q, t-Catalan numbers and generators for the radical ideal defining the diagonal locus of $(\mathbb{C}^2)^n$

Kyungyong Lee*
Department of Mathematics
University of Connecticut
Storrs, CT 06269, U.S.A.
kyungyong.lee@uconn.edu

Li Li
Department of Mathematics and Statistics
Oakland University
Rochester, MI 48309, U.S.A.
li2345@oakland.edu

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Abstract

Let I be the ideal generated by alternating polynomials in two sets of n variables. Haiman proved that the q, t-Catalan number is the Hilbert series of the bi-graded vector space $M(= \bigoplus_{d_1,d_2} M_{d_1,d_2})$ spanned by a minimal set of generators for I. In this paper we give simple upper bounds on dim M_{d_1,d_2} in terms of number of partitions, and find all bi-degrees (d_1, d_2) such that $\dim M_{d_1,d_2}$ achieve the upper bounds. For such bi-degrees, we also find explicit bases for M_{d_1,d_2}.

1 Introduction

In [6], Garsia and Haiman introduced the q, t-Catalan number $C_n(q, t)$, and showed that $C_n(q, 1)$ agrees with the q-Catalan number defined by Carlitz and Riordan [3]. To be more precise, take the $n \times n$ square whose southwest corner is $(0, 0)$ and northeast corner is (n, n). Let $\mathcal{D}_n$ be the collection of Dyck paths, i.e. lattice paths from $(0, 0)$ to (n, n) that proceed by NORTH or EAST steps and never go below the diagonal. For any Dyck path Π, define area(Π) to be the number of lattice squares below Π and strictly above the diagonal. Then

$$C_n(q, 1) = \sum_{\Pi \in \mathcal{D}_n} q^{\mathrm{area}(\Pi)}.$$

The q, t-Catalan number $C_n(q, t)$ also has a combinatorial interpretation using Dyck paths. Given a Dyck path Π, let $a_i(\Pi)$ be the number of squares in the i-th row that lie in the region bounded by Π and the diagonal, and define

$$\mathrm{dinv}(\Pi) := \big|\{(i, j) \,|\, i < j \text{ and } a_i(\Pi) = a_j(\Pi)\}\big| + \big|\{(i, j) \,|\, i < j \text{ and } a_i(\Pi) + 1 = a_j(\Pi)\}\big|.$$

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In [4, §1] and [5, Theorem I.2], Garsia and Haglund showed the following combinatorial formula [1],

$$C_n(q,t) = \sum_{\Pi \in \mathcal{D}_n} q^{\mathrm{area}(\Pi)} t^{\mathrm{dinv}(\Pi)}. \tag{1.1}$$

A natural question is to find the coefficient of $q^{d_1}t^{d_2}$ in $C_n(q,t)$ for each pair (d_1, d_2). In other words, the question is to count the Dyck paths with the same pair of statistics (area, dinv). It is well-known that the sum $\mathrm{area}(\Pi) + \mathrm{dinv}(\Pi)$ is at most $\binom{n}{2}$. In this paper we find coefficients of $q^{d_1}t^{d_2}$ in $C_n(q,t)$ when $\binom{n}{2} - d_1 - d_2$ is relatively small.

Denote by $p(k)$ the number of partitions of k and by convention $p(0) = 1$ and $p(k) = 0$ for $k < 0$. Denote by $p(b,k)$ the number of partitions of k with at most b parts, and by convention $p(0,k) = 0$ for $k > 0$, $p(b,0) = 1$ for $b \geq 0$. Our first theorem is as follows, which contains a result of Bergeron and Chen [1, Corollary 8.3.1] as a special case.

Theorem 1. *Let n be a positive integer, and d_1, d_2, k be non-negative integers such that $k = \binom{n}{2} - d_1 - d_2$. Define $\delta = \min(d_1, d_2)$. Then the coefficient of $q^{d_1}t^{d_2}$ in $C_n(q,t)$ is less than or equal to $p(\delta, k)$, and the equality holds if and only if one of the following conditions holds:*

- *$k \leq n-3$, or*
- *$k = n-2$ and $\delta = 1$, or*
- *$\delta = 0$.*

As a consequence, we recover a special case of a result of Loehr and Warrington with $C_n(q,t)$ replaced by any rational or irrational slope q,t-Catalan number (see [12, Theorem 3]. The result was probably first discovered by Mark Haiman according to their paper).

Corollary 2 (Haiman, Loehr–Warrington)**.** *In the formal power series ring $\mathbb{C}[[q^{-1}, t]]$, we have*

$$\lim_{n\to\infty} \frac{C_n(q,t)}{q^{\binom{n}{2}}} = \sum_{k,b\geq 0} p(b,k) q^{-k-b} t^b = \prod_{i=1}^{\infty} \frac{1}{1-q^{-i}t},$$

where the left hand side becomes a well-defined formal power series in the sense that, for any integers $i \leq 0$ and $j \geq 0$, the coefficient of $q^i t^j$ eventually becomes stationary.

And here is another corollary of Theorem 1.

Corollary 3.

$$C_n(q,q) = \sum_{k=0}^{n-3} \left(p(k)\left(\binom{n}{2} - 3k + 1\right) + 2\sum_{i=1}^{k-1} p(i,k) \right) q^{\binom{n}{2}-k} + (\textit{lower degree terms}).$$

[1] To be more precise, they showed $C_n(q,t) = \sum q^{\mathrm{area}(\Pi)} t^{\mathrm{maj}(\beta(\Pi))}$. The right hand side is equal to $\sum q^{\mathrm{dinv}(\Pi)} t^{\mathrm{area}(\Pi)}$ ([7, Theorem 3.15], where $\mathrm{maj}(\beta(\Pi))$ is the same as $\mathrm{bounce}(\Pi)$), and is then equal to $\sum q^{\mathrm{area}(\Pi)} t^{\mathrm{dinv}(\Pi)}$ [7, (3.52)].

We feel that the coefficient of $q^{d_1}t^{d_2}$ for general k can also be expressed in terms of numbers of partitions, although the expression might be complicated. For example, we give the following conjecture which is verified for $6 \le n \le 10$.

Conjecture 4. *Let n, d_1, d_2, δ, k be as in Theorem 1. If $n-2 \le k \le 2n-8$ and $\delta \ge k$, then the coefficient of $q^{d_1}t^{d_2}$ in $C_n(q,t)$ is equal to*

$$p(k) - 2[p(0) + p(1) + \cdots + p(k-n+1)] - p(k-n+2).$$

From the perspective of commutative algebra, the q,t-Catalan number is closely related to the diagonal ideal I that we are about to define. Let n be a positive integer. The set of all n-tuples of points in $\mathbb{C}^2$ forms an affine space $(\mathbb{C}^2)^n$ with coordinate ring $\mathbb{C}[\mathbf{x},\mathbf{y}] := \mathbb{C}[x_1, y_1, ..., x_n, y_n]$. We define the *diagonal ideal* $I \subset \mathbb{C}[\mathbf{x},\mathbf{y}]$ to be

$$I := \bigcap_{1 \le i < j \le n} (x_i - x_j, y_i - y_j).$$

(We define $I = (1)$ if $n = 1$.) Geometrically, I is the radical ideal defining the diagonal locus of $(\mathbb{C}^2)^n$ where at least two points coincide. Blowing up the ideal I gives the well-known isospectral Hilbert scheme discovered by Haiman in his proof of the $n!$ conjecture and the positivity conjecture for the Kostka-Macdonald coefficients [8, §3.4].

Let $M := I/(\mathbf{x},\mathbf{y})I$, where $(\mathbf{x},\mathbf{y})$ is the maximal ideal $(x_1, y_1, \ldots, x_n, y_n)$. The vector space M is naturally bi-graded as $\bigoplus_{d_1,d_2} M_{d_1,d_2}$ with respect to x- and y- degrees. A basis of the $\mathbb{C}$-vector space M corresponds to a minimal set of generators of I. Haiman discovered that the q,t-Catalan number $C_n(q,t)$ is exactly the Hilbert series of M [9, Corollary 3.3]:

$$C_n(q,t) = \sum_{d_1,d_2} q^{d_1}t^{d_2} \dim_{\mathbb{C}} M_{d_1,d_2}. \tag{1.2}$$

In the special case of $q = t = 1$, (1.2) implies that $\dim_{\mathbb{C}} M = \frac{1}{n+1}\binom{2n}{n} = C_n$, which is the usual Catalan number.

A natural question, posed by Haiman, is to study a minimal set of generators of the ideal I [10, §1]. There is a set of generators of the diagonal ideal I defined as follows. Denote by $\mathbb{N}$ the set of nonnegative integers. Let $\mathfrak{D}_n$ be the collection of sets $D = \big\{(a_1, b_1), ..., (a_n, b_n)\big\}$ of n distinct points in $\mathbb{N} \times \mathbb{N}$. For each $D \in \mathfrak{D}_n$, define

$$\Delta(D) = \Delta\big((a_1, b_1), ..., (a_n, b_n)\big) := \det[x_i^{a_j} y_i^{b_j}] = \begin{vmatrix} x_1^{a_1}y_1^{b_1} & x_1^{a_2}y_1^{b_2} & \ldots & x_1^{a_n}y_1^{b_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1}y_n^{b_1} & x_n^{a_2}y_n^{b_2} & \ldots & x_n^{a_n}y_n^{b_n} \end{vmatrix}.$$

Although $\Delta(D)$ depends on the order of $(a_1, b_1), ..., (a_n, b_n)$, $\Delta(D)$ is well-defined up to sign. Actually, we will fix a certain order as in §2.3. Then $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$ form a basis for the vector space $\mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$ of alternating polynomials. In [8, Corollary 3.8.3], Haiman proved that I is generated by $\mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$. An immediate consequence is that I is generated by $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$. But this set of generators is infinite and is far from being a minimal set, which should contain exactly C_n elements.

In general, it is difficult to construct a basis of M (or equivalently, a minimal set of generators of I). Meanwhile, not much is known about each graded piece M_{d_1,d_2}. In this paper, we give an explicit combinatorial basis for the subspace M_{d_1,d_2} of $I/(\mathbf{x},\mathbf{y})\cdot I$ for certain d_1 and d_2.

Theorem 5 (Main Theorem)**.** *Let n be a positive integer, and d_1, d_2, k be non-negative integers such that $k = \binom{n}{2} - d_1 - d_2$. Define $\delta = \min(d_1, d_2)$. Then $\dim M_{d_1,d_2} \leq p(\delta, k)$, and the equality holds if and only if one of the following conditions holds:*

- $k \leq n-3$, *or*
- $k = n-2$ *and* $\delta = 1$, *or*
- $\delta = 0$.

In case the equality holds, there is an explicit construction of a basis for M_{d_1,d_2}.

The Main Theorem follows immediately from Theorem 44 in §6.2 and Theorem 55 in §7. The construction of the basis for M_{d_1,d_2} consists of two parts: the easier part is to show

$$\dim M_{d_1,d_2} \leq p(\delta, k)$$

using a new characterization of q,t-Catalan numbers given in §5.1; the more difficult part is to construct $p(\delta, k)$ linearly independent elements in M_{d_1,d_2}. It seems difficult (at least to the authors) to test directly whether a given set of elements in M_{d_1,d_2} are linearly independent. Instead, we study a map φ sending an alternating polynomial $f \in \mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$ to a polynomial in a polynomial ring $\mathbb{C}[\rho] := \mathbb{C}[\rho_1, \rho_2, \dots]$ with countably many variables. The map φ has two desirable properties: (i) for many f, $\varphi(f)$ can be easily computed, and (ii) for each bi-degree (d_1, d_2), φ induces a well-defined morphism $\bar{\varphi} : M_{d_1,d_2} \longrightarrow \mathbb{C}[\rho]$. Therefore, in order to prove linear independence of a set of elements in M_{d_1,d_2}, it is sufficient (and necessary if Conjecture 48 holds) to prove linear independence of the images of those elements in $\mathbb{C}[\rho]$ under the map $\bar{\varphi}$. The latter is much easier.

The structure of the paper is as follows. After introducing the notations in §2, we study the asymptotic behavior in §3, then we define and study the map φ in §4. In §5 and §6 we give the upper bound and the lower bound of $\dim M_{d_1,d_2}$. Finally, we finish the proof of the main result in §7.

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2 Notations

2.1 General notations

- We adopt the convention that $\mathbb{N}$ is the set of natural numbers including zero, and $\mathbb{N}^+$ is the set of positive integers.

- For $n \in \mathbb{N}^+$, denote by S_n the symmetric group on the set $\{1, ..., n\}$.

2.2 Notations related to partitions and the ring $\mathbb{C}[\rho]$

- Let $k, b \in \mathbb{N}^+$. Denote the set of partitions of k by Π_k, and the set of partitions of k into at most b parts by $\Pi_{b,k}$. To be more precise,

  $$\Pi_k := \{\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) |\, \nu_i \in \mathbb{N}^+,\, \nu_1 \leq \nu_2 \leq \cdots \leq \nu_\ell,\, \nu_1 + \nu_2 + \cdots + \nu_\ell = k\}.$$
  $$\Pi_{b,k} := \{\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \Pi_k |\, \ell \leq b\}.$$

  A partition $\nu = (\underbrace{j_1, \ldots, j_1}_{m_1}, \underbrace{j_2, \ldots, j_2}_{m_2}, \ldots, \underbrace{j_r, \ldots, j_r}_{m_r})$ is also written as $\sum_{i=1}^{r} m_i j_i$.

  Define the number of partitions $p(k) = \#\Pi_k$ and $p(b, k) = \#\Pi_{b,k}$. By convention $p(0) = 1$, $p(0, k) = 0$ for $k > 0$, $p(b, 0) = 1$ for all $b \geq 0$.

- For a partition $\nu \in \Pi_k$, define $|\nu| := \sum \nu_i = k$.

- Define a partial order on the set of partitions Π_k as follows: for two partitions $\mu = (\mu_1, \cdots, \mu_s)$ and $\nu = (\nu_1, \cdots, \nu_t)$ in Π_k, define $\mu \preceq \nu$ if there is a partition of the set $\{1, \ldots, s\}$ with t nonempty parts $I_1, \ldots, I_t$, such that $\sum_{j \in I_i} \mu_j = \nu_i$ for $i = 1, \ldots, t$. Define $\mu \prec \nu$ if $\mu \prec \nu$ and $\mu \neq \nu$.

- Let $\mathbb{C}[\rho] := \mathbb{C}[\rho_1, \rho_2, \ldots]$ be the polynomial ring with countably many variables ρ_i, $i \in \mathbb{N}^+$. As a convention, we set $\rho_0 = 1$. For a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \Pi_k$, define $\rho_\nu := \rho_{\nu_1} \rho_{\nu_2} \cdots \rho_{\nu_\ell} \in \mathbb{C}[\rho]$. Define the *weight* of a monomial $c\rho_{i_1} \cdots \rho_{i_\ell}$ $(c \in \mathbb{C} \setminus \{0\})$ to be $i_1 + \cdots + i_\ell$. For $w \in \mathbb{N}$, define $\mathbb{C}[\rho]_w$ to be the subspace of $\mathbb{C}[\rho]$ spanned by monomials of weight w. For $f \in \mathbb{C}[\rho]$, there is a unique expression $f = \sum_{w=0}^{\infty} \{f\}_w$ with $\{f\}_w \in \mathbb{C}[\rho]_w$, and we call $\{f\}_w$ the *weight-w part of f*.

2.3 Notations on ordered sequences D of n points in $\mathbb{N} \times \mathbb{N}$

- For $P = (a, b) \in \mathbb{N} \times \mathbb{N}$, denote $|P| = a + b$, $|P|_x = a$, $|P|_y = b$.

- For $n \in \mathbb{N}^+$, define

  $$\mathfrak{D}_n := \{D = (P_1, \ldots, P_n) |\, P_i \in \mathbb{N} \times \mathbb{N}, \text{ for all } i = 1, \ldots, n\},$$
  $$\mathfrak{D}'_n := \{D = (P_1, \ldots, P_n) \big|\, |P_i|_x \in \mathbb{Z},\, |P_i|_y \in \mathbb{N},\, |P_i| \geq 0, \text{ for all } i = 1, \ldots, n\}.$$

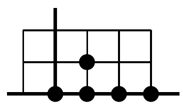
Define $\mathfrak{D} := \cup_{n=1}^{\infty}\mathfrak{D}_n$ and $\mathfrak{D}' = \cup_{n=1}^{\infty}\mathfrak{D}'_n$. For $D = (P_1, \dots, P_n)$ in $\mathfrak{D}_n$ or $\mathfrak{D}'_n$, we let (a_i, b_i) be the coordinates of P_i, $i = 1, \dots, n$. Unless otherwise specified, we assume throughout the paper that $P_1, \dots, P_n$ in D are in standard order, meaning that

$$P_1 < P_2 < \cdots < P_n, \tag{2.1}$$

where the relation "<" is defined as follows:

$$(a, b) < (a', b') \text{ if } a + b < a' + b', \text{ or if } a + b = a' + b' \text{ and } a < a'.$$

For D in standard order, we often use a square grid graph together with n dots to visualize it. For example, in the following picture, the horizontal and vertical bold lines represent x- and y-axes, respectively, and $D = \big((0, 0), (1, 0), (1, 1), (2, 0), (3, 0)\big)$.



- Given $D = (P_1, \dots, P_n) \in \mathfrak{D}_n$, we define the x-degree, y-degree and bi-degree of D to be $\sum_{i=1}^n |P_i|_x$, $\sum_{i=1}^n |P_i|_y$, and $(\sum_{i=1}^n |P_i|_x, \sum_{i=1}^n |P_i|_y)$, respectively.

2.4 Notations related to the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$

- The diagonal ideal I of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ and the graded $\mathbb{C}$-vector space $M = \oplus_{d_1,d_2} M_{d_1,d_2}$ are defined in §1. The ideal generated by homogeneous elements in I of degrees less than d is denoted by $I_{<d}$.

- Given a monomial $f = x_1^{a_1} y_1^{b_1} \cdots x_n^{a_n} y_n^{b_n} \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$, we define the bi-degree of f to be the pair $(\sum_{i=1}^n a_i, \sum_{i=1}^n b_i)$. We say that a polynomial in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ has bi-degree (d_1, d_2) if all its monomials have the same bi-degree (d_1, d_2).

- For $D \in \mathfrak{D}_n$, the alternating polynomial $\Delta(D) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is defined in §1. It is easy to see that the bi-degree of $\Delta(D)$ is equal to the bi-degree of D.

- Given two polynomials $f, g \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ of the same bi-degree (d_1, d_2), let $\bar{f}, \bar{g}$ be the corresponding elements in M_{d_1,d_2}. We say that

$$f \equiv g \quad \text{(modulo lower degrees)}$$

if $\bar{f} = \bar{g}$ in M_{d_1,d_2}, or, equivalently, if $f - g$ is in $I_{<d_1+d_2}$.

3 The asymptotic behavior

The goal of this section is to prove Theorem 14 which gives explicit bases for certain M_{d_1,d_2} under restrictive conditions on n, d_1, d_2. Roughly speaking, we study the behavior of M_{d_1,d_2} for $d_1 + d_2$ close enough to $\binom{n}{2}$, the highest degree of M, under the condition that d_1 and d_2 are not too small. We call this behavior the asymptotic behavior, because if we fix a positive integer k, let n, d_1, d_2 grow and satisfy $d_1 + d_2 = \binom{n}{2} - k$, then a simple pattern of behavior of M_{d_1,d_2} will appear when n, d_1, d_2 are sufficiently large. Such an asymptotic study provides the foundation for the whole paper.

3.1 Staircase forms and block diagonal forms

Definition-Proposition 6. Let $D = (P_1, \dots, P_n) \in \mathfrak{D}_n$, $P_i = (a_i, b_i)$ be as in §2. Define $k = \binom{n}{2} - \sum_i |P_i|$. Then there is an $n \times n$ matrix S whose (i, j)-th entry is

$$\begin{cases} 0, & \text{if } i \le |P_j|; \\ z_{i1} z_{i2} \cdots z_{i,|P_j|}, \text{ where } z_{i\ell} \text{ is either } x_i - x_\ell \text{ or } y_i - y_\ell, & \text{otherwise,} \end{cases}$$

for all $1 \le i, j \le n$, such that $\det S \equiv \Delta(D)$ (modulo lower degrees). We call S a *staircase form* of D.

Proof. Let $x_{ij} := x_i - x_j$ and $y_{ij} := y_i - y_j$ for $1 \le i, j \le n$. If $a_1 > 0$, the first column of the matrix $[x_i^{a_j} y_i^{b_j}]$ is equal to the following (where T means taking transpose of a matrix)

$$x_1[x_1^{a_1-1} y_1^{b_1}, \dots, x_n^{a_1-1} y_n^{b_1}]^T + [0, x_2^{a_1-1} x_{21} y_2^{b_1}, \dots, x_n^{a_1-1} x_{n1} y_n^{b_1}]^T.$$

Therefore

$$\Delta(D) = x_1 \begin{vmatrix} x_1^{a_1-1} y_1^{b_1} & \cdots & x_1^{a_n} y_1^{b_n} \\ \vdots & \ddots & \vdots \\ x_n^{a_1-1} y_n^{b_1} & \cdots & x_n^{a_n} y_n^{b_n} \end{vmatrix} + \begin{vmatrix} 0 & x_1^{a_2} y_1^{b_2} & \cdots & x_1^{a_n} y_1^{b_n} \\ x_2^{a_1-1} x_{21} y_2^{b_1} & x_2^{a_2} y_2^{b_2} & \cdots & x_2^{a_n} y_2^{b_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1-1} x_{n1} y_n^{b_1} & x_n^{a_2} y_n^{b_2} & \cdots & x_n^{a_n} y_n^{b_n} \end{vmatrix}$$

The first summand is a polynomial in $I_{<d}$, so $\Delta(D)$ is equivalent to the second summand modulo $I_{<d}$. If furthermore $a_1 - 1 > 0$, the first column $[0, x_2^{a_1-1} x_{21} y_2^{b_1}, \dots, x_n^{a_1-1} x_{n1} y_n^{b_1}]^T$ in the second determinant can be written as a sum of two vectors

$$x_2[0, x_2^{a_1-2} x_{21} y_2^{b_1}, \dots, x_n^{a_1-2} x_{n1} y_n^{b_1}]^T + [0, 0, x_3^{a_1-2} x_{32} x_{31} y_3^{b_1}, \dots, x_n^{a_1-2} x_{n2} x_{n1} y_n^{b_1}]^T.$$

Then by a similar argument as above, $\Delta(D)$ is equivalent to the determinant

$$\begin{vmatrix} 0 & x_1^{a_2} y_1^{b_2} & \cdots & x_1^{a_n} y_1^{b_n} \\ 0 & x_2^{a_2} y_2^{b_2} & \cdots & x_2^{a_n} y_2^{b_n} \\ x_3^{a_1-2} x_{32} x_{31} y_3^{b_1} & x_3^{a_2} y_3^{b_2} & \cdots & x_3^{a_n} y_3^{b_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1-2} x_{n2} x_{n1} y_n^{b_1} & x_n^{a_2} y_n^{b_2} & \cdots & x_n^{a_n} y_n^{b_n} \end{vmatrix}$$

modulo $I_{<d}$. If $b_1 > 0$, we apply similar operation as above. Eventually the first column becomes

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{|P_1|+1,1}x_{|P_1|+1,2}\cdots x_{|P_1|+1,a_1}y_{|P_1|+1,a_1+1}y_{|P_1|+1,a_1+2}\cdots y_{|P_1|+1,|P_1|} \\ x_{|P_1|+2,1}x_{|P_1|+2,2}\cdots x_{|P_1|+2,a_1}y_{|P_1|+2,a_1+1}y_{|P_1|+2,a_1+2}\cdots y_{|P_1|+2,|P_1|} \\ \vdots \\ x_{n1}x_{n2}\cdots x_{n,a_1}y_{n,a_1+1}y_{n,a_1+2}\cdots y_{n,|P_1|} \end{bmatrix},$$

where the top $\min\{|P_1|, n\}$ entries are 0. Note that if we use a different order of operations with respect to x_i or y_i, we may end up with a different first column.

Applying this procedure for every column, we get a matrix with $\min\{|P_j|, n\}$ zeros at the j-th column for $1 \leq j \leq n$. The resulting matrix is an expected staircase form S. $\square$

Corollary 7. *Let $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$ such that $|P_j| > j - 1$ for some $1 \leq j \leq n$. Then $\Delta(D) \equiv 0$ (modulo lower degrees).*

Proof. Let S be a staircase form of D. It is easy to check that $\det S = 0$, hence $\Delta(D) \equiv \det S = 0$ (modulo lower degrees) by Definition-Proposition 6. $\square$

Definition 8. Let D and S be defined as in Definition-Proposition 6. Consider the set $\{j \big| \, |P_j| = j - 1\} = \{r_1 < r_2 < \cdots < r_\ell\}$ and define $r_{\ell+1} = n + 1$. For $1 \leq t \leq \ell$, define the t-th block B_t of S to be the square submatrix of S of size $(r_{t+1} - r_t)$ whose upper left corner is the (r_t, r_t)-entry. Define the *block diagonal form* $B(S)$ of S to be the block diagonal matrix $\operatorname{diag}(B_1, \ldots, B_\ell)$.

Remark 9. It is easy to see that $\det B(S) = \det S$.

Example 10. Let $D = \big((0,0), (1,0), (0,2), (1,1), (3,1)\big)$. Then $\Delta(D)$ and a staircase form S are

$$\Delta(D) = \begin{vmatrix} 1 & x_1 & y_1^2 & x_1y_1 & x_1^3y_1 \\ 1 & x_2 & y_2^2 & x_2y_2 & x_2^3y_2 \\ 1 & x_3 & y_3^2 & x_3y_3 & x_3^3y_3 \\ 1 & x_4 & y_4^2 & x_4y_4 & x_4^3y_4 \\ 1 & x_5 & y_5^2 & x_5y_5 & x_5^3y_5 \end{vmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & x_{21} & 0 & 0 & 0 \\ 1 & x_{31} & y_{31}y_{32} & x_{31}y_{32} & 0 \\ 1 & x_{41} & y_{41}y_{42} & x_{41}y_{42} & 0 \\ 1 & x_{51} & y_{51}y_{52} & x_{51}y_{52} & x_{51}y_{52}x_{53}x_{54} \end{bmatrix},$$

and the block diagonal form of S is

$$B(S) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x_{21} & 0 & 0 & 0 \\ 0 & 0 & y_{31}y_{32} & x_{31}y_{32} & 0 \\ 0 & 0 & y_{41}y_{42} & x_{41}y_{42} & 0 \\ 0 & 0 & 0 & 0 & x_{51}y_{52}x_{53}x_{54} \end{bmatrix}.$$

Definition 11. Suppose that $\mu = \sum m_i j_i \in \Pi_k$ is a partition of k, where j_i are distinct positive integers. We say that a nonzero staircase form S is of *partition type* μ, if for each i the block diagonal form $B(S)$ contains exactly m_i blocks that have j_i nonzero entries strictly above the diagonal. We say that $D \in \mathfrak{D}_n$ is of partition type μ if its staircase form is of partition type μ. Furthermore, if

$$\text{(the entry in the } i\text{-th row and } j\text{-th column in } S) = 0 \text{ for each pair } (i,j),\ j > i+1, \tag{3.1}$$

then S is called a *minimal* staircase form of partition type μ. We call a block *minimal* if the block satisfies condition (3.1).

Remark 12. Let S be a staircase form of $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$. Then S is a minimal staircase form if and only if $|P_i| = i-1$ or $i-2$ for every $1 \le i \le n$. In this case, the partition type of S is

$$(i_1 - 1, i_2 - i_1 - 1, i_3 - i_2 - 1, \ldots, i_\ell - i_{\ell-1} - 1, n - i_\ell),$$

where $\{i_1 < i_2 < \cdots < i_\ell\}$ is the set of i's such that $|P_i| = i-1$.

For example, if $n = 8$, $D = (P_1, \ldots, P_8)$ and $(|P_1|, \ldots, |P_8|) = (0,1,1,2,4,4,5,6)$, then the staircase form S of D is a minimal staircase form. The set $\{i \mid |P_i| = i-1\}$ is $\{1,2,5\}$. The positive integers in the sequence $(1-1, 2-1-1, 5-2-1, 8-5)$ are $(2,3)$, so the partition type of D is $(2,3)$. □

Example 13. Suppose $n = 11$, $k = 7$, $D = (P_1, \ldots, P_{11})$ such that $(|P_1|, \ldots, |P_{11}|) = (0,1,2,2,4,4,4,7,7,8,9)$. Then a staircase form of D is of partition type $(1,3,3)$ but is not a minimal staircase form because there is a nonzero entry in the fifth row and seventh column. (In the matrices below, a "$*$" means a nonzero entry.)

$$S = \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \end{bmatrix}, \quad B(S) = \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

3.2 Theorem on asymptotic behavior of M_{d_1,d_2} and the proof

The main theorem of this section is the following.

Theorem 14. *Let k, n, d_1, d_2 be integers satisfying $n \ge 8k+5$, $d_1, d_2 \ge (2k+1)n$, and $d_1 + d_2 = \binom{n}{2} - k$. Then*

$$\dim_{\mathbb{C}} M_{d_1,d_2} = p(k).$$

Moreover, for each $\mu \in \Pi_k$, let S_μ be an arbitrary minimal staircase form of bi-degree (d_1, d_2) and of partition type μ. Then $\{\det S_\mu\}_{\mu \in \Pi_k}$ form a basis of M_{d_1,d_2}.

We need to establish a few lemmas before proving the above theorem.

Lemma 15 (Transfactor Lemma). *Let $D = (P_1, \dots, P_n) \in \mathfrak{D}_n$ and $P_i = (a_i, b_i)$ be as in §2. Let i, j be two integers satisfying $1 \le i \ne j \le n$, $|P_i| = i - 1$, $|P_{i+1}| = i$, $|P_j| = j - 1$, $|P_{j+1}| = j$, $b_i > 0$, $a_j > 0$ (we define $|P_{n+1}| = n$). Let D' be obtained from D by moving P_i to southeast and P_j to northwest, i.e.,*

$$D' = \big(P_1, \dots, P_{i-1}, P_i + (1, -1), P_{i+1}, \dots, P_{j-1}, P_j + (-1, 1), P_{j+1}, \dots, P_n\big).$$

Then $\Delta(D) \equiv \Delta(D')$ (modulo lower degrees).

Proof. By performing appropriate operations as in the proof of Definition-Proposition 6, we can obtain a staircase form S of D (resp. a staircase form S' of D'), such that the (i,i)-entry and (j,j)-entry of S (resp. S') are $y_{i1}\prod_{t=2}^{i-1} z_{it}$ and $x_{j1}\prod_{t=2}^{j-1} z_{jt}$ (resp. $x_{i1}\prod_{t=2}^{i-1} z_{it}$ and $y_{j1}\prod_{t=2}^{j-1} z_{jt}$). The block diagonal forms of S and S' only differ at two blocks of size 1 located at the (i,i)-entry and (j,j)-entry. Let f_0 be the product of determinants of all blocks of $B(S)$ except the (i,i)-entry and (j,j)-entry. Then $\Delta(D) - \Delta(D')$ is equivalent to the following (modulo lower degrees)

$$\begin{aligned}
\det(S) - \det(S') &= \Big(y_{i1}\prod_{t=2}^{i-1} z_{it}\Big)\Big(x_{j1}\prod_{t=2}^{j-1} z_{jt}\Big) f_0 - \Big(x_{i1}\prod_{t=2}^{i-1} z_{it}\Big)\Big(y_{j1}\prod_{t=2}^{j-1} z_{jt}\Big) f_0 \\
&= -\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{bmatrix} \Big(\prod_{t=2}^{i-1} z_{it}\Big)\Big(\prod_{t=2}^{j-1} z_{jt}\Big) f_0.
\end{aligned}$$

Without loss of generality, assume $i < j$. Then $(\det(S) - \det(S'))/z_{ji}$ is

$$-\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{bmatrix} \Big(\prod_{t=2}^{i-1} z_{it}\Big)\Big(\prod_{\substack{2\le t\le j-1 \\ t\ne i}} z_{jt}\Big) f_0.$$

This polynomial vanishes on the diagonal locus, so is in $I_{<d}$, and then the lemma follows. □

The Transfactor Lemma implies the following lemma, which is the base case $k = 0$ of the inductive proof of Proposition 23.

Lemma 16. *Let d_1, d_2 be two non-negative integers such that $d_1 + d_2 = \binom{n}{2}$. Let S be an arbitrary staircase form with bi-degree (d_1, d_2) and assume that $\det S \ne 0$. Then the $\mathbb{C}$-vector space M_{d_1,d_2} is spanned by $\det S$.*

Proof. Because $d_1 + d_2 = \binom{n}{2}$, there are $\binom{n}{2}$ zeros in the staircase form S. Since $\det S \ne 0$, S and its block diagonal form $B(S)$ must be of the following forms

$$S = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}, \quad B(S) = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & 0 \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}.$$

By repeatedly applying Lemma 15 we can easily deduce the following assertion: if S and S' are staircase forms of D and D', respectively, such that both S and S' have bi-degree (d_1, d_2), then $\det B(S') \equiv \det B(S)$ modulo $I_{<n(n-1)/2}$. The lemma follows from this assertion. $\square$

Lemma 17 (Minors Permuting Lemma)**.** *Let $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$ and $P_i = (a_i, b_i)$ as in §2. Suppose $h, \ell, m \in \mathbb{N}^+$ satisfy $2 \le h < h + \ell + m \le n + 1$, $|P_h| = h - 1, |P_{h+\ell}| = h + \ell - 1$, $|P_{h+\ell+m}| = h + \ell + m - 1$ (if $h + \ell + m = n + 1$ then we assume that $|P_{h+\ell+m}| = h + \ell + m - 1$ is vacuously true). Suppose that $a_{h+\ell}, \ldots, a_{h+\ell+m-1} \ge \ell$. Define D' by*

$$D' = \big(P_1, P_2, \ldots, P_{h-1}, P_{h+\ell} - (\ell, 0), P_{h+\ell+1} - (\ell, 0), \ldots, P_{h+\ell+m-1} - (\ell, 0),\\ P_h + (m, 0), P_{h+1} + (m, 0), \ldots, P_{h+\ell-1} + (m, 0), P_{h+\ell+m}, \ldots, P_n\big).$$

Then $\Delta(D) \equiv \Delta(D')$ (modulo lower degrees).

Proof. By performing appropriate operations as in the proof of Definition-Proposition 6 and using the assumption that $a_{h+\ell}, \ldots, a_{h+\ell+m-1} \ge \ell$, we can obtain a staircase form S of D whose (u, v)-entry contains the factor $\prod_{j=h}^{h+\ell-1} x_{uj} = \prod_{j=h}^{h+\ell-1} (x_u - x_j)$ for every pair (u, v) satisfying $h + \ell \le u, v \le h + \ell + m - 1$. Let $B(S) = \mathrm{diag}(B_1, B_2, \ldots, B_s)$ be the block diagonal form of S, and let B_r (resp. B_{r+1}) be the block of size ℓ (resp. m) whose upper left corner is the (h, h)-entry (resp. $(h + \ell, h + \ell)$-entry). Then by our choice of S, all entries in the i-th row ($1 \le i \le m$) of B_{r+1} contain $\prod_{j=h}^{h+\ell-1} x_{i+h+\ell-1,j}$ as a factor. Dividing the i-th row of B_{r+1} by $\prod_{j=h}^{h+\ell-1} x_{i+h+\ell-1,j}$ for $1 \le i \le m$ and multiplying the i'-th row of B_r by $\prod_{j=h+\ell}^{h+\ell+m-1} x_{i'+h-1,j}$ for $1 \le i' \le \ell$, we obtain a new block diagonal matrix $B' = \mathrm{diag}(B_1, \ldots, B_{r-1}, B'_r, B'_{r+1}, B_{r+2}, \ldots, B_s)$. Since

$$\prod_{j=h}^{h+\ell-1} x_{i+h+\ell-1,j} = (-1)^{\ell m} \prod_{j=h}^{h+\ell-1} x_{i+h+\ell-1,j},$$

we have $(-1)^{\ell m} \det B' = \det B = \det S$. Now interchange the two blocks B'_r and B'_{r+1} in B' and then change the indices $1, \ldots, n$ to

$$1, \ldots, (\ell - 1), (\ell + h), \ldots, (\ell + h + m - 1), \ell, \ldots, (\ell + h - 1), (\ell + h + m), \ldots, n.$$

The resulting matrix is the block diagonal matrix of a staircase form of D'. Note that when we change the indices, the determinant of the resulting matrix is equal to $(-1)^{\ell m} \det B'$. Therefore $\Delta(D) \equiv \Delta(D')$ (modulo lower degrees). $\square$

Example 18. Suppose $n = 11$, $k = 7$, $D = (P_1, \ldots, P_{11})$ such that $(|P_1|, \ldots, |P_{11}|) = (0, 1, 2, 2, 4, 4, 4, 7, 7, 8, 9)$, and $|P_8|_x, \ldots, |P_{11}|_x \ge 3$. Lemma 17 asserts that permuting the two blocks (as framed in the following figure) in the block diagonal form does not change the determinant modulo $I_{<d}$.

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix} \longrightarrow \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \end{bmatrix}$$

Lemma 19. *For $p, q \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$, we have*

$$A(Sym(p)q) = Sym(p)A(q),$$

where $Sym(p)$ denotes the symmetric sum $\sum_{\sigma \in S_n} \sigma(p)$, and $A(p)$ denotes the alternating sum $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma(p)$.

Proof. $A(Sym(p)q) = \sum_\sigma \operatorname{sgn}(\sigma) Sym(p)\sigma(q) = Sym(p)A(q)$. □

Lemma 20. *For $(a_i, b_i) \in \mathbb{N} \times \mathbb{N}$ $(1 \le i \le n)$ and $c, e \in \mathbb{N}$,*

$$(\sum_{i=1}^{n} x_i^c y_i^e) \cdot \begin{vmatrix} x_1^{a_1} y_1^{b_1} & x_1^{a_2} y_1^{b_2} & \cdots & x_1^{a_n} y_1^{b_n} \\ x_2^{a_1} y_2^{b_1} & x_2^{a_2} y_2^{b_2} & \cdots & x_2^{a_n} y_2^{b_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{a_1} y_n^{b_1} & x_n^{a_2} y_n^{b_2} & \cdots & x_n^{a_n} y_n^{b_n} \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} x_1^{a_1} y_1^{b_1} & \cdots & x_1^{a_i+c} y_1^{b_i+e} & \cdots & x_1^{a_n} y_1^{b_n} \\ x_2^{a_1} y_2^{b_1} & \cdots & x_2^{a_i+c} y_2^{b_i+e} & \cdots & x_2^{a_n} y_2^{b_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n^{a_1} y_n^{b_1} & \cdots & x_n^{a_i+c} y_n^{b_i+e} & \cdots & x_n^{a_n} y_n^{b_n} \end{vmatrix}.$$

As a consequence, we have

$$\sum_{i=1}^{n} \Delta\big((a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_i + c, b_i + e), (a_{i+1}, b_{i+1}), \ldots, (a_n, b_n)\big) \equiv 0$$

modulo lower degrees.

Proof. Plug in $p = x_1^c y_1^e$ and $q = x_1^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots x_n^{a_n} y_n^{b_n}$ in Lemma 19. □

The following definition involves minimal staircase forms defined in Definition 11.

Definition 21. Suppose that n, d_1, d_2, k are positive numbers satisfying $k = \binom{n}{2} - d_1 - d_2$, and μ is a partition of k. Define $J_{d_1,d_2}^{\prec\mu}$ (resp. $J_{d_1,d_2}^{\preceq\mu}$) to be the ideal of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ generated by the determinants of all minimal staircase forms of bi-degree (d_1, d_2) and partition type $\prec \mu$ (resp. $\preceq \mu$).

Lemma 22. *Let $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$, $P_i = (a_i, b_i)$ be as in §2 but we allow $P_i = P_j$ for $i \ne j$. Let (d_1, d_2) be the bi-degree of D. Let S be a staircase form of D of partition type μ, and $B(S)$ be the block diagonal form of S. Denote the number of nonzero entries strictly above the diagonal in the last block by j_r. If D satisfies the assumption that the last block of $B(S)$ is of size $t_0 \ge 2$, the first $(j_r + 2)$ blocks of $B(S)$ are of size 1, $P_2 = (1, 0)$,* *and* $b_{j_r+2} \geq 1$*, then for an integer* t *such that* $1 \leq t \leq t_0$ *and* $a_{n-t+1}, b_{n-t+1} \geq 1$*, we have* $2\Delta(D) \equiv \Delta(D^{\nwarrow}) + \Delta(D^{\searrow})$ *modulo the ideal* $I_{<d} + J^{\prec\mu}_{d_1,d_2}$*, where*

$$D^{\nwarrow} := \big(P_1, \dots, P_{j_r+1}, P_{j_r+2} + (1,-1), P_{j_r+3}, \dots, P_{n-t}, P_{n-t+1} + (-1,1), P_{n-t+2}, \dots, P_n\big),$$
$$D^{\searrow} := \big(P_1, (0,1), P_3, \dots, P_{n-t}, P_{n-t+1} + (1,-1), P_{n-t+2}, \dots, P_n\big).$$

Moreover, if the last block of $B(S)$ *is not minimal or if* $|P_{n-t+1}| > n - t_0$*, then* $\Delta(D) \equiv \Delta(D^{\searrow})$ *modulo* $I_{<d} + J^{\prec\mu}_{d_1,d_2}$*.*

Proof. Throughout the proof, "$\equiv$" means equivalence modulo the ideal $I_{<d} + J^{\prec\mu}_{d_1,d_2}$. We use the notation $(P_1, \dots, \widehat{P}_i, \dots, P_n)$ to denote $(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n)$. Note that the condition (2.1) does not always hold in the proof.

Suppose that the partition type of S is $\sum_{i=1}^r m_i j_i$. Applying Lemma 20 to

$$(\sum x_i^{a_{n-t+1}} y_i^{b_{n-t+1}-1}) \cdot \Delta(\big(P_1, (0,1), P_2, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big)),$$

we get a sum of n determinants. The first determinant is in $I_{<d}$ because all entries in the first row of the staircase form are zero. The second determinant is

$$\Delta\big(P_1, P_{n-t+1}, P_2, P_3, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big) = (-1)^{n-t-1}\Delta(D). \tag{3.2}$$

The i-th determinant for $i \geq 3$ is

$$\Delta\big(P_1, (0,1), P_2, P_3, \dots, P_{i-2}, P_{i-1} + P_{n-t+1} - (0,1), P_i, P_{i+1}, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big).$$

When $3 \leq i \leq j_r + 3$, its partition type is $\preceq m_1 j_1 + \dots + m_{r-1} j_{r-1} + (m_r - 1) j_r + (i - 3) + (j_r - i + 3)$. The latter partition is $\prec$ the partition type of S when $4 \leq i \leq j_r + 2$. If $i > j_r + 3$, the i-th determinant is equivalent to 0. So modulo $I_{<d} + J^{\prec\mu}_{d_1,d_2}$, the sum of (3.2) and the following two determinants

$$\Delta\big(P_1, (0,1), P_2 + P_{n-t+1} - (0,1), P_3, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big), \tag{3.3}$$

$$\Delta\big(P_1, (0,1), P_2, \dots, P_{j_r+1}, P_{j_r+2} + P_{n-t+1} - (0,1), P_{j_r+3}, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big) \tag{3.4}$$

is equivalent to 0.

Similarly, applying Lemma 20 to

$$(\sum x_i^{a_{n-t+1}-1} y_i^{b_{n-t+1}})\Delta\big(P_1, (0,1), P_2, \dots, P_{j_r+1}, P_{j_r+2} + (1,-1), P_{j_r+3}, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big),$$

we conclude that the sum of the following three determinant is equivalent to 0:

$$\Delta\big(P_1, P_{n-t+1} + (-1,1), P_2, \dots, P_{j_r+1}, P_{j_r+2} + (1,-1), P_{j_r+3}, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big), \tag{3.5}$$

$$\Delta\big(P_1, (0,1), P_{n-t+1}, P_3, \dots, P_{j_r+1}, P_{j_r+2} + (1,-1), P_{j_r+3}, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big), \tag{3.6}$$

$$\Delta\big(P_1, (0,1), P_2, \dots, P_{j_r+1}, P_{j_r+2} + P_{n-t+1} - (0,1), P_{j_r+3}, \dots, \widehat{P}_{n-t+1}, \dots, P_n\big), \tag{3.7}$$

Now we have two equations:

$$\begin{cases} (3.2) + (3.3) + (3.4) \equiv 0, \\ (3.5) + (3.6) + (3.7) \equiv 0. \end{cases}$$

By Transfactor Lemma (Lemma 15), the polynomial (3.6) is equivalent to

$$\begin{aligned} &\Delta\big(P_1, P_2, P_{n-t+1}, P_3, \ldots, P_{j_r+1}, P_{j_r+2}, P_{j_r+3}, \ldots, \widehat{P}_{n-t+1}, \ldots, P_n\big) \\ &= (-1)^{n-t-2}\Delta(D) = -(3.2), \end{aligned}$$

and we also have (3.4)=(3.7), therefore

$$(3.5) \equiv -(3.6) - (3.7) \equiv (3.2) - (3.4) \equiv 2(3.2) + (3.3).$$

Since (3.5)=$(-1)^{n-t-1}\Delta(D^{\nwarrow})$ and (3.3)= $(-1)^{n-t-2}\Delta(D^{\searrow})$, the lemma follows.

Note that since $|P_{n-t+1}| \geq |P_{n-t_0+1}| = n - t_0$, we have

$$|P_{j_r+2} + P_{n-t+1} - (0,1)| \geq (j_r + 1) + (n - t_0) - 1 = j_r + n - t_0$$

which is greater than $n-1$ if $j_r \geq t_0$. But this is always the case if the last block of $B(S)$ is not minimal. In this case, (3.4)$\equiv 0$ and therefore $(3.2) + (3.3) \equiv 0$. Of course we still have $(3.2) + (3.3) \equiv 0$ if $|P_{n-t+1}| > n - t_0$. □

Proposition 23. *Let $k \in \mathbb{N}$, $n, d_1, d_2 \in \mathbb{N}^+$ satisfy $n \geq 8k + 5$ and $d_1, d_2 \geq (2k + 1)n$. Let $\mu = \sum m_i j_i$ be a partition of k. Suppose that $D_1, D_2 \in \mathfrak{D}_n$ have the same bi-degree (d_1, d_2) and the same partition type μ, and suppose that staircase forms of D_1 and D_2 are both minimal. Then $\Delta(D_1) \equiv \Delta(D_2)$ modulo $I_{<d} + J^{\prec\mu}_{d_1,d_2}$.*

Proof. The conditions $d_1 + d_2 \leq \binom{n}{2}$ and $d_1, d_2 \geq (2k + 1)n$ imply $\binom{n}{2} \geq 2(2k + 1)n$, or equivalently, $n \geq 8k + 5$.

We prove the proposition by induction on k. The base case $k = 0$ is proved in Lemma 16. Suppose the proposition holds for $k < k_0$, and we need to prove the case $k = k_0$.

Let $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$, and let S be a minimal staircase form of D of partition type μ. Without loss of generality, we assume that the last block of $B(S)$ is of size greater than 1. (Otherwise, the last block corresponding to P_n is of size 1. Let M be the last block of size greater than 1. Since $d_1 \geq (2k + 1)n$, there are sufficiently many size-1 blocks in $B(S)$, such that by successively moving a P_i corresponding to a size-1 block to northwest direction and moving P_n to southeast direction using Transfactor Lemma 15, we can assume $P_n = (a_n, 0)$. Then we apply Minors Permuting Lemma 17 to permute the last block with the blocks in its northwest direction until it moves to the northwest of M. This procedure moves M to the southeast direction. Repeat the procedure until M becomes the last block.)

Because of Transfactor Lemma 15, Minors Permuting Lemma 17 and the condition $n \geq 8k + 5$, we can assume that the first $(k + 2)$ blocks of $B(S)$ are of size 1.

Now we apply Lemma 22. Denote by t_0 the size of the last block in $B(S)$. By Transfactor Lemma 15 we may assume $P_2 = (1, 0)$. If there is an integer t, such that

$1 \le t \le t_0$ and $|P_{n-t+1}| > n - t_0$, then $D \equiv D^{\searrow}$. Therefore we may assume that $|P_i|_y = 0$ for $i > n - t + 2$.

Define $a(D) = |P_{n-t_0+2}|_x - |P_{n-t_0+1}|_x$ and define $a(D^{\nwarrow})$ and $a(D^{\searrow})$ similarly. Then $a(D^{\nwarrow}) - 1 = a(D) = a(D^{\searrow}) + 1$. Consider the special case when $P_{n-t_0+1} = P_{n-t_0+2}$. In this case $\Delta(D) = 0$, hence $\Delta(D^{\nwarrow}) \equiv -\Delta(D^{\searrow})$, $a(D^{\nwarrow}) = 1$ and $a(D^{\searrow}) = -1$. Let D'' be the set obtained from $D^{\searrow}$ by interchanging the $(n - t_0 + 1)$-th and $(n - t_0 + 2)$-th points. Now compare $D^{\nwarrow} = (P'_1, \dots, P'_n)$ with $D'' = (P''_1, \dots, P''_n)$:

- they both give minimal staircase forms of the same partition type as S,
- $a(D^{\nwarrow}) = a(D'') = 1$,
- $\Delta(D^{\nwarrow}) \equiv \Delta(D'')$,
- $P''_i = \begin{cases} P'_i + (1, -1), & \text{for } i = n - t + 1, n - t + 2; \\ P'_i + (-1, 1), & \text{for } i = 2, j_r + 2; \\ P'_i, & \text{otherwise.} \end{cases}$

In other words, we can move P'_{n-t+1} and P'_{n-t+2} of $D^{\nwarrow}$ to southeast direction and move two size-1 blocks of $D^{\nwarrow}$ to northwest direction simultaneously without changing $\Delta(D^{\nwarrow})$ (modulo the equivalence relation). Repeat the movement until the y-coordinates of the $(n - t + 1)$-th and $(n - t + 2)$-th points become 1 and 0, respectively. Then apply the inductive assumption for the first $n - t$ points, we can draw the following conclusion: if $D_1, D_2 \in \mathfrak{D}_n$, such that

(i) they both have minimal staircase forms,

(ii) they have the same partition type,

(iii) they have the same bi-degree,

(iv) $a(D_1) = a(D_2) = 1$,

then $\Delta(D_1) \equiv \pm\Delta(D_2)$. This implies the proposition under the extra condition (iv). For the rest of the proof, we show how to remove the condition (iv). Note that, if (ii) is replaced by a stronger condition:

(ii)′ they are both in standard order and their block diagonal forms are of the same shape (in the sense that the size of the i-th blocks in the two block diagonal forms are the same for every i),

then $\Delta(D_1) \equiv \Delta(D_2)$.

By Lemma 22, we can show that, assuming (i) (ii)′ (iii) and $a(D_1), a(D_2) > 0$, we have

$$\frac{1}{a(D_1)}\Delta(D_1) \equiv \frac{1}{a(D_2)}\Delta(D_2). \tag{3.8}$$

Indeed, it is sufficient to show that

$$\text{if conditions (i)(ii)}' \text{ (iii) hold and } a(D_1) = 1, \text{ then } a(D_2)\Delta(D_1) \equiv \Delta(D_2). \tag{3.9}$$

This can be proved by induction on $a(D_2)$. The case $a(D_2) = 0$ is trivial since in this case $\Delta(D_2) = 0$. The case $a(D_2) = 1$ is already shown. Now by induction we assume that (3.9) is true for $a(D_2) = m-1$ and m. Suppose $a(D_2) = m+1$. Take $D \in \mathfrak{D}_n$ such that $D^{\nwarrow} \equiv D_2$. (This is always possible, since we can modify D_2 using Transfactor Lemma and Minors Permuting Lemma if necessary.) Then Lemma 22 asserts that $2\Delta(D) \equiv \Delta(D^{\nwarrow}) + \Delta(D^{\searrow})$. The inductive assumption implies $\Delta(D) \equiv m\,\Delta(D_1)$ and $\Delta(D^{\searrow}) \equiv (m-1)\Delta(D_1)$, hence

$$\Delta(D_2) \equiv \Delta(D^{\nwarrow}) \equiv 2m\,\Delta(D_1) - (m-1)\Delta(D_1) = (m+1)\Delta(D_1).$$

This completes the inductive proof of (3.9). □

Proposition 24. *Suppose that $n \geq 8k+5$, $d_1, d_2 \geq (2k+1)n$, and $\mu = \sum m_i j_i$ is a partition of k. If $D \in \mathfrak{D}_n$ has a nonzero staircase form S of type μ and of bi-degree (d_1, d_2), then $\Delta(D)$ is in the ideal $I_{<d} + J^{\preceq\mu}_{d_1,d_2}$.*

Proof. Assume $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$, S is a staircase form of D and is not minimal. By Transfactor Lemma 15 and Minors Permuting Lemma 17 , we can assume without loss of generality that, in the block diagonal form $B(S) = \operatorname{diag}(B_1, \ldots, B_s)$, all the size-1 blocks are in the northwest of the blocks of size greater than 1. In particular, the size t_0 of the last block of $B(S)$ is greater than 1.

First note that if the assumption of Lemma 22 is satisfied and the last block of $B(S)$ is not minimal, the conclusion easily follows. Indeed, in this case the equivalence $\Delta(D) \equiv \Delta(D^{\searrow})$ in Lemma 22 implies that we may move any point P_i in the last block of $B(S)$ to $P_i + (1, -1)$. Suppose P_i has the same degree as P_{i+1} for some i, $n - t_0 + 1 \leq i \leq n-1$. Keep on moving P_i to southeast direction until it collides with P_{i+1}. Then the determinant will be 0.

Now we show that we can always assume the assumption of Lemma 22 holds and the last block of $B(S)$ is not minimal. Indeed, since there are sufficiently many size-1 blocks in $B(S)$, we can apply Minors Permuting Lemma and Transfactor Lemma to move the points in D until the assumption of Lemma 22 is satisfied. To see the latter, let us assume on the contrary that the last block B_s of $B(S)$ is minimal. Define $n' = n - t_0$, $D' = (P_1, \ldots, P_{n-t_0}) \in \mathfrak{D}_{n'}$, $d' = \sum_{i=1}^{n-t_0} |P_i|$, $d_1' = \sum_{i=1}^{n-t_0} a_i$, $d_2' = \sum_{i=1}^{n-t_0} b_i$, $k' = \binom{n'}{2} - d'$, and let μ' be the partition type of D'. Then $k \geq k' + t_0 - 1$, and

$$\begin{aligned}
&n' \geq 8k + 5 - t_0 \geq 8(k' + t_0 - 1) + 5 - t_0 \geq 8k' + 5,\\
&d_1' > d_1 - t_0 n \geq (2k+1)n - t_0 n \geq (2k' + t_0 - 1)n \geq (2k'+1)n \geq (2k'+1)n'.
\end{aligned}$$

Similarly, $d_2' \geq (2k'+1)n'$. By inductive assumption, $\Delta(D')$ is in the ideal $I_{<d'} + J^{\preceq\mu'}_{d_1',d_2'}$, so $\Delta(D) = \Delta(D') \cdot \det(B_s)$ is in the ideal $I_{<d} + J^{\preceq\mu}_{d_1,d_2}$. Hence in the case when B_s is minimal, there is nothing to prove. □

Lemma 25. *Suppose that $n, k, u \in \mathbb{N}$ satisfy $k \le u \le n-2$. Define $v = n-1-u$, $d_1 = u(u+1)/2$, $d_2 = v(v+1)/2 + uv - k$. Then $d_1, d_2 \ge 0$, $k = \binom{n}{2} - d_1 - d_2$, and* $\dim M_{d_1,d_2} \ge p(k)$.

Proof. The only nontrivial statement, which we shall prove, is the last inequality. Consider a partition

$$\lambda = (\lambda) = (u+\varepsilon_0,\ u-1+\varepsilon_1,\ u-2+\varepsilon_2,\ \ldots,\ 1+\varepsilon_{u-1}, \underbrace{0,\ 0,\ \ldots,\ 0}_{v+1}),$$

where $\varepsilon_0, \ldots, \varepsilon_{u-1} \in \{0,1\}$ satisfy

$$\sum_{i=0}^{u-1} \varepsilon_i = k \quad \text{and} \quad \sum_{i=0}^{u-1} i\varepsilon_i = k(k+1)/2. \tag{3.10}$$

The partition λ determines a Dyck path Π with $a_i(\Pi) = n - i - \lambda_i$ for $i = 1, \ldots, n$. It is easy to check that $\mathrm{area}(\Pi) = v(v+1)/2 + uv - k$ and $\mathrm{dinv}(\Pi) = u(u+1)/2$. Since there are $p(k)$ number of solutions for the system (3.10), we have $\dim M_{d_1,d_2} \ge p(k)$ due to (1.2). $\square$

Finally, we are ready to prove Theorem 14.

Proof of Theorem 14. It follows from Proposition 24 and Proposition 23 that the $\mathbb{C}$-vector space M_{d_1,d_2} is spanned by $\{\det S_\mu\}_{\mu \in \Pi_k}$. In particular, $\dim M_{d_1,d_2} \le p(k)$. So we only need to show that $\dim M_{d_1,d_2} \ge p(k)$. Lemma 25 proves this inequality for special values of d_1 and d_2. For general d_1 and d_2, we add sufficiently many appropriate size-1 blocks and apply Lemma 25. To be more precise, choose a sufficiently large number $\tilde{n} \gg n$ such that there are positive integers u and v satisfying $k \le u \le \tilde{n} - 2$, $1 + u + v = \tilde{n}$, $u(u+1)/2 \ge (2k+1)\tilde{n}$, and $v(v+1)/2 + uv - k \ge (2k+1)\tilde{n}$. Choose $(\tilde{n} - n)$ points $P_i = (a_i, b_i) \in \mathbb{N} \times \mathbb{N}$ for $n+1 \le i \le \tilde{n}$, such that

$$\begin{aligned} &a_i + b_i = i - 1 \quad \text{for } n+1 \le i \le \tilde{n}, \\ &\tilde{d}_1 := \sum_{i=1}^{\tilde{n}} a_i = u(u+1)/2, \\ &\tilde{d}_2 := \sum_{i=1}^{\tilde{n}} b_i = v(v+1)/2 + uv - k, \end{aligned}$$

(which is always possible). By our choice of $P_{n+1}, \ldots, P_{\tilde{n}}$, if $D = (P_1, \ldots, P_n)$ has a minimal staircase form of partition type μ, then $\tilde{D} = (P_1, \ldots, P_n, P_{n+1}, \ldots, P_{\tilde{n}})$ also has a minimal staircase form of the same partition type μ. Let $\tilde{S}$ be the staircase form of $\tilde{D}$ and $B(\tilde{S})$ the block diagonal form of $\tilde{S}$. Denote by f_0 the product of the last $(\tilde{n}-n)$ size-1 minors in $B(\tilde{S})$. Define $\tilde{I} = \cap_{1 \le i < j \le \tilde{n}} (x_i - x_j, y_i - y_j)$ to be an ideal of $\mathbb{C}[x_1, y_1, \ldots, x_{\tilde{n}}, y_{\tilde{n}}]$, and define $\tilde{M} = \tilde{I}/(\mathbf{x},\mathbf{y})\tilde{I}$ which is doubly graded as $\oplus_{\tilde{d}_1,\tilde{d}_2}\tilde{M}_{\tilde{d}_1,\tilde{d}_2}$. Then we have a $\mathbb{C}$-linear map:

$$\begin{aligned} L: M_{d_1,d_2} \to\ & \tilde{M}_{\tilde{d}_1,\tilde{d}_2} \\ f \mapsto\ & f \cdot f_0. \end{aligned}$$

For each partition μ of k, if D_μ has a minimal staircase form S_μ of partition type μ, then $L(\det S_\mu) = \det \tilde{S}_\mu$, where $\tilde{S}_\mu$ is of partition type μ and is a minimal staircase form of $\tilde{D}_\mu = D_\mu \cup (P_{n+1}, \ldots, P_{\tilde{n}})$. Hence $\{L(\det S_\mu)\}_{\mu\in\Pi_k}$ form a basis for $\tilde{M}_{\tilde{d}_1,\tilde{d}_2}$, and the map L is surjective, which implies $\dim M_{d_1,d_2} \geq \dim \tilde{M}_{\tilde{d}_1,\tilde{d}_2} \geq p(k)$. $\square$

4 Map φ

4.1 Definition and properties of φ

In this subsection we define and study the map φ which naturally arises when we look for a minimal set of generators of the diagonal ideal I.

Definition 26. (a) For $D = \big((a_1,b_1),...,(a_n,b_n)\big) \in \mathfrak{D}'_n$, let $k = \binom{n}{2} - \sum_{i=1}^n (a_i + b_i)$ and define

$$\varphi(D) = \varphi\big((a_1,b_1),...,(a_n,b_n)\big) := (-1)^k \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \Big(\sum \rho_{w_1}\rho_{w_2}\cdots\rho_{w_{b_i}}\Big),$$

where $(w_1,\ldots,w_{b_i})$ in the sum $\sum \rho_{w_1}\rho_{w_2}\cdots\rho_{w_{b_i}}$ runs through the set

$$\{(w_1,\ldots,w_{b_i}) \in \mathbb{N}^{b_i} \mid w_1 + \cdots + w_{b_i} = \sigma(i) - 1 - a_i - b_i\}, \tag{4.1}$$

with the convention that

$$\sum \rho_{w_1}\cdots\rho_{w_{b_i}} = \begin{cases} 0 & \text{if } \sigma(i) - 1 - a_i - b_i < 0; \\ 0 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i > 0; \\ 1 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i = 0. \end{cases}$$

This defines a map $\varphi : \mathfrak{D}'_n \to \mathbb{C}[\rho]$ and we denote its restriction $\varphi|_{\mathfrak{D}_n} : \mathfrak{D}_n \to \mathbb{C}[\rho]$ also by φ.

(b) We give an equivalent definition of $\varphi(D)$. For $b \in \mathbb{N}$, $\mathrm{w} \in \mathbb{Z}$, define

$$h(b,\mathrm{w}) := \big\{(1 + \rho_1 + \rho_2 + \cdots)^b\big\}_{\mathrm{w}}.$$

Then

$$\varphi(D) = (-1)^k \begin{vmatrix} h(b_1,-|P_1|) & h(b_1,1-|P_1|) & h(b_1,2-|P_1|) & \cdots & h(b_1,n-1-|P_1|) \\ h(b_2,-|P_2|) & h(b_2,1-|P_2|) & h(b_2,2-|P_2|) & \cdots & h(b_2,n-1-|P_2|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(b_n,-|P_n|) & h(b_n,1-|P_n|) & h(b_n,2-|P_n|) & \cdots & h(b_n,n-1-|P_n|) \end{vmatrix}.$$

(c) We extend the definition of φ linearly: given a formal sum $\sum_{i=1}^{\ell} c_i D_i$, where $D_1, \dots, D_\ell \in \mathfrak{D}'_n$ and $c_1, \dots c_\ell \in \mathbb{C}$, we define

$$\varphi(\sum_{i=1}^{\ell} c_i D_i) := \sum_{i=1}^{\ell} c_i\, \varphi(D_i).$$

For any bi-homogeneous alternating polynomial $f \in \mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$, since $\{\Delta(D)\}_{D\in\mathfrak{D}_n}$ is a basis of $\mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$, there is a unique expression $f = \sum_{i=1}^{\ell} c_i\, \Delta(D_i)$, where $D_i \in \mathfrak{D}_n$. We define

$$\varphi(f) := \varphi(\sum_{i=1}^{\ell} c_i D_i) = \sum_{i=1}^{\ell} c_i\, \varphi(D_i).$$

This induces a map $\varphi : \mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon \to \mathbb{C}[\rho]$.

Example 27. The equivalence of (a) and (b) is not obvious but follows from a straightforward computation, so we will not go through the proof here. Instead, we give the following example: let $n = 4$, $D = \big((0,0),(0,1),(1,0),(0,2)\big)$. Then $k = \binom{4}{2} - (0+1+1+2) = 2$. We first consider the definition (a). There are only two $\sigma \in S_4$ that contribute to the sum: 1324 and 1423. For $\sigma = 1324$, the sum $\sum \rho_{w_1}\rho_{w_2}\cdots\rho_{w_{b_i}}$ are 1, ρ_1, 1, $2\rho_1$ for $i = 1,2,3,4$, respectively, so $\mathrm{sgn}(\sigma)\prod_{i=1}^{n}\Big(\sum \rho_{w_1}\rho_{w_2}\cdots\rho_{w_{b_i}}\Big) = (-1)1\cdot\rho_1\cdot 1\cdot 2\rho_1 = -2\rho_1^2$. Similarly, for $\sigma = 1423$, $\mathrm{sgn}(\sigma)\prod_{i=1}^{n}\Big(\sum \rho_{w_1}\rho_{w_2}\cdots\rho_{w_{b_i}}\Big) = (+1)1\cdot\rho_2\cdot 1\cdot 1 = \rho_2$. Therefore $\varphi(D) = (-1)^2(-2\rho_1^2+\rho_2) = -2\rho_1^2+\rho_2$. On the other hand, the definition (b) gives

$$\varphi(D) = (-1)^2\begin{vmatrix} h(0,0) & h(0,1) & h(0,2) & h(0,3)\\ h(1,-1) & h(1,0) & h(1,1) & h(1,2)\\ h(0,-1) & h(0,0) & h(0,1) & h(0,2)\\ h(2,-2) & h(2,-1) & h(2,0) & h(2,1)\end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & \rho_1 & \rho_2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\rho_1\end{vmatrix} = -2\rho_1^2+\rho_2.$$

Lemma 28. *Let $n \in \mathbb{N}^+$, $D = (P_1,\dots,P_n) \in \mathfrak{D}'_n$, where $P_1 < \cdots < P_n$ as in (2.1).*

(i) *If $|P_i| \ge i$ for some $1 \le i \le n$, then $\varphi(D) = 0$.*

(ii) *Suppose $m \in \mathbb{N}^+$ and $Q_1,\dots,Q_m \in \mathbb{Z}\times\mathbb{N}$ satisfy $|Q_i| = i-1$ for $1 \le i \le m$. Let $\tilde{D} = (Q_1,\dots,Q_m,P_1+(m,0),P_2+(m,0),\dots,P_n+(m,0)) \in \mathfrak{D}'_{m+n}$. Then $\varphi(\tilde{D}) = \varphi(D)$.*

(iii) *Let $t \in \mathbb{N}^+$, $Q = (-t,t)$ and $\tilde{D} = (P_1+Q,P_2+Q,\dots,P_n+Q)$. Then $\varphi(\tilde{D}) = \varphi(D)$.*

(iv) *Let $S = \{i \mid |P_i| := a_i+b_i = i-1\} = \{i_1 < \cdots < i_\ell\}$ and assume $i_1 = 1$. We call $(P_{i_r},\dots,P_{i_{r+1}-1})$* the r-th block *of D, for $1 \le r \le \ell$ (assuming $P_{i_{\ell+1}} = n+1$). Then $\varphi(D) = \prod_{r=1}^{\ell}\varphi\big(P_{i_r}-(i_r-1,0),P_{i_r+1}-(i_r-1,0),\dots,P_{i_{r+1}-1}-(i_r-1,0)\big)$.*

(v) *Suppose $|P_i| = 0$ for $1 \le i \le n$. Then $\varphi(D) = c\cdot\rho_1^{\binom{n}{2}}$, where $c = \frac{\prod_{i<j}(b_i-b_j)}{1!2!\cdots(n-1)!}$ is a positive integer.*

(vi) *For $s \in \mathbb{N}^+$, let $D = \big((-1,1),(0,0),(1,0),\dots,(s-1,0)\big)$. Then $\varphi(D) = \rho_s$.*

Proof. (i) follows from the convention stated after (4.1), and (vi) follows from Definition 26 (b).

(ii) By definition, $\varphi(\tilde{D}) = (-1)^{\tilde{k}} \sum_{\tilde{\sigma}\in S_{m+n}} \operatorname{sgn}(\tilde{\sigma}) \prod_{i=1}^{m+n} \left(\sum \rho_{w_1} \cdots \rho_{w_{b_i}} \right)$, where w_1, $\ldots$, $w_{b_i} \in \mathbb{N}$ and

$$w_1 + \cdots + w_{b_i} = \tilde{\sigma}(i) - 1 - a_i - b_i = \begin{cases} \tilde{\sigma}(i) - i, & \text{if } i \le m; \\ \tilde{\sigma}(i) - 1 - m - |P_{i-m}|, & \text{if } i > m. \end{cases}$$

If $\tilde{\sigma}(i) < i$ for some $i \le m$, then none of $(w_1, \ldots, w_{b_i})$ in $\mathbb{N}^{b_i}$ satisfies the condition (4.1), hence $\prod_{i=1}^{m+n} (\sum \rho_{w_1} \cdots \rho_{w_{b_i}}) = 0$, and the summand corresponding to $\tilde{\sigma}$ does not contribute to $\varphi(\tilde{D})$. So we only need to consider those $\tilde{\sigma}$ with $\tilde{\sigma}(i) = i$ $(1 \le i \le m)$. Each such $\tilde{\sigma}$ corresponds to a permutation of $\{m+1, \ldots, m+n\}$. Define $\sigma \in S_n$ by $\sigma(i-m) = \tilde{\sigma}(i) - m, \quad m+1 \le i \le m+n$. Then $\tilde{\sigma}(i) - 1 - m - |P_{i-m}| = \sigma(i-m) - 1 - |P_{i-m}|$ for $m+1 \le i \le m+n$. Moreover,

$$\tilde{k} = \binom{n+m}{2} - \sum_{i=1}^{m} |Q_i| - \sum_{i=1}^{n} (|P_i| + m) = \binom{n}{2} - \sum_{i=1}^{n} |P_i| = k.$$

Comparing with the definition of $\varphi(D)$, we conclude that $\varphi(\tilde{D}) = \varphi(D)$.

(iii) It suffices to prove the case when $t = 1$. Define

$$\mathbf{v}_i = \begin{bmatrix} h(b_1, i - |P_1|) \\ h(b_2, i - |P_2|) \\ \vdots \\ h(b_n, i - |P_n|) \end{bmatrix}, \quad \mathbf{v}_i' = \begin{bmatrix} h(b_1 + 1, i - |P_1|) \\ h(b_2 + 1, i - |P_2|) \\ \vdots \\ h(b_n + 1, i - |P_n|) \end{bmatrix}, \quad 0 \le i \le n - 1.$$

By the definition of the map φ,

$$\varphi(D) = (-1)^k \det(\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}), \quad \varphi(\tilde{D}) = (-1)^k \det(\mathbf{v}_0', \ldots, \mathbf{v}_{n-1}').$$

By the definition of the function h, it is easy to deduce the relation

$$h(b+1, \mathrm{w}) = h(b, \mathrm{w}) + \rho_1 h(b, \mathrm{w} - 1) + \rho_2 h(b, \mathrm{w} - 2) + \cdots .$$

Since $|P_1|, \ldots, |P_n|$ are non-negative integers, the above relation implies

$$\mathbf{v}_i' = \mathbf{v}_i + \rho_1 \mathbf{v}_{i-1} + \rho_2 \mathbf{v}_{i-2} + \cdots + \rho_i \mathbf{v}_0, \quad 0 \le i \le n - 1,$$

hence

$$\varphi(D) = (-1)^k \det(\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}) = (-1)^k \det(\mathbf{v}_0', \ldots, \mathbf{v}_{n-1}') = \varphi(\tilde{D}).$$

(iv) Suppose that the summand in $\varphi(D)$ corresponding to $\sigma \in S_n$ does contribute. By the definition of $\varphi(D)$, it is necessary that $\sigma(j) - 1 - |P_j| \ge 0$ for each $1 \le j \le n$. For each integer $1 \le r \le \ell$, if $j \ge i_r$, then $\sigma(j) \ge 1 + |P_j| \ge 1 + |P_{i_r}| = i_r$. So σ maps the set $\{i_r, i_r + 1, \ldots, n\}$ to itself for every r. It follows that σ maps each block to itself. Let σ_r be the restriction of σ to $\{i_r, i_r+1, \dots, i_{r+1}-1\}$. Define $n_r = i_{r+1} - i_r$, $k_r = \sum_{j=i_r-1}^{i_{r+1}-2} j - \sum_{j=i_r}^{i_{r+1}-1} |P_j|$. Then by (ii) and a routine computation, we have

$$\begin{aligned}\varphi(D) &= (-1)^{k_1+\cdots+k_\ell} \sum_{\sigma_1,\dots,\sigma_\ell} \operatorname{sgn}(\sigma_1)\cdots\operatorname{sgn}(\sigma_\ell) \prod_{i=1}^{n_1+\cdots+n_\ell} \Big(\sum \rho_{w_1}\ldots\rho_{w_{b_i}}\Big) \\ &= \prod_{r=1}^{\ell}\bigg((-1)^{k_r}\sum_{\sigma_r}\operatorname{sgn}(\sigma_r)\prod_{i=1}^{n_r}\Big(\sum \rho_{w_1}\ldots\rho_{w_{b_i}}\Big)\bigg) \\ &= \prod_{r=1}^{\ell}\varphi\big(P_{i_r}-(i_r-1,0), P_{i_r+1}-(i_r-1,0), \dots, P_{i_{r+1}-1}-(i_r-1,0)\big).\end{aligned}$$

(v) We rewrite the definition of φ as

$$\varphi(D) = (-1)^k \sum_{(\sigma,\{w_j^{(i)}\})} \left(\operatorname{sgn}(\sigma)\prod_{i=1}^{n}\rho_{w_1^{(i)}}\rho_{w_2^{(i)}}\cdots\rho_{w_{b_i}^{(i)}}\right), \tag{4.2}$$

where $\{w_j^{(i)}\}$ is a set of nonnegative integers, $1 \le i \le n$, $1 \le j \le b_i$. For $1 \le i \le n$, since $|P_i| = 0$, those $w_j^{(i)}$ satisfy the condition $w_1^{(i)} + \cdots + w_{b_i}^{(i)} = \sigma(i) - 1$. Denote by Σ the set of all possible data $(\sigma, \{w_j^{(i)}\})$.

Let $\Sigma' \subset \Sigma$ be the subset consisting of those $(\sigma, \{w_j^{(i)}\})$ such that not all $w_j^{(i)}$ are 0 or 1. We shall define an automorphism $\iota : \Sigma' \to \Sigma'$ such that $\iota \circ \iota$ is the identity. For $(\sigma, \{w_j^{(i)}\}) \in \Sigma'$, define m_i to be the number of nonzero elements in $(w_1^i, \dots, w_{b_i}^{(i)})$. Then $m_1 + \cdots + m_n \le 0 + 1 + \cdots + (n-1) = \binom{n}{2}$. Since some $w_j^{(i)}$ is greater than 1, the inequality must be strict, therefore we can find a smallest pair (r, r') such that $r < r'$ and $m_r = m_{r'}$ (here $(r, r') < (s, s')$ if $r < s$, or $r = s$ and $r' < s'$). Define

$$\{j_1 < \cdots < j_{m_r}\} := \{j \mid w_j^{(r)} \neq 0\}, \quad \{j'_1 < \cdots < j'_{m_r}\} := \{j' \mid w_j^{(r')} \neq 0\}.$$

Define $\tilde\sigma \in S_n$ as $\tilde\sigma(r) = \sigma(r')$, $\tilde\sigma(r') = \sigma(r)$, and $\tilde\sigma(\ell) = \sigma(\ell)$ for $\ell \neq r, r'$. Define $\{\tilde w_j^{(i)}\}$ as follows: for $i \neq r, r'$, define $\tilde w_j^{(i)} = w_j^{(i)}$, $1 \le j \le b_i$; for $i = r$, define $\tilde w_{j_\ell}^{(r)} = w_{j'_\ell}^{(r')}$ for $1 \le \ell \le m_r$, and $\tilde w_j^{(r)} = 0$ for $j \neq j_1, \dots, j_{m_r}$; for $i = r'$, define $\tilde w_{j'_\ell}^{(r')} = w_{j_\ell}^{(r)}$ for $1 \le \ell \le m_r$, and $\tilde w_{j'}^{(r')} = 0$ for $j' \neq j'_1, \dots, j'_{m_r}$. Define the automorphism $\iota : (\sigma, \{w_j^{(i)}\}) \mapsto (\tilde\sigma, \{\tilde w_j^{(i)}\})$. It is easy to check that $\iota \circ \iota$ is the identity. Moreover, ι has no fixed point because $\sigma \neq \tilde\sigma$. Since $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\tilde\sigma)$, the summand in (4.2) corresponding to $(\sigma, \{w_j^{(i)}\})$ cancels with the summand corresponding to $(\tilde\sigma, \{\tilde w_j^{(i)}\})$.

Now we are left with the case when all $w_j^{(i)}$ are 0 or 1. Using Definition 26 (b), and using the fact that the monomial ρ_1^w in $h(b, \mathrm{w})$ has coefficient $\binom{b}{\mathrm{w}}$, we have

$$\varphi(D) = (-1)^{\binom{n}{2}} \begin{vmatrix} \binom{b_1}{0}\rho_1^0 & \binom{b_1}{1}\rho_1^1 & \cdots & \binom{b_1}{n-1}\rho_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{b_n}{0}\rho_1^0 & \binom{b_n}{1}\rho_1^1 & \cdots & \binom{b_n}{n-1}\rho_1^{n-1} \end{vmatrix} = \begin{vmatrix} \binom{b_n}{0} & \binom{b_n}{1} & \cdots & \binom{b_n}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{b_1}{0} & \binom{b_1}{1} & \cdots & \binom{b_1}{n-1} \end{vmatrix} \rho_1^{\binom{n}{2}} = c \cdot \rho_1^{\binom{n}{2}},$$

where c is the second determinant, which is an integer. Moreover,

$$c = \begin{vmatrix} \binom{b_n}{0} & \binom{b_n}{1} & \cdots & \binom{b_n}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{b_1}{0} & \binom{b_1}{1} & \cdots & \binom{b_1}{n-1} \end{vmatrix} = \begin{vmatrix} 1 & b_n & \frac{b_n^2}{2!} & \cdots & \frac{b_n^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & b_1 & \frac{b_1^2}{2!} & \cdots & \frac{b_1^{n-1}}{(n-1)!} \end{vmatrix} = \frac{\prod_{i<j}(b_i - b_j)}{1!2!\cdots(n-1)!},$$

by properties of Vandermonde matrices. Since $b_1 > b_2 > \cdots > b_n$, c is positive. $\square$

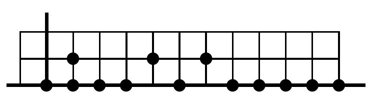
4.2 Relation between $\varphi(D)$ and $\Delta(D)$

Definition 29. The data $(m, n, (r_1, \ldots, r_m), (s_1, \ldots, s_m)) \in \mathbb{N} \times \mathbb{N}^+ \times \mathbb{N}^m \times \mathbb{N}^m$ satisfying $1 \le r_1 < r_2 < \cdots < r_m < r_{m+1} := n$ and $0 \le s_i \le r_{i+1} - r_i - 1$ $(1 \le i \le m)$ determines an element $D \in \mathfrak{D}_n$, which is obtained by sorting the set

$$\{(0,0),(1,0),\cdots,(n-1,0)\} \cup \{(r_1-1,1),(r_2-1,1),\ldots,(r_m-1,1)\} \\ \setminus \{(r_1+s_1,0),(r_2+s_2,0),\ldots,(r_m+s_m,0)\}$$

in increasing order as in (2.1). A staircase form of such D is called a *special minimal staircase form*.

Remark 30. It is easy to see that a special minimal staircase form is indeed a minimal staircase form. Using the notation in the definition, the partition type of a special minimal staircase form D is obtained from $(s_1, s_2, \ldots, s_m)$ by eliminating 0's and sorting the sequence if necessary. The following figure gives a typical example of D which has a special minimal staircase form, where $m = 3$, $n = 13$, $(r_1, r_2, r_3) = (2, 5, 7)$, $(s_1, s_2, s_3) = (2, 1, 5)$, and the partition type is $(1, 2, 5)$.



Lemma 31. (i) *Let $n \in \mathbb{N}^+$, $d_1, d_2, k \in \mathbb{N}$ and $d_1 + d_2 = \binom{n}{2} - k$. Define*

$$\Pi'_k = \left\{ \mu \in \Pi_k \,\middle|\, \begin{array}{l} \text{there exists } F_\mu \in \mathfrak{D}_n \text{ whose staircase form is minimal,} \\ \text{of partition type } \mu \text{ and of bi-degree } (d_1, d_2) \end{array} \right\}.$$

If there are coefficients $\{c_\mu \in \mathbb{C}\}_{\mu \in \Pi'_k}$ satisfying

$$\sum_{\mu \in \Pi'_k} c_\mu \, \Delta(F_\mu) \equiv 0 \quad \text{(modulo lower degrees)},$$

then $c_\mu = 0$ for every $\mu \in \Pi'_k$. In other words, $\{\Delta(F_\mu)\}_{\mu \in \Pi'_k}$ form a linearly independent set in M_{d_1,d_2}.

(ii) *If $D_1, D_2 \in \mathfrak{D}_n$ have the same partition type and the same bi-degree,and both have special minimal staircase forms, then $D_1 \equiv D_2$ (modulo lower degrees).*

Proof. (i) Choose a sufficiently large $N \in \mathbb{N}$ and choose $(N-n)$ points $P_{n+1}, \ldots, P_N \in \mathbb{N} \times \mathbb{N}$ such that $|P_i| = i-1$ for $n+1 \leq i \leq N$ and

$$|P_{n+1}|_x + \cdots + |P_N|_x \geq (2k+1)N, \quad |P_{n+1}|_y + \cdots + |P_N|_y \geq (2k+1)N.$$

Define $F'_\mu = F_\mu \cup (P_{n+1}, P_{n+2}, \ldots, P_N)$. Theorem 14 asserts that $\{\Delta(F'_\mu)\}_{\mu \in \Pi'_k}$ are linearly independent modulo lower degrees. Since $\Delta(F'_\mu)$ is equivalent to $\Delta(F_\mu) \cdot f_0$ for a polynomial f_0 independent of μ, the linear independence of $\{\Delta(F'_\mu)\}_{\mu \in \Pi'_k}$ implies the linear independence of $\{\Delta(F_\mu)\}_{\mu \in \Pi'_k}$.

(ii) The claim follows immediately from Minors Permuting Lemma 17. $\square$

Proposition 32. *Let $n \in \mathbb{N}^+$, $D = (P_1, \ldots, P_n) \in \mathfrak{D}_n$ and $k = \binom{n}{2} - \sum_{i=1}^n |P_i| \geq 0$. Suppose that $N \in \mathbb{N}^+$ satisfies $N > N_0 := (\sum_{i=1}^n |P_i|_y)(k+1)$. Define*

$$\tilde{D} := \big((0,0), (1,0), \ldots, (N-1,0), P_1 + (N,0), \ldots, P_n + (N,0)\big) \in \mathfrak{D}_{N+n}.$$

Let $d_2 = \sum_i |P_i|_y$ be the y-degree of D (which is also the y-degree of $\tilde{D}$). For $\mu \in \Pi_{d_2,k}$, suppose that $F_\mu \in \mathfrak{D}_n$ is of partition type μ, of the same bi-degree as $\tilde{D}$ and has a special minimal staircase form. Then there exist unique integers c_μ ($\mu \in \Pi_{d_2,k}$) such that

$$\Delta(\tilde{D}) \equiv \sum_{\mu \in \Pi_{d_2,k}} c_\mu \cdot \Delta(F_\mu) \quad \text{(modulo lower degrees)}.$$

Moreover, the integers c_μ satisfy

$$\varphi(D) = \sum_{\mu \in \Pi_{d_2,k}} c_\mu \rho_\mu. \tag{4.3}$$

Proof. Throughout the proof, we do not require the standard order (2.1) for elements in $\mathfrak{D}$.

The uniqueness of c_μ follows from the fact that $\{\Delta(F_\mu)\}_{\mu \in \Pi'_k}$ form a linearly independent set in M_{d_1,d_2} which is proved in Lemma 31. For the existence of c_μ, we shall give an algorithm showing that those c_μ are exactly the integers satisfying (4.3).

Define $Q_s^{(0)} = (s-1, 0)$ for $1 \leq s \leq N$, $P_t^{(0)} = P_t + (N,0)$ for $1 \leq t \leq n$ and define

$$D^{(0)} := \tilde{D} = (Q_1^{(0)}, Q_2^{(0)}, \ldots, Q_N^{(0)}, P_1^{(0)}, P_2^{(0)}, \ldots, P_n^{(0)}).$$

Partition the sequence of N_0 points $(Q_2^{(0)}, Q_3^{(0)}, \ldots, Q_{N_0+1}^{(0)})$ into $(\sum_{i=1}^n |P_i|_y)$ parts of length $(k+1)$: for $1 \leq r \leq \sum_{i=1}^n |P_i|_y$, the r-th part is $(Q^{(0)}_{(r-1)(k+1)+2}, \ldots, Q^{(0)}_{r(k+1)+1})$. Define a sequence A of length $\sum_{t=1}^n |P_t|_y$ as $A = \big((1, |P_1|_y), \ldots, (1,2), (1,1), (2, |P_2|_y), \ldots, (2,2), (2,1), \ldots, (n, |P_n|_y), \ldots, (n,2), (n,1)\big)$.

Given a set of nonnegative integers $\mathbf{w} = \{w_{j'}^{(i')}\}_{(i',j') \in A}$, we construct

$$D_{\mathbf{w}}^{(r)} = (Q_1^{(r)}, \ldots, Q_N^{(r)}, P_1^{(r)}, \ldots, P_n^{(r)})$$

inductively on $r \in [1, \sum_{\ell=1}^{n} |P_\ell|_y]$. Suppose $D_{\mathbf{w}}^{(r-1)}$ has been constructed, and the r-th pair in the sequence A is (i,j). Then $D_{\mathbf{w}}^{(r)}$ is constructed as follows.

$$\begin{aligned}
&Q^{(r)}_{(r-1)(k+1)+2+w_j^{(i)}} = ((r-1)(k+1), 1),\\
&Q_\ell^{(r)} = Q_\ell^{(r-1)}, \quad \text{ for } 1 \le \ell \le N \text{ and } \ell \neq (r-1)(k+1) + 2 + w_j^{(i)},\\
&P_i^{(r)} = P_i^{(r-1)} + (w_j^{(i)} + 1, -1),\\
&P_\ell^{(r)} = P_\ell^{(r-1)}, \quad \text{ for } 1 \le \ell \le n \text{ and } \ell \neq i.
\end{aligned}$$

The following equivalence can be proved inductively on r from 1 to n:

$$\Delta(\tilde{D}) \equiv (-1)^r \sum_{\mathbf{w}} \Delta(D_{\mathbf{w}}^{(r)}), \quad \text{(modulo lower degrees)}, \tag{4.4}$$

where $\mathbf{w}$ runs through all sets of integers $\{w_{j'}^{(i')}\}_{(i',j') \le (i,j)}$ with $w_{j'}^{(i')} \in [0,k]$. To illustrate the idea, we only go through the first two steps $r = 1$ and $r = 2$ and leave the details to the interested reader. For $r = 1$, we can assume $|P_1|_y > 0$ because otherwise we can take the smallest h with $|P_h|_y > 0$ and the argument will be similar. Denote $w = w_{|P_1|_y}^{(1)}$ for simplicity. We need to show $\Delta(\tilde{D}) + \sum_{0 \le w \le k} \Delta\big((0,0), \ldots, (w,0), (0,1), (w+2,0), \ldots, (N-1,0), P_1 + (w+1,-1), P_2, \ldots, P_n\big)$ is equivalent to 0 modulo lower degrees. This follows immediately from Lemma 20 by plugging in $(c,e) = P_1^{(0)} - (0,1)$ and $\big((a_1,b_1), \ldots, (a_{N+n}, b_{N+n})\big) = \big((0,0), (1,0), \ldots, (N-1,0), (0,1), P_2^{(0)}, P_3^{(0)}, \ldots, P_n^{(0)}\big)$. Here we can assume $w \le k$ because otherwise the total degree of the polynomial $\Delta(\tilde{D})$ is strictly greater than $\binom{N+n}{2}$, which implies $\Delta(\tilde{D}) \equiv 0$ modulo lower degrees. For $r = 2$, we only consider the case $|P_1|_y \ge 2$ since other cases are similar. By induction,

$$\Delta(\tilde{D}) \equiv - \sum_{0 \le w_{|P_1|_y}^{(1)} \le k} \Delta(D^{(1)}_{\{w_{|P_1|_y}^{(1)}\}}), \quad \text{(modulo lower degrees)}.$$

A similar argument as in the case $r = 1$ gives

$$\Delta(D^{(1)}_{\{w_{|P_1|_y}^{(1)}\}}) \equiv - \sum_{0 \le w_{|P_1|_y - 1}^{(1)} \le k} \Delta(D^{(1)}_{\{w_{|P_1|_y}^{(1)}, w_{|P_1|_y-1}^{(1)}\}}), \quad \text{(modulo lower degrees)}.$$

Combining the above two equivalences together, we have

$$\Delta(\tilde{D}) \equiv (-1)^2 \sum_{0 \le w_{|P_1|_y}^{(1)}, w_{|P_1|_y-1}^{(1)} \le k} \Delta(D^{(1)}_{\{w_{|P_1|_y}^{(1)}, w_{|P_1|_y-1}^{(1)}\}}), \quad \text{(modulo lower degrees)}.$$

An induction similar to the above argument gives the proof of (4.4).

Take $r = r_0 = \sum_{\ell=1}^{n} |P_\ell|_y$ into (4.4). Now we have $|P_1^{(r_0)}|_y = \cdots = |P_n^{(r_0)}|_y = 0$. Assume

$$\{|P_1^{(r_0)}|_x, |P_2^{(r_0)}|_x, \ldots, |P_n^{(r_0)}|_x\} \text{ is a permutation of } \{N, N+1, \ldots, N+n-1\},$$

because it is a necessary condition for $\Delta(D_{\mathbf{w}}^{(r_0)}) \not\equiv 0$. Let $\sigma \in S_n$ be the permutation satisfying $|P_i^{(r_0)}|_x = \sigma(i) + N - 1$. Since $P_i^{(r_0)} = P_i^{(0)} + \sum_{j=1}^{|P_i|_y} w_j^{(i)}$, we have

$$\sum_{j=1}^{|P_i|_y} w_j^{(i)} = P_i^{(r_0)} - P_i^{(0)} = (\sigma(i) + N - 1) - (N + |P_i|) = \sigma(i) - 1 - |P_i| = \sigma(i) - 1 - a_i - b_i,$$

which is exactly the condition in the definition of $\varphi(D)$ (see Definition 26(a)). Next, we shall figure out the correct sign. For this, we rearrange the order of points in $D_{\mathbf{w}}^{(r_0)}$ to satisfy the condition (2.1). For $1 \leq r \leq \sum_{\ell=1}^{n} |P_\ell|_y$, the r-th part

$$\big((r-1)(k+1)+1, 0\big), \big((r-1)(k+1)+2, 0\big), \dots, \big((r-1)(k+1)+1+w_j^{(i)}, 0\big), \dots, \big(r(k+1), 0\big)$$

is modified to

$$\big((r-1)(k+1)+1, 0\big), \big((r-1)(k+1)+2, 0\big), \dots, \big((r-1)(k+1), 1\big), \dots, \big(r(k+1), 0\big).$$

The only change is that the point $\big((r-1)(k+1)+1+w_j^{(i)}, 0\big)$ is replaced by $\big((r-1)(k+1), 1\big)$. To rearrange this part into standard order, we need to move the $(1 + w_j^{(i)})$-th point in front of the first point, so the change of sign is $(-1)^{w_j^{(i)}}$. On the other hand, rearranging $(P_1^{(r_0)}, \dots, P_1^{(r_0)})$ to the standard order incurs a sign change $\mathrm{sgn}(\sigma)$. So the total change of sign is

$$(-1)^{\sum_{i=1}^{n} \sum_{j=1}^{|P_i|_y} w_j^{(i)}} \cdot \mathrm{sgn}(\sigma) = (-1)^{\sum_{i=1}^{n} (\sigma(i) - 1 - |P_i|)} \cdot \mathrm{sgn}(\sigma) = (-1)^k \mathrm{sgn}(\sigma),$$

which coincides with the sign in the definition of $\varphi(D)$ (Definition 26(a)).

Finally, note that $D_{\mathbf{w}}^{(r_0)}$ (after rearranging it to the standard order) has a special minimal staircase form. The partition type of $D_{\mathbf{w}}^{(r_0)}$ is $(w_j^{(i)})_{i,j}$, which is compatible with the definition (4.2) of $\varphi(D)$. Thus we have finished the proof of Proposition 32. $\square$

5 The upper bound of $\dim M_{d_1,d_2}$

5.1 A characterization of the q,t-Catalan number

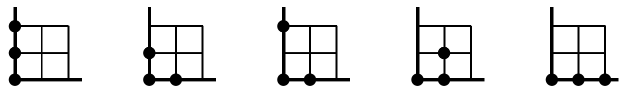
We give the following conjecture, which is equivalent to a conjecture by Mahir Can and Nick Loehr in their unpublished work [2].

Conjecture 33. *Let Λ_n be the set of integer sequences $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ satisfying $\lambda_i \leq n - i$ for all $i \in [1, n]$. For any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$, let*

$$a_i = n - i - \lambda_i, \quad b_i = \#\{j |\, i < j \leq n, \lambda_i - \lambda_j + i - j \in \{0, 1\}\},$$

and define $D(\lambda) = \big((a_1, b_1), \dots, (a_n, b_n)\big)$. Then $\{\Delta(D(\lambda))\}_{\lambda \in \Lambda_n}$ generates the ideal I.

Example 34. For $n = 3$, Λ_3 consists of $(2,1,0),(1,1,0),(2,0,0),(1,0,0),(0,0,0)$, the corresponding $D(\lambda)$ are as shown in the following figure.



Definition 35. Let $\mathfrak{D}_n^{catalan}$ be the set which consists of $D \in \mathfrak{D}_n$ satisfying the following conditions:

(a) If $(p,0)$ is in D, then $(i,0)$ is in D for all $i \in [0,p]$.

(b) For every $p \in \mathbb{N}$,

$$\#\{j \,|\, (p+1,j) \in D\} + \#\{j \,|\, (p,j) \in D\} \geq \max\{j \,|\, (p,j) \in D\} + 1.$$

(If $\{j \,|\, (p,j) \in D\} = \emptyset$, then we require that no point $(i,j) \in D$ satisfies $i \geq p$.)

Proposition 36. *The map $\theta : \Lambda_n \to \mathfrak{D}_n^{catalan}$ defined by sending λ to $D(\lambda)$ is a bijection.*

Proof. We first show that $D(\lambda)$ is in $\mathfrak{D}_n^{catalan}$. By the definition of $D(\lambda)$, suppose $a_i = a_{i'}$ for some $i, i' \in [1,n]$, then $i \leq i'$ if and only if $b_i \geq b_{i'}$. Indeed, suppose $i \leq i'$. Since $a_i = a_{i'}$ implies $(\lambda_i + i) = (\lambda_{i'} + i')$, we have

$$\{j |\, i < j \leq n, (\lambda_i + i) - (\lambda_j + j) \in \{0,1\}\} \supseteq \{j |\, i' < j \leq n, (\lambda_{i'} + i') - (\lambda_j + j) \in \{0,1\}\},$$

hence $b_i \geq b_{i'}$. For (a), suppose $(a_\ell, b_\ell) = (p,0) \in D(\lambda)$ and $(p-1,0) \notin D(\lambda)$. Since

$$a_i - a_{i+1} = (n - i - \lambda_i) - (n - i - 1 - \lambda_{i+1}) = 1 - (\lambda_i - \lambda_{i+1}) \leq 1, \quad \text{for all } i \in [1, n-1]$$

and $a_n = 0$, there exists $i \in [\ell+1, n]$ such that $a_i = p - 1$. Suppose i_0 is maximal among all such i. Since $(a_\ell, b_\ell) = (p, 0)$, we have $a_i < p$ for all $i > \ell$. Therefore

$$b_{i_0} = \#\{j |\, i_0 < j \leq n,\ a_j - a_{i_0} \in \{0,1\}\} = \#\{j |\, i_0 < j \leq n,\ a_j \in \{p-1, p\}\} = 0,$$

$(a_{i_0}, b_{i_0}) = (p-1, 0)$, which contradicts our assumption that $(p-1,0) \notin D(\lambda)$. For (b), if $\{j \,|\, (p,j) \in D\} = \emptyset$, then since $a_i - a_{i+1} \leq 1$ for all $i \in [1, n-1]$, there is no point in D whose x-coordinate is greater than or equal to p. Therefore we assume $\{j \,|\, (p,j) \in D\} \neq \emptyset$. Define $q = \max\{j \,|\, (p,j) \in D\}$, and assume (p,q) is the ℓ-th pair (a_ℓ, b_ℓ) in D. Then

$$q = b_\ell = \#\{j |\, \ell < j \leq n,\ a_j - a_\ell \in \{0,1\}\} = \#\{j |\, \ell < j \leq n,\ a_j = p \text{ or } p+1\},$$

$$\#\{j \,|\, (p+1,j) \in D\} + \#\{j \,|\, (p,j) \in D\} \geq q + 1.$$

So $D(\lambda)$ is in $\mathfrak{D}_n^{catalan}$.

To show that $\theta : D \mapsto D(\lambda)$ is a bijection, it suffices to construct a map θ^{-1} sending $D(\lambda)$ back to λ. We give an inductive construction on n. Let $p \in \mathbb{N}$ be the minimal integer such that

$$\#\{j \,|\, (p+1,j) \in D\} + \#\{j \,|\, (p,j) \in D\} \leq \max\{j \,|\, (p,j) \in D\} + 1.$$

Let $q = \max\{j \mid (p, j) \in D\}$ and take $(a_1, b_1) = (p, q) \in D$. Now let D' be obtained from D by deleting the point (a_1, b_1). It is easy to check that D' is in $\mathfrak{D}_{n-1}^{catalan}$. By induction we have $\theta^{-1}(D') = (\lambda'_1, \lambda'_2, \cdots, \lambda'_{n-1})$. Then we define

$$\theta^{-1}(D) := (n - 1 - p, \lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}).$$

To check that it is in Λ_n, we need to show $n-1-p \geq \lambda'_1$, i.e., $p \leq (n-1) - \lambda'_1 = a'_1 + 1$, where a'_1 is the minimal integer that

$$\#\{j \mid (a'_1 + 1, j) \in D'\} + \#\{j \mid (a'_1, j) \in D'\} \leq \max\{j \mid (a'_1, j) \in D'\} + 1.$$

But D and D' coincide on column $0, 1, \ldots, p-1$, therefore $a'_1 \geq p - 1$.

To check that θ and θ^{-1} are inverse to each other is routine and we shall skip. □

Remark 37. The above proposition is discovered independently by Alexander Woo [13].

Corollary 38. *The dimension of M_{d_1,d_2}, which is also the coefficient of $q^{d_1} t^{d_2}$ in the q,t-Catalan number $C_n(q,t)$, is equal to the number of $D \in \mathfrak{D}_n^{catalan}$ with bi-degree (d_1, d_2). In particular, the usual Catalan number $\frac{1}{n+1}\binom{n}{2}$ is equal to $|\mathfrak{D}_n^{catalan}|$.*

Proof. It follows immediately from Proposition 36 and (1.1). □

5.2 The upper bound of $\dim M_{d_1,d_2}$

In order to compare M_{d_1,d_2} for different n, we use $M^{(n)}_{d_1,d_2}$ to specify which n we are considering.

Proposition 39. *Let $\ell, n \in \mathbb{N}^+$, $d_1, d_2 \in \mathbb{N}$, and $k = \binom{n}{2} - d_1 - d_2 \geq 0$. Then we have*

$$\dim M^{(n)}_{d_1,d_2} \leq \dim M^{(n+\ell)}_{d_1+\binom{\ell}{2}+n\ell,\, d_2}.$$

In particular, $\dim M^{(n)}_{d_1,d_2} \leq p(d_2, k)$.

Proof. For $D^{(n)} \in \mathfrak{D}_n^{catalan}$ of bi-degree (d_1, d_2), let $D^{(n+\ell)} = \big((0,0),(1,0),\ldots,(\ell-1,0)\big) \cup \big(D^{(n)} + (\ell, 0)\big)$, where $D^{(n)} + (\ell, 0)$ means translating the set $D^{(n)}$ by the vector $(\ell, 0)$. It is easy to verify that $D^{(n+\ell)}$ is in the set $\mathfrak{D}_{n+\ell}^{catalan}$ and has bi-degree $(d_1 + \binom{\ell}{2} + n\ell, d_2)$. Then Corollary 38 implies the first assertion.

For any $D^{(n)} \in \mathfrak{D}_n$ of bi-degree (d_1, d_2), by taking sufficiently large ℓ and applying Proposition 32, we get $\Delta(D^{(n+\ell)}) \equiv \sum_{\mu \in \Pi_{d_2,k}} c_\mu \cdot \Delta(F_\mu)$ (modulo lower degrees), where $F_\mu \in \mathfrak{D}_{n+\ell}$ have special minimal staircase forms of bi-degree $(d_1 + \binom{\ell}{2} + n\ell, d_2)$ and of partition type μ. This implies $\dim M^{(n+\ell)}_{d_1+\binom{\ell}{2}+n\ell,\, d_2} \leq p(d_2, k)$ and $\dim M^{(n)}_{d_1,\, d_2} \leq p(d_2, k)$. □

6 The lower bound of $\dim M_{d_1,d_2}$

6.1 A homogeneous term order.

Definition 40. Let k be a positive integer.

(a) For two partitions $\nu = (\nu_1 \leq \cdots \leq \nu_m), \mu = (\mu_1 \leq \cdots \leq \mu_n) \in \Pi_k$, we define $\rho_\nu < \rho_\mu$ if there is a positive integer $j \leq \min(m,n)$ such that $\nu_i = \mu_i$ for $1 \leq i \leq j-1$ and $\nu_j < \mu_j$. This defines a total order on the monomials in $\mathbb{C}[\rho]_k$.

(b) For a nonzero polynomial $f = \sum c_\nu \rho_\nu \in \mathbb{C}[\rho]_k$, where $c_\nu \in \mathbb{C}$, the leading monomial of f is defined as

$$\textsc{lm}(f) := \max\{\rho_\nu | c_\nu \neq 0\},$$

and the leading term of f is $\textsc{lt}(f) := c_\nu \rho_\nu$, where $\rho_\nu = \textsc{lm}(f)$. For $c \in \mathbb{C} \setminus \{0\}$, define $\textsc{lt}(c) = 1$ and $\textsc{lm}(c) = c$.

Example 41. Let $f = 2\rho_1\rho_2\rho_7 - 5\rho_4\rho_6 \in \mathbb{C}[\rho]_{10}$. Then $\textsc{lm}(f) = \rho_4\rho_6$, $\textsc{lt}(f) = -5\rho_4\rho_6$.

Lemma 42. *(a) Let ρ_ν be a fixed monomial in $\mathbb{C}[\rho]$. Then two monomials $\rho_\mu, \rho_{\mu'}$ in $\mathbb{C}[\rho]_k$ satisfy $\rho_\mu \leq \rho_{\mu'}$ if and only if $\rho_\mu\rho_\nu \leq \rho_{\mu'}\rho_\nu$.*

(b) Let ρ_μ, ρ_ν be monomials in $\mathbb{C}[\rho]_k$ and $\rho_{\mu'}, \rho_{\nu'}$ be monomials in $\mathbb{C}[\rho]_{k'}$ such that $\rho_\mu \leq \rho_\nu$ and $\rho_{\mu'} \leq \rho_{\nu'}$. Then $\rho_\mu\rho_{\mu'} \leq \rho_\nu\rho_{\nu'}$ in $\mathbb{C}[\rho]_{k+k'}$.

(c) If $k, k' \in \mathbb{N}$, $f \in \mathbb{C}[\rho]_k$, and $g \in \mathbb{C}[\rho]_{k'}$, then

$$\textsc{lm}(fg) = \textsc{lm}(f)\textsc{lm}(g), \quad \textsc{lt}(fg) = \textsc{lt}(f)\textsc{lt}(g).$$

Proof. The proof is easy and is left as an exercise to the interested reader. $\square$

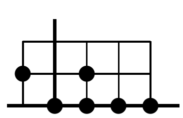
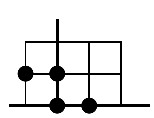
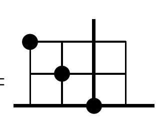
6.2 The theorems on the lower bound of $\dim M_{d_1,d_2}$

Theorem 43. *Suppose that $n, d_1, d_2 \in \mathbb{N}^+$ and $k \in \mathbb{N}$ satisfy $k \leq n-4$, $d_1 + d_2 = \binom{n}{2} - k$ and $d_2 \leq d_1$. Then for each $\nu \in \Pi_{d_2,k}$, there exists a $D_\nu \in \mathfrak{D}_n$ such that $\Delta(D_\nu)$ has bi-degree (d_1, d_2), and that $\textsc{lm}(\varphi(D_\nu)) = \rho_\nu$.*

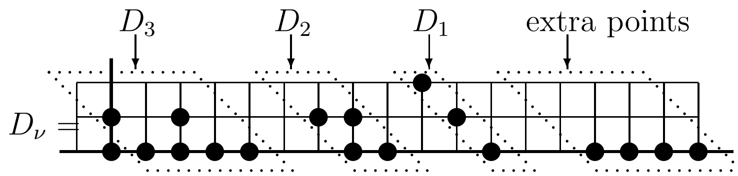
Theorem 44. *Suppose that $n, d_1, d_2 \in \mathbb{N}^+$ and $k \in \mathbb{N}$ satisfy $k \leq n-3$, $d_1 + d_2 = \binom{n}{2} - k$, $d_2 \leq d_1$. Then for each $\nu \in \Pi_{d_2,k}$, there exists an alternating polynomial f_ν of bi-degree (d_1, d_2), either of the form $\Delta(D)$ for some $D \in \mathfrak{D}^n$, or of the form $\Delta(D) - \Delta(D')$ for some $D, D' \in \mathfrak{D}_n$, such that $\textsc{lm}(\varphi(f_\nu)) = \rho_\nu$. Moreover,* $\dim M_{d_1,d_2} = p(d_2, k)$.

Remark 45. Theorem 43 and Theorem 44 are proved using the same idea. In the proofs, we give explicit constructions for D_ν (in Theorem 43) and f_ν (in Theorem 44).

Example 46. We give an example of D_ν in Theorem 43. Let $n = 18$, $k = 14$, $(d_1, d_2) = (84, 7)$, $\nu = (1, 1, 1, 2, 2, 3, 4)$. Divide ν into 3 sub-partitions $\tilde{\nu}_1 = (1, 1, 1)$, $\tilde{\nu}_2 = (2, 2)$, $\tilde{\nu}_3 = (3, 4)$. Construct $D_i \in \mathfrak{D}'_{|\tilde{\nu}_i|}$ such that $\textsc{lm}(\varphi(D_i)) = \rho_{\tilde{\nu}_i}$ for $i = 1, 2, 3$.

$D_3 =$  $D_2 =$  $D_1 =$ 

Then assemble D_3, D_2 and D_1 together with appropriate extra points:



This way we get a D_ν satisfying $\textsc{lm}(\varphi(D_\nu)) = \rho_{\tilde{\nu}_1}\rho_{\tilde{\nu}_2}\rho_{\tilde{\nu}_3} = \rho_\nu$.

6.3 Proof of Theorem 43 and Theorem 44

The following crucial lemma provides an effective method to verify if a set of alternating polynomials is linearly independent.

Lemma 47. *Fix $(d_1, d_2) \in \mathbb{N} \times \mathbb{N}$. Let $f \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^\epsilon$ be a bi-homogeneous alternating polynomial of bi-degree (d_1, d_2). If $\varphi(f) \neq 0$, then $f \not\equiv 0$ (modulo lower degrees). As a consequence, φ induces a well-defined linear map*

$$\bar{\varphi} : M_{d_1,d_2} \longrightarrow \mathbb{C}[\rho_1, \rho_2, ...]_k.$$

Proof. Suppose $\varphi(f) \neq 0$. By Proposition 32, after replacing n by a sufficiently large integer if necessary, we can assume that $f \equiv \sum_\mu c_\mu F_\mu$ (modulo lower degrees). Since $\varphi(f) \neq 0$, Proposition 32 guarantees $c_\mu \neq 0$ for some μ. Using the fact that $\{\Delta(F_\mu)\}_\mu$ are linearly independent in M_{d_1,d_2}, we conclude that $f \not\equiv 0$ (modulo lower degrees). □

The map $\bar{\varphi}$ is useful in the study of M_{d_1,d_2}. Theorem 44 implies that, for $k = \binom{n}{2} - d_1 - d_2 \leq n-3$ and $d_2 \leq d_1$, the map $\bar{\varphi}$ is injective and the image is spanned by $\{\rho_\nu\}_{\nu \in \Pi_{d_2,k}}$. In fact, all the computations we did so far support the following conjecture.

Conjecture 48. *The linear map $\bar{\varphi}$ is injective.*

Proposition 49. *Conjecture 33 implies Conjecture 48.*

Proof. Suppose that $f \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^\epsilon$ is a bi-homogeneous alternating polynomial of bi-degree (d_1, d_2) satisfying $\bar{\varphi}(f) = 0$. Conjecture 33 implies that the elements of $\mathfrak{D}_n^{catalan}$ with bi-degree (d_1, d_2) form a basis of M_{d_1,d_2}, so we can express f as a linear combination $\sum_i c_i \Delta(D_i)$, $D_i \in \mathfrak{D}_n^{catalan}$. Define $D_i' \in \mathfrak{D}_{n+\ell}^{catalan}$ as in the proof of Proposition 39. Then $\bar{\varphi}(\sum_i c_i \Delta(D_i')) = \bar{\varphi}(\sum_i c_i \Delta(D_i)) = \bar{\varphi}(f) = 0$. Since $\bar{\varphi} : M_{d_1+\ell,d_2} \to \mathbb{C}[\rho]_k$ is injective, $\sum_i c_i \Delta(D_i') = 0$, which implies $c_i = 0$ for all i and therefore $f \equiv \sum_i c_i \Delta(D_i) = 0$. □

Lemma 50. *Let $w \geq 2 \in \mathbb{N}$. Suppose $D = (P_1, \ldots, P_{w+1}) \in \mathfrak{D}_{w+1}'$, where P_i are all distinct and $|P_1| = |P_2| = 0$, $|P_i| = i-2$ $(3 \leq i \leq w+1)$. Then $\textsc{lt}(\varphi(D)) = (|P_1|_y - |P_2|_y)\rho_w$ and $\textsc{lm}(\varphi(D)) = \rho_w$.*

Proof. It immediately follows from the definition of $\varphi(D)$. □

Lemma 51. *Let $v, w \in \mathbb{N}$ and $2 \leq v \leq w$. Suppose that $D = (P_1, ..., P_{w+2}) \in \mathfrak{D}'_{w+2}$, where P_i are all distinct and*

$$|P_i| = \begin{cases} 0, & \text{if } i = 1, 2; \\ i-2, & \text{if } 3 \leq i \leq w-v+3; \\ i-3, & \text{if } w-v+4 \leq i \leq w+2. \end{cases}$$

Then $\textsc{lt}(\varphi(D)) = -(|P_1|_y - |P_2|_y)(|P_{w-v+3}|_y - |P_{w-v+4}|_y)\rho_v\rho_w$ *and* $\textsc{lm}(\varphi(D)) = \rho_v\rho_w$.

Proof. Let $\varphi(D) = \sum c_\mu \rho_\mu$. First, we show that if $c_\mu \neq 0$ then $\rho_\mu \leq \rho_v\rho_w$. There exist integers $\{w_j^{(i)}\}$ and $\sigma \in S_{w+2}$ such that $\left(\mathrm{sgn}(\sigma)\prod_{i=1}^n \rho_{w_1^{(i)}}\rho_{w_2^{(i)}}\cdots\rho_{w_{b_i}^{(i)}}\right)$ in (4.2) is not zero, and

$$\rho_\mu = \prod_{i=1}^{n} \rho_{w_1^{(i)}}\rho_{w_2^{(i)}}\cdots\rho_{w_{b_i}^{(i)}}. \tag{6.1}$$

Because of condition (4.1), we have $\sigma(i)-1-a_i-b_i \geq 0$, for all $i \in [1, w+2]$. In particular, $\sigma(w-v+3) \geq w-v+2$, $\sigma(w-v+4) \geq w-v+2$. Since σ is a permutation, $\sigma(w-v+3)$ and $\sigma(w-v+4)$ are different from each other, hence at least one of them is greater than or equal to $w-v+3$. Let u be $w-v+3$ or $w-v+4$ such that $\sigma(u) \geq w-v+3$. Since $\sigma(u) \leq w+2$ and $|P_u|(= a_u + b_u) = w-v+1$, we have $1 \leq \sigma(u) - 1 - |P_u| \leq v$. By condition (4.1), $w_1^{(u)} + ... + w_{b_u}^{(u)} = \sigma(u) - 1 - a_u - b_u \in [1, v]$. Take $j \in \mathbb{N}^+$, $1 \leq j \leq b_u$ such that $w_j^{(u)} \neq 0$. Then $\rho_{w_j^{(u)}}$ is a factor of ρ_μ by (6.1). Since $w_j^{(u)} \leq v \leq w$, we conclude that $\rho_\mu \leq \rho_v\rho_w$.

Now we show that $c_\mu \neq 0$ for $\mu = (v, w)$. Assume the monomial $\rho_v\rho_w$ appears in (6.1). By the argument in the above paragraph, we have $\sigma(u) - 1 - |P_u| = v$, which implies $\sigma(u) = w+2$. Define $\delta := u - (w-v+3) \in \{0, 1\}$. Since $\sigma(1)$ and $\sigma(2)$ cannot be 1, we may assume $\sigma(1+\epsilon) \neq 1$ for $\epsilon \in \{0, 1\}$. Then $\sigma(1+\epsilon)-1-|P_{1+\epsilon}| = w$ and $\sigma(1+\epsilon) = w+1$. For every positive integer $i \leq w+2$ that $i \neq 1+\epsilon$ and $i \neq u$, we must have $\sigma(i) = 1+|P_i|$. So $\sigma \in S_n$ must be one of the four permutations $\sigma_{\epsilon,\delta}$ for $(\epsilon, \delta) = (0, 0)$, $(0, 1)$, $(1, 0)$ or $(1, 1)$, where

$$\sigma_{\epsilon,\delta}(i) = \begin{cases} 1, & \text{if } i = 2-\epsilon; \\ w+1, & \text{if } i = 1+\epsilon; \\ i-1, & \text{if } \epsilon+2 \leq i \leq w-v+2+\delta; \\ w+2, & \text{if } i = w-v+3+\delta; \\ i-2, & \text{if } w-v+4+\delta \leq i \leq w+2. \end{cases}$$

By routine computation, we get

ϵ	δ	coefficient of $\rho_v\rho_w$ corresponding to σ
0	0	$-\lvert P_1\rvert_y\lvert P_{w-v+3}\rvert_y$
0	1	$+\lvert P_1\rvert_y\lvert P_{w-v+4}\rvert_y$
1	0	$+\lvert P_2\rvert_y\lvert P_{w-v+3}\rvert_y$
1	1	$-\lvert P_2\rvert_y\lvert P_{w-v+4}\rvert_y$

Adding the coefficients gives $c_\mu = c_{(v,w)} = -(|P_1|_y - |P_2|_y)(|P_{w-v+3}|_y - |P_{w-v+4}|_y) \neq 0$. $\square$

Example 52. Let $v = 2$, $w = 3$, $D = \big((-1,1),(0,0),(0,1),(0,2),(1,1)\big)$. Computation shows that $\varphi(D) = -\rho_2\rho_3 + \rho_1\rho_4 + \rho_1\rho_2^2 - 2\rho_1^2\rho_3 + 2\rho_1^3\rho_2 - \rho_1^5$. So $\textsc{lt}(\varphi(D)) = -(1-0)(2-1)\rho_2\rho_3 = -\rho_2\rho_3$ as asserted in Lemma 51.

Definition 53. To a sequence $\nu = (\nu_1 \le \nu_2 \le \cdots \le \nu_n)$ of positive integers, we associate a finite sequence $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \dots)$ of subsequences of ν as follows, where c is the number of 1's in ν and $m = n - c$:

$$\tilde{\nu}_i = \begin{cases} (1,1,1), & 1 \le i \le \lceil \frac{c}{3} \rceil - 1; \\ (\underbrace{1,\dots,1}_{c+3-3\lceil \frac{c}{3} \rceil}), & i = \lceil \frac{c}{3} \rceil; \\ (\nu_{c+2(i-\lceil \frac{c}{3} \rceil)-1}, \nu_{c+2(i-\lceil \frac{c}{3} \rceil)}), & \lceil \frac{c}{3} \rceil + 1 \le i \le \lceil \frac{c}{3} \rceil + \lceil \frac{m}{2} \rceil - 1; \\ (\nu_{c+2\lceil \frac{m}{2} \rceil - 1}, \dots, \nu_{c+m}), & i = \lceil \frac{c}{3} \rceil + \lceil \frac{m}{2} \rceil. \end{cases}$$

Example 54. If $\nu = (9)$ then $\tilde{\nu} = ((9))$.
If $\nu = (1,1,1,1)$ then $\tilde{\nu} = ((1,1,1),(1))$.
If $\nu = (1,1,1,1,10)$ then $\tilde{\nu} = ((1,1,1),(1),(10))$.
If $\nu = (1,1,1,1,1,1,1,1,3,3,5,5)$ then $\tilde{\nu} = ((1,1,1),(1,1,1),(1,1),(3,3),(5,5))$.
If $\nu = (1,1,1,2,2,2,3,3,7,7)$ then $\tilde{\nu} = ((1,1,1),(2,2),(2,3),(3,7),(7))$.

Proof of Theorem 43. The following table is the building block of our proof.

μ	$E_\mu \in \mathfrak{D}'$	$\lvert\mu\rvert$	$\#E_\mu$
(1,1,1)	(P_1,P_2,P_3), $\lvert P_1\rvert = \lvert P_2\rvert = \lvert P_3\rvert = 0$	3	3
(1,1)	(P_1,P_2,P_3,P_4), $\lvert P_1\rvert = \lvert P_2\rvert = 0, \lvert P_3\rvert = \lvert P_4\rvert = 2$	2	4
(1)	(P_1,P_2), $\lvert P_1\rvert = \lvert P_2\rvert = 0$	1	2
(v,w) $2 \le v \le w$	$(P_1,\dots,P_{w+2}) \in \mathfrak{D}'_{w+2}$, such that $\lvert P_i\rvert = \begin{cases} 0, & \text{if } 1 \le i \le 2; \\ i-2, & \text{if } 3 \le i \le w-v+3; \\ i-3, & \text{if } w-v+4 \le i \le w+2. \end{cases}$	$v+w$	$w+2$
(w) $w \ge 2$	$(P_1,\dots,P_{w+1})$, such that $\lvert P_1\rvert = \lvert P_2\rvert = 0, \lvert P_i\rvert = i-2$ $(3 \le i \le w+1)$	w	$w+1$

(6.2)

We claim that $\textsc{lm}(\varphi(E_\mu)) = \rho_\mu$ in the above table: the case $\mu = (1,1,1)$ or (1) follows from Lemma 28 (v); the case $\mu = (1,1)$ follows from Lemma 28 (iv)(v); the case $\mu = (v,w)$ follows from Lemma 51; the case $\mu = (w)$ follows from Lemma 50.

Let $\tilde{\nu} = \{\tilde{\nu}_1, \dots, \tilde{\nu}_m\}$ be defined as in Definition 53. The idea of the construction of D_ν is to take the union of translations of $E_{\tilde{\nu}_1}, \dots, E_{\tilde{\nu}_m}$ together with some points in $\mathbb{N} \times \mathbb{N}$ that do not affect the value of φ. Consider the following two cases separately.

CASE 1: $\tilde{\nu}_m \ne (1,1,1)$.

Define translating vectors $T_1, \dots, T_m \in \mathbb{N} \times \mathbb{N}$ as follows. $T_m = (1,0)$,

$$T_i = (1 + \#E_{\tilde{\nu}_{i+1}} + \#E_{\tilde{\nu}_{i+2}} + \cdots + \#E_{\tilde{\nu}_m}, 0), \quad \text{for all } i \in [1, m-1].$$

Define $n_0 = 1 + \#E_{\tilde{\nu}_1} + \#E_{\tilde{\nu}_2} + \cdots + \#E_{\tilde{\nu}_m}$. Then $n_0 \leq (1 + |\tilde{\nu}_1| + |\tilde{\nu}_2| + \cdots + |\tilde{\nu}_m|) + 3 = k + 4 \leq n$. Choose $P_j \in \mathbb{N} \times \mathbb{N}$ such that $|P_j| = j - 1$ for $j \in [n_0 + 1, n]$. Define $D \in \mathfrak{D}'$ as

$$\begin{aligned} D = {} & \text{The sequence obtained by sorting } \{(0,0)\} \cup \cup_{i=1}^{m} (E_{\tilde{\nu}_i} + T_i) \cup \cup_{j=n_0+1}^{n} \{P_j\} \\ & \text{in increasing order as in (2.1).} \end{aligned} \tag{6.3}$$

Claim. Fix $\nu \in \Pi_{d_2,k}$. For an integer d_2' satisfying $\#\nu \leq d_2' \leq \binom{n}{2} - k - (\#\nu)$, define $d_1' = \binom{n}{2} - k - d_2'$. Then we can make choices of $E_{\tilde{\nu}_i}$ and P_j such that the D in (6.3) has bi-degree (d_1', d_2'), and the x-coordinates of the points in D are non-negative, i.e. $D \in \mathfrak{D}$.

Proof of Claim. We give the exact lower and upper bounds of the y-degree of D, and show that every integer between these bounds can be the y-degree of some D.

For the exact lower bound, let $P_j = (j-1, 0)$ and $E_{\tilde{\nu}_i}$ be as in the following table.

$\tilde{\nu}_i$	$E_{\tilde{\nu}_i} \in \mathfrak{D}'$	y-degree of $E_{\tilde{\nu}_i}$
(1,1,1)	$\big((-2,2),(-1,1),(0,0)\big)$	3
(1,1)	$\big((-1,1),(0,0),(1,1),(0,2)\big)$	2
(1)	$\big((-1,1),(0,0)\big)$	1
(v,w) $2 \leq v \leq w$	$\big((-1,1),(0,0),(1,0),\ldots,(w-v,0),$ $(w-v,1),(w-v+1,0),\ldots,(w-1,0)\big)$	2
(w) $w \geq 2$	$\big((-1,1),(0,0),(1,0),\ldots,(w-1,0)\big)$	1

Denote the resulting D by $D_{\min y}$. The y-degree of $E_{\tilde{\nu}_i}$ is equal to $\#\tilde{\nu}_i$ for every $\tilde{\nu}_i$ in the table. So the y-degree of $D_{\min y}$ is $\sum_{i=1}^{m} (\#\tilde{\nu}_i) = (\#\nu)$.

For the exact upper bound, we note that if $D \in \mathfrak{D}_n$ can be constructed as (6.3), then the transpose of D (i.e. swap the x and y coordinates of each point in D) can also be constructed as (6.3) for some choices of P_j and $E_{\tilde{\nu}_i}$. In particular, the transpose of $D_{\min y}$, denoted by $D_{\max y}$, can be constructed as (6.3). The y-degree of $D_{\max y}$ is $\binom{n}{2} - k - (\#\nu)$, and is the maximal y-degree for all possible $D \in \mathfrak{D}_n$ constructed in (6.3).

Finally, if we move an appropriate point of D to the north-west direction, the y-degree will increase by 1. So every integer between $\#\nu$ and $\binom{n}{2} - k - (\#\nu)$ is the y-degree of some D. This completes the proof of Claim. $\square$

Now we continue the proof of CASE 1. By assumption, $d_2 \leq d_1$, $d_1 + d_2 = \binom{n}{2} - k$, and $(\#\nu) \leq d_2$ since ν is a partition of k with at most d_2 parts. Then $(\#\nu) \leq d_2 \leq \binom{n}{2} - k - (\#\nu)$. The above claim asserts that d_2 is the y-degree of some $D \in \mathfrak{D}$ constructed as (6.3). This D, denote it by D_ν, has bi-degree (d_1, d_2) . Lemma 28 (ii)(iii)(iv) imply $\varphi(D_\nu) = \prod_{i=1}^{m} \varphi(E_{\tilde{\nu}_i})$, therefore $\textsc{lm}(\varphi(D_\nu)) = \prod_{i=1}^{m} \textsc{lm}(\varphi(E_{\tilde{\nu}_i})) = \prod_{i=1}^{m} \rho_{\tilde{\nu}_i} = \rho_\nu$ using Lemma 42 (c).

CASE 2: $\tilde{\nu}_m = (1,1,1)$.

In this case, $(\#\nu) = k = 3m$. Choose a $D \in \mathfrak{D}$ such that $|P_j| = j - 1$ for $1 \leq j \leq n - 3m$, $|P_{n-3m+3j-2}| = |P_{n-3m+3j-1}| = |P_{n-3m+3j}| = n = 3m + 3j - 3$. By assumption, we have $\nu \in \Pi_{d_2,k}$ and therefore $d_2 \geq k$. It is straightforward to verify that we can choose such a D with bi-degree (d_1, d_2). This completes the proof of Theorem 43. $\square$

Proof of Theorem 44. The proof is almost identical with the one of Theorem 43. We only need to modify the row $\mu = (1,1)$ in the table (6.2). Instead of using $E_{(1,1)} \in \mathfrak{D}'$ (which contains 4 points), we use two elements $E'_{(1,1)} = \big((-a-1, a+1), (-a, a), (a+1, a)\big)$ and $E''_{(1,1)} = \big((-a-1, a+1), (-a, a), (a, a+1)\big)$ in D'. A simple computation shows $\varphi(E'_{(1,1)}) = \rho_2$, $\varphi(E''_{(1,1)}) = -\rho_1^2 + \rho_2$, so $\varphi(E'_{(1,1)}) - \varphi(E''_{(1,1)}) = \rho_1^2$. Here we need to be cautious because the bi-degree of $E'_{(1,1)}$ and $E''_{(1,1)}$ are not the same. This will not bring any problem, since we can move points in other E_μ to adjust the total bi-degree. Eventually, suppose that ℓ is the integer that $\tilde{\nu}_\ell = (1,1)$, we can construct $D'_\nu, D''_\nu \in \mathfrak{D}$ both of bi-degree (d_1, d_2) such that

$$\varphi(D'_\nu) = \varphi(E'_{(1,1)}) \prod_{i \neq \ell} \varphi(E_{\tilde{\nu}_i}), \quad \varphi(D''_\nu) = \varphi(E''_{(1,1)}) \prod_{i \neq \ell} \varphi(E_{\tilde{\nu}_i}).$$

Then $f := \Delta(D'_\nu) - \Delta(D''_\nu)$ satisfies $\textsc{lm}(\varphi(f)) = \rho_\nu$.

Now for each $\nu \in \Pi_{d_2,k}$, we can construct f_ν such that $\textsc{lm}(\varphi(f)) = \rho_\nu$. If we write down the coefficient matrix for $\varphi(f_\nu)$ with basis $\{\rho_\mu\}_{\mu \in \Pi_k}$ arranged in decreasing order, we obtain a row echelon form with rank $p(d_2, k)$. So $\dim M_{d_1,d_2} \geq p(d_2,k)$ by Lemma 47. Combining the upper bound obtained in Proposition 39, we conclude that $\dim M_{d_1,d_2} = p(d_2,k)$. $\square$

7 The condition for the equality $\dim M_{d_1,d_2} = p(d_2,k)$ to hold

In Proposition 39 we showed the inequality $\dim M_{d_1,d_2} \leq p(d_2,k)$, then in Theorem 44 we showed that "=" holds for $k \leq n-3$. In this section, we show that the condition $k \leq n-3$ is the best we can hope, in the sense of the following theorem.

Theorem 55. *Let n, d_1, d_2, k be as in Theorem 5, and $d_2 \leq d_1$. Then $\dim M_{d_1,d_2} \leq p(d_2,k)$. Moreover, the equality holds if and only if "$k \leq n-3$", or "$k = n-2$ and $d_2 = 1$", or "$d_2 = 0$".*

Proof. The inequality is proved in Proposition 39, so we are left to show the second statement. The "if" part follows easily from Theorem 44. For the "only if" part, It is straightforward to check the cases $d_2 = 0$ or 1, so we can focus on the case $d_2 \geq 2$. By Proposition 39, it suffices to show that $\dim M^{(n)}_{d_1,d_2} < \dim M^{(n+1)}_{d_1+n,d_2}$ for $k = n-2$. It is easy to check that $(d_1 - n + 3) \geq 0$, therefore $(d_1 - n + 3)$ and $(d_2 - 2)$ are two non-negative integers that add up to $\binom{n}{2} - k - n + 1 = \binom{n-2}{2}$. We know that $\dim M^{(n-2)}_{d_1-n+3,d_2-2} = 1$. Choose $D^{(n-2)} \in \mathfrak{D}^{Catalan}_{n-2}$ which has bi-degree $(d_1 - n + 3, d_2 - 2)$. Define $D^{(n+1)} = \big((0,0),(1,0),(0,2)\big) \cup \big(D^{(n-2)} + (2,0)\big)$. Then $D^{(n+1)} \in \mathfrak{D}^{Catalan}_{n+1}$ is of bi-degree (d_1+n, d_2). On the other hand, every $D^{(n)} \in \mathfrak{D}^{Catalan}_{n}$ of bi-degree (d_1, d_2) determines an element $\big((0,0)\big) \cup \big(D^{(n)} + (1,0)\big)$ in $\mathfrak{D}^{Catalan}_{n+1}$ of bi-degree (d_1+n, d_2) and is not the same as $D^{(n+1)}$. Therefore $\dim M^{(n)}_{d_1,d_2} < \dim M^{(n+1)}_{d_1+n,d_2}$. $\square$

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