all real, it follows that $Sp(G^{\sigma})=\mathbf{i}Sp(G)$ if and only if $P(G;x)=\sum_{i=0}^{n}a_ix^{n-i}=x^{n-2r}\prod_{i=1}^{r}(x^2-\lambda_i^2)$ and $P(G;x)=\sum_{i=0}^{n}c_ix^{n-i}=x^{n-2r}\prod_{i=1}^{r}(x^2+\lambda_i^2)$ for some non-zero real numbers $\lambda_1,\lambda_2,...,\lambda_r$ if and only if

$$a_{2i}=(-1)^ic_{2i},a_{2i+1}=c_{2i+1}=0, \tag{3.1}$$

where $i=0,1,...,\lfloor\frac{n}{2}\rfloor$.

Let G^{σ} be an orientation of a graph G. An even cycle $C_{2\ell}$ is said to be *oriented uniformly* if $C_{2\ell}$ is oddly (resp., evenly) oriented relative to G^{σ} when ℓ is odd (resp., even).

Lemma 3.1 *Let G be a bipartite graph and G^{σ} be an orientation of G. If every even cycle is oriented uniformly then $Sp(G^{\sigma})=\mathbf{i}Sp(G)$.*

Proof. Since G is bipartite, all cycles in G are even and all linear subgraphs are even. Then $a_{2i+1}=0$ for all i. Since every even cycle is oriented uniformly, for every cycle $C_{2\ell}$ with length 2ℓ, $C_{2\ell}$ is evenly oriented relative to G^{σ} if and only if ℓ is even. Thus $(-1)^{p_e(C_{2\ell})}=(-1)^{\ell+1}$.

By Eqs (2.2) and (2.9), we have

$$(-1)^ia_{2i}=m(G,i)+\sum_{L\in\mathcal{CL}_{2i}}(-1)^{p_1(L)+i}(-2)^{p_2(L)}, \tag{3.2}$$

$$c_{2i}=m(G,i)+\sum_{L\in\mathcal{CL}_{2i}}(-2)^{p_e(L)}2^{p_o(L)}, \tag{3.3}$$

where $m(G,i)$ is the number of matchings with i edges and $\mathcal{CL}_{2i}$ is the set of all linear subgraphs with $2i$ vertices of G and with at least one cycle.

For a linear subgraph $L\in\mathcal{CL}_{2i}$ of G, assume that L contains the cycles $C_{2\ell_1},...,C_{2\ell_{p_2}}$. Then the number of components of L that are single edge is $p_1(L)=i-\sum_{j=1}^{p_2(L)}\ell_j$. Hence

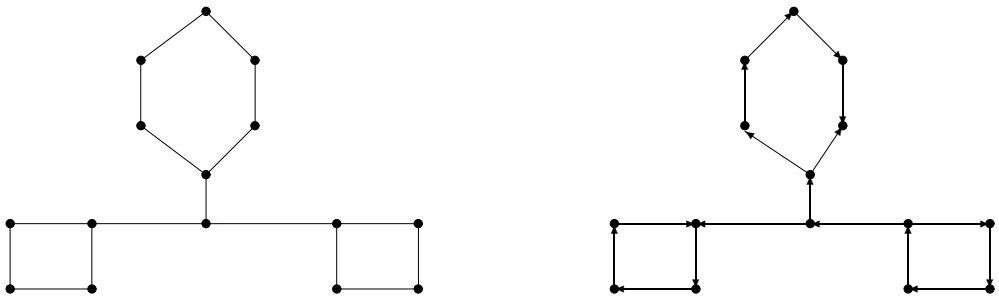

Figure 1: A graph G and an orientation G^σ with all even cycles oriented uniformly.

$(-1)^{p_1(L)+i} = (-1)^{\sum_{j=1}^{p_2(L)} \ell_j}$. Therefore L contributes $(-1)^{\ell_1+1}\cdots(-1)^{\ell_{p_2}+1}(-2)^{p_2(L)} = (-1)^{p_1(L)+i}(-2)^{p_2(L)}$ in c_{2i}. Thus $(-1)^i a_{2i} = c_{2i}$ by Eqs. (3.2) and (3.3) and the proof is completed. □

The following corollary provides a new family of oriented bipartite graphs G^σ with $Sp(G^\sigma) = \mathbf{i}Sp(G)$ and hence $\mathcal{E}(G^\sigma) = \mathcal{E}(G)$.

Corollary 3.2 *Let G be a graph whose blocks are K_2 or even cycles. If all even cycles of G are oriented uniformly in G^σ then $Sp(G^\sigma) = \mathbf{i}Sp(G)$ and hence $\mathcal{E}(G^\sigma) = \mathcal{E}(G)$.*

The following two results appeared in [10]. The proofs there are based on matrix theory. Now we give proofs that are more combinatorial.

Theorem 3.3 *A graph G is bipartite if and only if there is an orientation σ such that $Sp(G^\sigma) = \mathbf{i}Sp(G)$.*

Proof. (Sufficiency) If there is an orientation σ such that $Sp(G^\sigma) = \mathbf{i}Sp(G)$ then $a_{2i+1} = c_{2i+1} = 0$. Hence G is bipartite.

(Necessity) If G is a bipartite graph with vertices partition $V = V_1 \cup V_2$. Let G^σ be the orientation such that all arcs are from V_1 to V_2. Then $a_{2i+1} = 0$ for all i, and every even cycle is oriented uniformly relative to G^σ. Thus $Sp(G^\sigma) = \mathbf{i}Sp(G)$ by Lemma 3.1 □

We call a graph G *acyclic* (or a forest) if G contains no cycles. A tree is a connected and acyclic graph.

Theorem 3.4 *Let G be a graph. Then $\mathbf{i}Sp(G) = Sp(G^\sigma)$ for any orientation G^σ if and only if G is acyclic.*

Proof. (Sufficiency) If G is acyclic, then $a_{2i+1} = 0$ and $a_{2i} = (-1)^i m(G,i)$ and $c_{2i} = m(G,i)$ and hence $\mathbf{i}Sp(G) = Sp(G^\sigma)$ by the first paragraph of this section.

(Necessity) Suppose that G is not acyclic, then G contains at least a cycle. G is bipartite by $\mathbf{i}Sp(G) = Sp_S(G^\sigma)$ and Theorem 3.3. Let the length of shortest cycle of G be g, then g is even, say $g = 2r$. Then $(-1)^r a_g = m(G,r) + (-1)^{r+1}2n(G,C_g)$ and

$c_g = m(G,r) + 2n_o(G^\sigma, C_g) - 2n_e(G^\sigma, C_g)$, where $n_o(G^\sigma, C_g)$ ($n_e(G^\sigma, C_g)$) is the number of oddly (resp., evenly) oriented cycles in G of length g relative to G^σ and $n(G, C_g)$ is the number of cycles in G of length g. Note that $n_o(G^\sigma, C_g) + n_e(G^\sigma, C_g) = n(G, C_g)$. As in the proof of Theorem 3.3, let G have the orientation G^σ where all edges are directed from V_1 to V_2. For this orientation, $2n_o(G^\sigma, C_g) - 2n_e(G^\sigma, C_g)$ equals $2n_o(G^\sigma, C_g)$ if r odd and $-2n_e(G^\sigma, C_g)$ if r is even. Thus reversing the direction of an edge that is on at least one cycle of length g must change $2n_o(G^\sigma, C_g) - 2n_e(G^\sigma, C_g)$ and so must change c_g. Hence $(-1)^{r+1}2n(G, C_g) \neq 2n_o(G^\sigma, C_g) - 2n_e(G^\sigma, C_g)$. That is, $(-1)^r a_{2r} \neq c_{2r}$, which is contradiction with $\mathbf{i}Sp(G) \neq Sp(G^\sigma)$. □

From the above Theorem 3.4, if T is a tree and $\overrightarrow{T}$ is any orientation of T then $Sp(\overrightarrow{T}) = \mathbf{i}Sp(T)$. In what follows we provide another interesting family of oriented graphs G^σ with $Sp(G^\sigma) = \mathbf{i}Sp(G)$ and hence with $\mathcal{E}(G^\sigma) = \mathcal{E}(G)$

Let T be a tree with a perfect matching $\mathcal{M}$ (in this case, T has a unique perfect matching) and $\overrightarrow{T}$ be an orientation of T. Note that the adjacency matrix $A(T)$ of T and skew-adjacency matrix $S(\overrightarrow{T})$ of $\overrightarrow{T}$ are nonsingular if and only if T has a perfect matching. In order to describe the inverses of $A(T)$ and $S(T)$, the following definition of an *alternating path* is taken from Buckley, Doty and Harary [5, p.156].

Definition 3.5 Let G be a graph with a perfect matching $\mathcal{M}$. A path in $G : P(i,j) = i_1 i_2 \cdots i_{2k}$ (where $i_1 = i, i_{2k} = j$) from a vertex i to a vertex j is said to be an *alternating path* if the edges $i_1 i_2, i_3 i_4, \cdots, i_{2k-1} i_{2k}$ are edges in the perfect matching $\mathcal{M}$.

For a tree with a perfect matching, there is at most one alternating path between any pair of vertices. Note that if $P(i,j)$ is an alternating path between vertices i and j, then the number of edges in $P(i,j)$ which are not in $\mathcal{M}$ is $\frac{|P(i,j)|-1}{2}$, where $|P(i,j)|$ is the number of the edges in the path $P(i,j)$.

Proposition 3.6 *(Buckley, Doty and Harary [5, Theorem 3]) Let T be a nonsingular tree on n vertices and A be its adjacency matrix. Let $B = (b_{i,j})$, where*

$$b_{i,j} = \begin{cases} (-1)^{\frac{|P(i,j)|-1}{2}}, & \text{if there is an alternating path } P(i,j); \\ 0, & \text{otherwise.} \end{cases}$$

Then $B = A^{-1}$.

Let T be a nonsingular tree with vertices $1, 2, \cdots, n$. Let T^{-1} denote the graph with vertex set $\{1, 2, \cdots, n\}$, where vertices i and j are adjacent in T^{-1} if and only if there is an alternating path between i and j in T. We call the graph T^{-1} the *inverse graph* of the nonsingular tree T. It is shown in [3] that the graph T^{-1} is connected and bipartite, see [3] for more detail.

Corollary 3.7 *(Barik, Neumann and Pati [3, Lemma 2.3]) Let T be a nonsingular tree and T^{-1} be its inverse graph. Then the inverse matrix of the adjacency matrix of T is similar to the adjacency matrix of T^{-1} via a diagonal matrix of ± 1.*

Let $P(i,j)$ be an alternating path from vertex i to vertex j of $\overrightarrow{T}$ and let $|\overrightarrow{P}(i,j)|$ be the number of oriented edges in $P(i,j)$ whose orientation agrees with the routing from i to j. Note that if the alternating path $P(i,j) = i_1 i_2 i_3 \cdots i_{2k}$ (where $i_1 = i, i_{2k} = j$) then $(-1)^{|\overrightarrow{P}(i,j)|} = (-s_{i_1,i_2})(-s_{i_2,i_3}) \cdots (-s_{i_{2k-1},i_{2k}})$, where $S = (s_{i,j})$ is the skew-adjacency matrix of $\overrightarrow{T}$.

Although we are concerned with trees here, it should be mentioned that Proposition 3.6 and Corollary 3.7 have been generalized to bipartite graphs with a unique perfect matching (see [3, Lem. 2.1] and [2, Thm. 5 and Cor.5]. Also, the inverse graph T^{-1} is presented as an example of a graph inverse G^+ defined in [11] (see Thm 3.2 there).

Using a technique similar to that in [3, Lemma 2.1], we obtain the following combinatorial description of the inverse of the skew-adjacency matrix of a tree with a perfect matching.

Lemma 3.8 *Let $\overrightarrow{T}$ be an orientation of a nonsingular tree T on n vertices and S be its skew-adjacency matrix. Let $R = (r_{i,j})$, where*

$$r_{i,j} = \begin{cases} (-1)^{|\overrightarrow{P}(i,j)|}, & \text{if there is an alternating path } P(i,j), \\ 0, & \text{otherwise.} \end{cases}$$

Then $R = S^{-1}$.

Proof. The (i,j)-th entry of SR is given by

$$(SR)_{i,j} = \sum_{k=1}^{n} s_{i,k} r_{k,j} = \sum_{k \sim i} s_{i,k} r_{k,j}.$$

Thus for each $i = 1, 2, ..., n$,

$$(SR)_{i,i} = \sum_{k \sim i} s_{i,k} r_{k,i} = s_{i,i'}(-s_{i',i}) = 1,$$

as there exists exactly one vertex, say i', such that the edge $i'i \in \mathcal{M}$.

Now let i, j be two distinct vertices in T. Suppose that for each vertex v adjacent to i, there is no alternating path from v to j, then $r_{v,j} = 0$. Thus we have $(SR)_{i,j} = 0$. Moreover, v is unique, otherwise there would be a cycle in T containing the vertex i.

Assume now that there is a vertex $v \neq i'$ adjacent to i such that $P(v,j) = v x_2 \cdots x_{m-1} j$ is an alternating path from v to j. In this case, $P' = i' i v x_2 \cdots x_{m-1} j$, that is, $i'iP(v,j)$ is also an alternating path from i' to j. Conversely, if there is an alternating path $P(i',j)$ from i' to j, it must have the form $i'ivx_2 \cdots x_{m-1}j$. Thus there must exist a vertex $v \neq i'$ adjacent to i such that an alternating path from v to j exists.

We have just seen that the alternating path from i' to j is of the form $i'iP(v,j)$, where $P(v,j)$ is the alternating path from v to j. Hence

$$(SR)_{i,j} = s_{i,i'} r_{i',j} + s_{i,v} r_{v,j} = s_{i,i'}(-s_{i',i})(-s_{i,v}) r_{v,j} + s_{i,v} r_{v,j} = 0$$

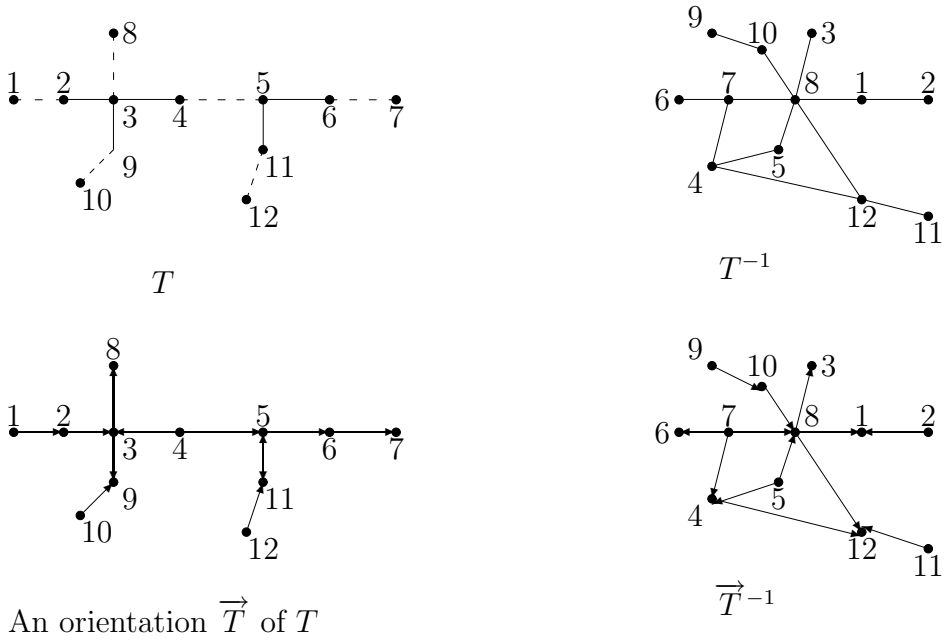

Figure 2: A tree and its inverses

and the proof is done. □

From Lemma 3.8, we see that if S^{-1} is the skew-adjacency matrix of an orientation $\overrightarrow{T}$ of a tree T with a perfect matchings, then S^{-1} is also a skew symmetric matrix with entries $0, -1$, or 1. Thus S^{-1} is the skew-adjacency matrix of some oriented graph, we use the notation $\overrightarrow{T}^{-1}$ for this oriented graph and call $\overrightarrow{T}^{-1}$ the *inverse oriented graph* of $\overrightarrow{T}$. Because of $|b_{ij}| = |r_{ij}|$ in Proposition 3.6 and Lemma 3.8, it follows that $\overrightarrow{T}^{-1}$ is an orientation of the inverse graph T^{-1} of T. See Fig. 2 for an example based on Fig. 1 in [3]. The dotted lines represent the edges in the perfect matching $\mathcal{M}$.

Proposition 3.9 *Let T be a tree with a perfect matching and $\overrightarrow{T}$ be any orientation of T. Then $Sp(\overrightarrow{T}^{-1}) = \mathbf{i}Sp(T^{-1})$ and hence $\mathcal{E}(\overrightarrow{T}^{-1}) = \mathcal{E}(T^{-1})$.*

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be all eigenvalues of T. Then $\lambda_1, \lambda_2, ..., \lambda_n$ are non-zero as T is nonsingular and $Sp(T^{-1}) = \{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}\}$ by Corollary 3.7. As T is a tree, we have $Sp(\overrightarrow{T}) = \{\lambda_1\mathbf{i}, \lambda_2\mathbf{i}, \cdots, \lambda_n\mathbf{i}\}$ by Theorem 3.4. Thus $Sp_S(\overrightarrow{T}^{-1}) = \{-\frac{1}{\lambda_1}\mathbf{i}, -\frac{1}{\lambda_2}\mathbf{i}, \cdots, -\frac{1}{\lambda_n}\mathbf{i}\} = \{\frac{1}{\lambda_1}\mathbf{i}, \frac{1}{\lambda_2}\mathbf{i}, \cdots, \frac{1}{\lambda_n}\mathbf{i}\}$ for the skew-adjacency matrix of $\overrightarrow{T}^{-1}$ is the inverse of the skew-adjacency matrix of $\overrightarrow{T}$ and the negative of each eigenvalue of T is also an eigenvalue of T. Therefore $Sp(\overrightarrow{T}^{-1}) = \mathbf{i}Sp(T^{-1})$ and hence $\mathcal{E}(\overrightarrow{T}^{-1}) = \mathcal{E}(T^{-1})$. □

Acknowledgment

The authors would like to express their sincere gratitude to the referee for a very careful reading of the paper and for all his or her insightful comments and valuable suggestions, which make a number of improvements on this paper. The first author was supported by National Natural Science Foundation of China.

References

[1] C. Adiga, R. Balakrishnan and Wasin So, The skew energy of a digraph, *Linear Algebra and its Applications* 432 (2010) 1825-1835.

[2] S. Akbari and S. J. Kirkland, On unimodular graphs, *Linear Algebra and its Applications* 421 (2007) 3-15.

[3] S. Barik, M. Neumann and S. Pati, On nonsingular trees and a reciprocal eigenvalue property, *Linear and Mulitilinear Algebra* 54 (2006) 453–465.

[4] S. Barik, M. Nath, S. Pati and B. K. Sarma, Unicyclic graphs with the strong reciprocal eigenvalue property, *Electronic Journal of Linear Algebra 17 (2008) 139–153.*

[5] F. Buckley, L. L. Doty and F. Harary, On graphs with signed inverses, *Networks,* 18 (1988) 151–157.

[6] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory,* Cambridge University Press, Cambridge, 1991.

[7] D. Cvetkovic M. Doob and H. Sachs, *Spectra of Graphs,* Academic Press, New York, 1980.

[8] C. D. Godsil, *Algebraic Combinatorics,* Chapman & Hall, London, 1993.

[9] L. Lovász and M. Plummer, *Matching Theory,* Ann. of Discrete Math. 29, North-Holland, New York,1988.

[10] B. Shader and Wasin So, Skew spectra of oriented graphs, *The Electronic Journal of Combinatorics* 16 (2009), #N32.

[11] R. M. Tifenbach and S. J. Kirkland, Directed intervals and dual of a graph, *Linear Algebra and its Applications* 431 (2009) 792–807.