Vertex subsets with minimal width and dual width in Q-polynomial distance-regular graphs

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Abstract

We study Q-polynomial distance-regular graphs from the point of view of what we call *descendents*, that is to say, those vertex subsets with the property that the *width* w and *dual width* w^* satisfy $w + w^* = d$, where d is the diameter of the graph. We show among other results that a nontrivial descendent with $w \geqslant 2$ is convex precisely when the graph has classical parameters. The classification of descendents has been done for the 5 classical families of graphs associated with short regular semilattices. We revisit and characterize these families in terms of posets consisting of descendents, and extend the classification to all of the 15 known infinite families with classical parameters and with unbounded diameter.

1 Introduction

Q-polynomial distance-regular graphs are thought of as finite/combinatorial analogues of compact symmetric spaces of rank one, and are receiving considerable attention; see e.g., [2, 3, 15, 25] and the references therein. In this paper, we study these graphs further from the point of view of what we shall call *descendents*, that is to say, those (vertex) subsets with the property that the *width* w and *dual width* w^* satisfy $w + w^* = d$, where d is the diameter of the graph. See §2 for formal definitions. A typical example is a w-cube $H(w, 2)$ in the d-cube $H(d, 2)$ ($w \leqslant d$).

The width and dual width of subsets were introduced and discussed in detail by Brouwer, Godsil, Koolen and Martin [4], and descendents arise as a special, but very important, case of the theory [4, §5]. They showed among other results that every descendent

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is completely regular, and that the induced subgraph is a Q-polynomial distance-regular graph if it is connected [4, Theorems 1–3]. When the graph is defined on the top fiber of a *short regular semilattice* [13] (as is the case for the d-cube), each object of the semilattice naturally gives rise to a descendent [4, Theorem 5]. Hence we may also view descendents as reflecting intrinsic geometric structures of Q-polynomial distance-regular graphs. Incidentally, descendents have been applied to the Erdős–Ko–Rado theorem in extremal set theory [31, Theorem 3], and implicitly to the Assmus–Mattson theorem in coding theory [32, Examples 5.4, 5.5].

Associated with each Q-polynomial distance-regular graph Γ is a *Leonard system* [38, 39, 40], a linear algebraic framework for a famous theorem of Leonard [24], [2, §3.5] which characterizes the terminating branch of the Askey scheme [21] of (basic) hypergeometric orthogonal polynomials[1] by the duality properties of Γ. The starting point of the research presented in this paper is a result of Hosoya and Suzuki [20, Proposition 1.3] which gives a system of linear equations satisfied by the eigenmatrix of the induced subgraph Γ_Y of a descendent Y of Γ (when it is connected), and we reformulate this result as the existence of a *balanced* bilinear form between the underlying vector spaces of the Leonard systems associated with Γ and Γ_Y; see §4. Balanced bilinear forms were independently studied in detail in an earlier paper [33], and we may derive all the parametric information on descendents from the results of [33].

The contents of the paper are as follows. Sections 2 and 3 review basic notation, terminology and facts concerning Q-polynomial distance-regular graphs and Leonard systems. The concept of a descendent is introduced in §2. In §4, we relate descendents and balanced bilinear forms. We give a necessary and sufficient condition on Γ_Y to be Q-polynomial distance-regular (or equivalently, to be connected) in terms of the parameters of Γ (Proposition 4.2). In passing, we also show that if Γ_Y is connected then a nonempty subset of Y is a descendent of Γ_Y precisely when it is a descendent of Γ (Proposition 4.4), so that we may define a poset structure on the set of isomorphism classes of Q-polynomial distance-regular graphs in terms of isometric embeddings as descendents. It should be remarked that the parameters of Γ_Y in turn determine those of Γ, provided that the width of Y is at least three; see Proposition 4.3.

In §5, we suppose Γ is bipartite (with diameter d). The induced subgraph $\Gamma_d^2(x)$ of the distance-2 graph of Γ on the set $\Gamma_d(x)$ of vertices at distance d from a fixed vertex x is known [8] to be distance-regular and Q-polynomial. We show that if $\Gamma_d^2(x)$ has diameter $\lfloor d/2 \rfloor$ then for every descendent Y of a halved graph of Γ, $Y \cap \Gamma_d(x)$ is a descendent of $\Gamma_d^2(x)$ unless it is empty (Proposition 5.2). This result will be used in §8.

Section 6 establishes the main results of the present paper. Many classical examples of Q-polynomial distance-regular graphs have the property that their parameters are expressed in terms of the diameter d and three other parameters q, α, β [3, p. 193]. Such graphs are said to have *classical parameters* (d, q, α, β). There are many results characterizing this property in terms of substructures of graphs; see e.g., [42, Theorem 7.2], [41]. We show that a nontrivial descendent Y with width $w \geqslant 2$ is convex (i.e., geodetically closed) precisely when Γ has classical parameters (d, q, α, β) (Theorem 6.3). Moreover, if

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[1] We also allow the specialization $q \to -1$.

this is the case then Γ_Y has classical parameters (w, q, α, β) (Theorem 6.4).

In view of this connection with convexity, the remainder of the paper is concerned with graphs with classical parameters. Currently, there are 15 known infinite families of such graphs with unbounded diameter, and 5 of them are associated with short regular semilattices. The classification of descendents has been done for these 5 families; by Brouwer et al. [4, Theorem 8] for Johnson and Hamming graphs, and by the author [31, Theorem 1] for Grassmann, bilinear forms and dual polar graphs. It turned out that every descendent is isomorphic (under the full automorphism group of the graph) to one afforded by an object of the semilattice.

Section 7 is concerned with the 5 families of "semilattice-type" graphs. We show that if $d \geqslant 4$ then these graphs are characterized by the following properties: (1) Γ has classical parameters; and there is a family $\mathscr{P}$ of descendents of Γ such that (2) any two vertices, say, at distance i, are contained in a unique descendent in $\mathscr{P}$ with width i; and (3) the intersection of two descendents in $\mathscr{P}$ is either empty or a member of $\mathscr{P}$ (Theorem 7.19). We remark that if $\mathscr{P}$ is *the* set of descendents of Γ then (1), (2) imply (3) (Proposition 7.20). We shall in fact show that $\mathscr{P}$, together with the partial order defined by reverse inclusion, forms a *regular quantum matroid* [37]. The semilattice structure of Γ is then completely recovered from $\mathscr{P}$, and the characterization of Γ follows from the classification of nontrivial regular quantum matroids with rank at least four [37, Theorem 39.6].

Section 8 extends the classification of descendents to all of the 15 families. We make heavy use of previous work on (noncomplete) convex subgraphs [23, 26] and maximal cliques [16, 17, 5] in some of these families. We shall see a strong contrast between the distributions of descendents in the 5 families of "semilattice-type" and the other 10 families of "non-semilattice-type".

The paper ends with an appendix containing necessary data involving the *parameter arrays* (see §3 for the definition) of Leonard systems.

2 Q-polynomial distance-regular graphs

Let X be a finite set and $\mathbb{C}^{X \times X}$ the $\mathbb{C}$-algebra of complex matrices with rows and columns indexed by X. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of nonempty symmetric binary relations on X. For each i, let $A_i \in \mathbb{C}^{X \times X}$ be the adjacency matrix of the graph (X, R_i). The pair $(X, \mathcal{R})$ is a (*symmetric*) *association scheme with d classes* if

(AS1) $A_0 = I$, the identity matrix;

(AS2) $\sum_{i=0}^{d} A_i = J$, the all ones matrix;

(AS3) $A_i A_j \in \boldsymbol{A} := \langle A_0, A_1, \ldots, A_d \rangle$ for $0 \leqslant i, j \leqslant d$.

It follows from (AS1)–(AS3) that the $(d+1)$-dimensional vector space $\boldsymbol{A}$ is a commutative algebra, called the *Bose–Mesner algebra* of $(X, \mathcal{R})$. Since $\boldsymbol{A}$ is semisimple[2] (as it is closed under conjugate-transposition), there is a basis $\{E_i\}_{i=0}^{d}$ consisting of the primitive

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[2]We refer to [11, §3] for the background material on semisimple algebras.

idempotents of $\boldsymbol{A}$, i.e., $E_i E_j = \delta_{ij} E_i$, $\sum_{i=0}^{d} E_i = I$. We shall always set $E_0 = |X|^{-1} J$. By (AS2), $\boldsymbol{A}$ is also closed under entrywise multiplication, denoted $\circ$. The A_i are the primitive idempotents of $\boldsymbol{A}$ with respect to this multiplication, i.e., $A_i \circ A_j = \delta_{ij} A_i$, $\sum_{i=0}^{d} A_i = J$. For convenience, define $A_i = E_i = 0$ if $i < 0$ or $i > d$.

Let $\mathbb{C}^X$ be the Hermitean space of complex column vectors with coordinates indexed by X, so that $\mathbb{C}^{X \times X}$ acts on $\mathbb{C}^X$ from the left. For each $x \in X$ let $\hat{x}$ be the vector in $\mathbb{C}^X$ with a 1 in coordinate x and 0 elsewhere. The $\hat{x}$ form an orthonormal basis for $\mathbb{C}^X$.

We say $(X, \mathcal{R})$ is *P-polynomial with respect to the ordering* $\{A_i\}_{i=0}^{d}$ if there are integers a_i, b_i, c_i $(0 \leqslant i \leqslant d)$ such that $b_d = c_0 = 0$, $b_{i-1} c_i \neq 0$ $(1 \leqslant i \leqslant d)$ and

(2.1) $A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$ $(0 \leqslant i \leqslant d)$

where $b_{-1} = c_{d+1} = 0$. Such an ordering is called a *P-polynomial ordering*. It follows that (X, R_1) is regular of valency $k := b_0$, $a_i + b_i + c_i = k$ $(0 \leqslant i \leqslant d)$, $a_0 = 0$ and $c_1 = 1$. Note that $A := A_1$ generates $\boldsymbol{A}$ and hence has $d + 1$ distinct eigenvalues $\theta_0 := k, \theta_1, \ldots, \theta_d$ so that $A = \sum_{i=0}^{d} \theta_i E_i$. Note also that (X, R_i) is the distance-i graph of (X, R_1) for all i. Dually, we say $(X, \mathcal{R})$ is *Q-polynomial with respect to the ordering* $\{E_i\}_{i=0}^{d}$ if there are scalars a_i^*, b_i^*, c_i^* $(0 \leqslant i \leqslant d)$ such that $b_d^* = c_0^* = 0$, $b_{i-1}^* c_i^* \neq 0$ $(1 \leqslant i \leqslant d)$ and

(2.2) $E_1 \circ E_i = |X|^{-1}(b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1})$ $(0 \leqslant i \leqslant d)$

where $b_{-1}^* = c_{d+1}^* = 0$. Such an ordering is called a *Q-polynomial ordering*. It follows that $\operatorname{rank} E_1 = m := b_0^*$, $a_i^* + b_i^* + c_i^* = m$ $(0 \leqslant i \leqslant d)$, $a_0^* = 0$ and $c_1^* = 1$. Note that $|X|E_1$ generates $\boldsymbol{A}$ with respect to $\circ$ and hence has $d + 1$ distinct entries $\theta_0^* := m, \theta_1^*, \ldots, \theta_d^*$ so that $|X|E_1 = \sum_{i=0}^{d} \theta_i^* A_i$. We may remark that

(2.3) $\langle (E_1 \mathbb{C}^X) \circ (E_i \mathbb{C}^X) \rangle \subseteq E_{i-1} \mathbb{C}^X + E_i \mathbb{C}^X + E_{i+1} \mathbb{C}^X$ $(0 \leqslant i \leqslant d)$.

See e.g., [2, p. 126, Proposition 8.3].

A connected simple graph Γ with vertex set $V\Gamma = X$, diameter d and path-length distance ∂ is called *distance-regular* if the distance-i relations $(0 \leqslant i \leqslant d)$ together form an association scheme. Hence P-polynomial association schemes, with specified P-polynomial ordering, are in bijection with distance-regular graphs, and we shall say, e.g., that Γ *is Q-polynomial*, and so on. The sequence

(2.4) $\iota(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$

is called the *intersection array* of Γ. Given $x \in X$, we write $\Gamma_i(x) = \{y \in X : \partial(x, y) = i\}$, $k_i = |\Gamma_i(x)|$ $(0 \leqslant i \leqslant d)$. We abbreviate $\Gamma(x) = \Gamma_1(x)$.

We say Γ has *classical parameters* (d, q, α, β) [3, p. 193] if

(2.5) $b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right)$, $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \right)$ $(0 \leqslant i \leqslant d)$

where $\begin{bmatrix} i \\ j \end{bmatrix}_q$ is the q-binomial coefficient. By [3, Proposition 6.2.1], q is an integer $\neq 0, -1$. In this case Γ has a Q-polynomial ordering $\{E_i\}_{i=0}^{d}$ which we call *standard*, such that

(2.6) $\theta_i^* = \xi^* \begin{bmatrix} d-i \\ 1 \end{bmatrix}_q + \zeta^*$ $(0 \leqslant i \leqslant d)$

for some ξ^*, ζ^* with $\xi^* \neq 0$ [3, Corollary 8.4.2].

For the rest of this section, suppose further that Γ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$. For the moment fix a "base vertex" $x \in X$, and let $E_i^* = E_i^*(x) := \mathrm{diag}(A_i \hat{x})$, $A_i^* = A_i^*(x) := |X| \, \mathrm{diag}(E_i \hat{x})$ $(0 \leqslant i \leqslant d)$.[3] We abbreviate $A^* = A_1^*$. Note that $E_i^* E_j^* = \delta_{ij} E_i^*$, $\sum_{i=0}^d E_i^* = I$. The E_i^* and the A_i^* form two bases for the *dual Bose–Mesner algebra* $\boldsymbol{A}^* = \boldsymbol{A}^*(x)$ *with respect to* x. Note also that A^* generates $\boldsymbol{A}^*$ and $A^* = \sum_{i=0}^d \theta_i^* E_i^*$. The *Terwilliger* (or *subconstituent*) *algebra* $\boldsymbol{T} = \boldsymbol{T}(x)$ *of* Γ *with respect to* x is the subalgebra of $\mathbb{C}^{X \times X}$ generated by $\boldsymbol{A}$, $\boldsymbol{A}^*$ [34, 35, 36]. Since $\boldsymbol{T}$ is closed under conjugate-transposition, it is semisimple and any two nonisomorphic irreducible $\boldsymbol{T}$-modules in $\mathbb{C}^X$ are orthogonal.

Let Y be a nonempty subset of X and $\hat{Y} = \sum_{x \in Y} \hat{x}$ its characteristic vector. We let Γ_Y denote the subgraph of Γ induced on Y. Set $Y_i = \{x \in X : \partial(x, Y) = i\}$ $(0 \leqslant i \leqslant \rho)$, where $\rho = \max\{\partial(x, Y) : x \in X\}$ is the covering radius of Y. Note that $\sum_{i=0}^\rho \hat{Y}_i = \hat{X}$. We call Y *completely regular* if $\langle \hat{Y}_0, \hat{Y}_1, \ldots, \hat{Y}_\rho \rangle$ is an $\boldsymbol{A}$-module. Brouwer et al. [4] defined the *width* w and *dual width* w^* of Y as follows:

(2.7) $w = \max\{i : \hat{Y}^\mathsf{T} A_i \hat{Y} \neq 0\}, \quad w^* = \max\{i : \hat{Y}^\mathsf{T} E_i \hat{Y} \neq 0\}.$

They showed (among other results) that

(2.8) **Theorem**([4, §5]). *We have $w + w^* \geqslant d$. If equality holds then Y is completely regular with covering radius w^*, and $(Y, \mathcal{R}^Y)$ forms a Q-polynomial association scheme with w-classes, where $\mathcal{R}^Y = \{R_i \cap (Y \times Y) : 0 \leqslant i \leqslant w\}$.*

We call Y a *descendent* of Γ if $w + w^* = d$. The descendents with $w = 0$ are precisely the singletons, and X is the unique descendent with $w = d$; we shall refer to these cases as *trivial* and say *nontrivial* otherwise. By (2.8) it follows that

(2.9) **Theorem**([4, Theorem 3]). *Suppose $w + w^* = d$. If Γ_Y is connected then it is a Q-polynomial distance-regular graph with diameter w.*

We comment on the Q-polynomiality of $(Y, \mathcal{R}^Y)$ stated in (2.8). Suppose $w + w^* = d$ and let $\boldsymbol{A}'$ be the Bose–Mesner algebra of $(Y, \mathcal{R}^Y)$. For every $B \in \boldsymbol{A}$, let $\breve{B}$ be the principal submatrix of B corresponding to Y. Brouwer et al. [4, §4] observed

(2.10) $\breve{E}_i \breve{E}_j = 0$ if $|i - j| > w^*$,

and then showed that $\langle \breve{E}_0, \breve{E}_1, \ldots, \breve{E}_i \rangle$ is an ideal of $\boldsymbol{A}'$ for all i. Hence we get a Q-polynomial ordering $\{E_i'\}_{i=0}^w$ of the primitive idempotents of $\boldsymbol{A}'$ such that

(2.11) $\langle E_0', E_1', \ldots, E_i' \rangle = \langle \breve{E}_0, \breve{E}_1, \ldots, \breve{E}_i \rangle$ $(0 \leqslant i \leqslant w)$.

Throughout we shall adopt the following convention and retain the notation of §2:

(2.12) For the rest of this paper, we assume Γ is distance-regular with diameter $d \geqslant 3$ and is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$. Unless otherwise stated, Y will denote a nontrivial descendent of Γ with width w and dual width $w^* = d - w$.

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[3]For a complex matrix B, it is customary that B^* denotes the conjugate transpose of B. It should be stressed that we are *not* using this convention.

3   Leonard systems

Let d be a positive integer. Let $\mathfrak{A}$ be a $\mathbb{C}$-algebra isomorphic to the full matrix algebra $\mathbb{C}^{(d+1)\times(d+1)}$ and W an irreducible left $\mathfrak{A}$-module. Note that W is unique up to isomorphism and $\dim W = d+1$. An element $\mathfrak{a}$ of $\mathfrak{A}$ is called *multiplicity-free* if it has $d+1$ mutually distinct eigenvalues. Suppose $\mathfrak{a}$ is multiplicity-free and let $\{\theta_i\}_{i=0}^d$ be an ordering of the eigenvalues of $\mathfrak{a}$. Then there is a sequence of elements $\{\mathfrak{e}_i\}_{i=0}^d$ in $\mathfrak{A}$ such that (i) $\mathfrak{a}\mathfrak{e}_i = \theta_i\mathfrak{e}_i$; (ii) $\mathfrak{e}_i\mathfrak{e}_j = \delta_{ij}\mathfrak{e}_i$; (iii) $\sum_{i=0}^d \mathfrak{e}_i = \mathbb{1}$ where $\mathbb{1}$ is the identity of $\mathfrak{A}$. We call $\mathfrak{e}_i$ the *primitive idempotent* of $\mathfrak{a}$ associated with θ_i. Note that $\mathfrak{a}$ generates $\langle \mathfrak{e}_0, \mathfrak{e}_1, \ldots, \mathfrak{e}_d \rangle$.

A *Leonard system* in $\mathfrak{A}$ [38, Definition 1.4] is a sequence

$$(3.1)\quad \Phi = \big(\mathfrak{a}; \mathfrak{a}^*; \{\mathfrak{e}_i\}_{i=0}^d; \{\mathfrak{e}_i^*\}_{i=0}^d\big)$$

satisfying the following axioms (LS1)–(LS5):

(LS1) Each of $\mathfrak{a}, \mathfrak{a}^*$ is a multiplicity-free element in $\mathfrak{A}$.

(LS2) $\{\mathfrak{e}_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $\mathfrak{a}$.

(LS3) $\{\mathfrak{e}_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of $\mathfrak{a}^*$.

(LS4) $\mathfrak{e}_i^*\mathfrak{a}\mathfrak{e}_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

(LS5) $\mathfrak{e}_i\mathfrak{a}^*\mathfrak{e}_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

We call d the *diameter* of Φ. For convenience, define $\mathfrak{e}_i = \mathfrak{e}_i^* = 0$ if $i < 0$ or $i > d$. Observe

$$(3.2)\quad \mathfrak{e}_0^*W + \mathfrak{e}_1^*W + \cdots + \mathfrak{e}_i^*W = \mathfrak{e}_0^*W + \mathfrak{a}\mathfrak{e}_0^*W + \cdots + \mathfrak{a}^i\mathfrak{e}_0^*W \quad (0 \leqslant i \leqslant d).$$

A Leonard system Ψ in a $\mathbb{C}$-algebra $\mathfrak{B}$ is *isomorphic* to Φ if there is a $\mathbb{C}$-algebra isomorphism $\sigma : \mathfrak{A} \to \mathfrak{B}$ such that $\Psi = \Phi^\sigma := \big(\mathfrak{a}^\sigma; \mathfrak{a}^{*\sigma}; \{\mathfrak{e}_i^\sigma\}_{i=0}^d; \{\mathfrak{e}_i^{*\sigma}\}_{i=0}^d\big)$. Let $\xi, \xi^*, \zeta, \zeta^*$ be scalars with $\xi, \xi^* \neq 0$. Then

$$(3.3)\quad \big(\xi\mathfrak{a} + \zeta\mathbb{1}; \xi^*\mathfrak{a}^* + \zeta^*\mathbb{1}; \{\mathfrak{e}_i\}_{i=0}^d; \{\mathfrak{e}_i^*\}_{i=0}^d\big)$$

is a Leonard system in $\mathfrak{A}$, called an *affine transformation* of Φ. We say Φ, Ψ are *affine-isomorphic* if Ψ is isomorphic to an affine transformation of Φ. The *dual* of Φ is

$$(3.4)$$
$$\Phi^* = \big(\mathfrak{a}^*; \mathfrak{a}; \{\mathfrak{e}_i^*\}_{i=0}^d; \{\mathfrak{e}_i\}_{i=0}^d\big).$$

For any object f associated with Φ, we shall occasionally denote by f^* the corresponding object for Φ^*; an example is $\mathfrak{e}_i^*(\Phi) = \mathfrak{e}_i(\Phi^*)$. Note that $(f^*)^* = f$.

(3.5) **Example.** With reference to (2.12), fix the base vertex $x \in X$ and let $W = \boldsymbol{A}\hat{x} = \boldsymbol{A}^*\hat{X}$ be the *primary $\boldsymbol{T}$-module* [34, Lemma 3.6]. Set $\mathfrak{a} = A|_W$, $\mathfrak{a}^* = A^*|_W$, $\mathfrak{e}_i = E_i|_W$, $\mathfrak{e}_i^* = E_i^*|_W$ $(0 \leqslant i \leqslant d)$. Then $\Phi = \Phi(\Gamma; x) := \big(\mathfrak{a}; \mathfrak{a}^*; \{\mathfrak{e}_i\}_{i=0}^d; \{\mathfrak{e}_i^*\}_{i=0}^d\big)$ is a Leonard system. See [35, Theorem 4.1], [10]. We remark that $\Phi(\Gamma; x)$ does not depend on x up to isomorphism, so that we shall write $\Phi(\Gamma) = \Phi(\Gamma; x)$ where the context allows.

(3.6) **Example.** More generally, let W be any irreducible $\boldsymbol{T}$-module. We say W is *thin* if $\dim E_i^* W \leqslant 1$ for all i. Suppose W is thin. Then W is *dual thin*, i.e., $\dim E_i W \leqslant 1$ for all i, and there are integers ϵ (*endpoint*), ϵ^* (*dual endpoint*), δ (*diameter*) such that $\{i : E_i^* W \neq 0\} = \{\epsilon, \epsilon+1, \ldots, \epsilon+\delta\}$, $\{i : E_i W \neq 0\} = \{\epsilon^*, \epsilon^*+1, \ldots, \epsilon^*+\delta\}$ [34, Lemmas 3.9, 3.12].[4] Set $\mathfrak{a} = A|_W$, $\mathfrak{a}^* = A^*|_W$, $\mathfrak{e}_i = E_{\epsilon^*+i}|_W$, $\mathfrak{e}_i^* = E_{\epsilon+i}^*|_W$ $(0 \leqslant i \leqslant \delta)$. Then $\Phi = \Phi(W) := \left(\mathfrak{a}; \mathfrak{a}^*; \{\mathfrak{e}_i\}_{i=0}^{\delta}; \{\mathfrak{e}_i^*\}_{i=0}^{\delta}\right)$ is a Leonard system. Note that $\Phi(\Gamma; x) = \Phi(\boldsymbol{A}\hat{x})$.

For $0 \leqslant i \leqslant d$ let θ_i (resp. θ_i^*) be the eigenvalue of $\mathfrak{a}$ (resp. $\mathfrak{a}^*$) associated with $\mathfrak{e}_i$ (resp. $\mathfrak{e}_i^*$). By [38, Theorem 3.2] there are scalars φ_i $(1 \leqslant i \leqslant d)$ and a $\mathbb{C}$-algebra isomorphism $\natural : \mathfrak{A} \to \mathbb{C}^{(d+1)\times(d+1)}$ such that $\mathfrak{a}^\natural$ (resp. $\mathfrak{a}^{*\natural}$) is the lower (resp. upper) bidiagonal matrix with diagonal entries $(\mathfrak{a}^\natural)_{ii} = \theta_i$ (resp. $(\mathfrak{a}^{*\natural})_{ii} = \theta_i^*$) $(0 \leqslant i \leqslant d)$ and subdiagonal (resp. superdiagonal) entries $(\mathfrak{a}^\natural)_{i,i-1} = 1$ (resp. $(\mathfrak{a}^{*\natural})_{i-1,i} = \varphi_i$) $(1 \leqslant i \leqslant d)$. We let $\phi_i = \varphi_i(\Phi^\Downarrow)$ $(1 \leqslant i \leqslant d)$, where $\Phi^\Downarrow = \left(\mathfrak{a}; \mathfrak{a}^*; \{\mathfrak{e}_{d-i}\}_{i=0}^{d}; \{\mathfrak{e}_i^*\}_{i=0}^{d}\right)$.[5] The *parameter array* of Φ is

$$(3.7) \quad p(\Phi) = \left(\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d}; \{\varphi_i\}_{i=1}^{d}; \{\phi_i\}_{i=1}^{d}\right).$$

By [38, Theorem 1.9], the isomorphism class of Φ is determined by $p(\Phi)$. In [40], $p(\Phi)$ is given in closed form; see also (A.1). Note that the parameter array of (3.3) is given by

$$(3.8) \quad \left(\{\xi\theta_i + \zeta\}_{i=0}^{d}; \{\xi^*\theta_i^* + \zeta^*\}_{i=0}^{d}; \{\xi\xi^*\varphi_i\}_{i=1}^{d}; \{\xi\xi^*\phi_i\}_{i=1}^{d}\right).$$

Let u be a nonzero vector in $\mathfrak{e}_0 W$. Then $\{\mathfrak{e}_i^* u\}_{i=0}^{d}$ is a basis for W [39, Lemma 10.2]. Define the scalars a_i, b_i, c_i $(0 \leqslant i \leqslant d)$ by $b_d = c_0 = 0$ and

$$(3.9) \quad \mathfrak{a}\mathfrak{e}_i^* u = b_{i-1}\mathfrak{e}_{i-1}^* u + a_i \mathfrak{e}_i^* u + c_{i+1}\mathfrak{e}_{i+1}^* u \quad (0 \leqslant i \leqslant d)$$

where $b_{-1} = c_{d+1} = 0$. By [39, Theorem 17.7] it follows that

$$(3.10) \quad b_i = \varphi_{i+1}\frac{\tau_i^*(\theta_i^*)}{\tau_{i+1}^*(\theta_{i+1}^*)}, \quad c_i = \phi_i\frac{\eta_{d-i}^*(\theta_i^*)}{\eta_{d-i+1}^*(\theta_{i-1}^*)} \quad (0 \leqslant i \leqslant d)$$

where $\theta_{-1}^*, \theta_{d+1}^*$ are indeterminates, $\varphi_{d+1} = \phi_0 = 0$ and

$$(3.11) \quad \tau_i(\lambda) = \prod_{l=0}^{i-1}(\lambda - \theta_l), \quad \eta_i(\lambda) = \prod_{l=0}^{i-1}(\lambda - \theta_{d-l}) \quad (0 \leqslant i \leqslant d).$$

(3.12) **Example.** Let $\Phi = \Phi(\Gamma)$ be as in (3.5). Then $b_i(\Gamma) = b_i(\Phi)$, $b_i^*(\Gamma) = b_i^*(\Phi)$, $c_i(\Gamma) = c_i(\Phi)$, $c_i^*(\Gamma) = c_i^*(\Phi)$ $(0 \leqslant i \leqslant d)$. See [35, Theorem 4.1].

Let $\Phi' = \left(\mathfrak{a}'; \mathfrak{a}^{*\prime}; \{\mathfrak{e}_i'\}_{i=0}^{d'}; \{\mathfrak{e}_i^{*\prime}\}_{i=0}^{d'}\right)$ be another Leonard system with diameter $d' \leqslant d$ and $W' = W(\Phi')$ the vector space underlying Φ'. Given an integer ρ $(0 \leqslant \rho \leqslant d - d')$, a nonzero bilinear form $(\cdot|\cdot) : W \times W' \to \mathbb{C}$ is called *ρ-balanced with respect to Φ, Φ'* if

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[4]In [34, 35, 36], ϵ and ϵ^* are called the dual endpoint and endpoint of W, respectively.

[5]Viewed as permutations on all Leonard systems, $*$ and $\Downarrow$ generate a dihedral group with 8 elements which plays a fundamental role in the theory of Leonard systems.

(B1) $(\mathfrak{e}_i^* W | \mathfrak{e}_j^{*\prime} W') = 0$ if $i - \rho \neq j$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant d')$;

(B2) $(\mathfrak{e}_i W | \mathfrak{e}_j' W') = 0$ if $i < j$ or $i > j + d - d'$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant d')$.

We call Φ' a ρ-*descendent* of Φ whenever such a form exists. The ρ-*descendents* of Φ are completely classified; see (A.3). In particular, by (A.2), (A.3), (3.8) it follows that

(3.13) **Proposition.** *Let d, d', ρ be integers such that $1 \leqslant d' \leqslant d,\ 0 \leqslant \rho \leqslant d - d'$. Then a Leonard system with diameter d has at most one ρ-descendent with diameter d' up to affine isomorphism. Conversely, if $d' \geqslant 3$ then a Leonard system with diameter d' is a ρ-descendent of at most one Leonard system with diameter d up to affine isomorphism.*

4   Basic results concerning descendents

With reference to (2.12), we begin with the following observation (cf. [20, p. 73]):

(4.1) *With the notation of §2, for any i, j $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant w)$ we have*

$$
\breve{E}_i E_j' = \begin{cases} 0 & \text{if } i < j \text{ or } i > j + w^*, \\ \neq 0 & \text{if } i = j \text{ or } i = j + w^*. \end{cases}
$$

Proof. By (2.10), (2.11) it follows that $\breve{E}_i \in \langle E_{i-w^*}', \ldots, E_i' \rangle$, so that $\breve{E}_i E_j' = 0$ if $i < j$ or $i > j + w^*$. By (2.11) we also find $\breve{E}_j E_j' \neq 0$. Note that $E_{j+w^*} \circ E_j \in \langle E_{w^*}, \ldots, E_d \rangle$ and the coefficient of E_{w^*} in $E_{j+w^*} \circ E_j$ is nonzero. Hence $\text{trace}(\breve{E}_{j+w^*} \breve{E}_j) = \hat{Y}^\mathsf{T}(E_{j+w^*} \circ E_j)\hat{Y} \neq 0$. It follows that $\breve{E}_{j+w^*}\breve{E}_j \neq 0$ and therefore $\breve{E}_{j+w^*} E_j' \neq 0$ by (2.10), (2.11). $\qquad\square$

As mentioned in the introduction, Hosoya and Suzuki translated (4.1) into a system of linear equations satisfied by the eigenmatrix of $(Y, \mathcal{R}^Y)$; see [20, Proposition 1.3]. We now show how descendents are related to balanced bilinear forms:

(4.2) **Proposition.** *Pick any $x \in Y$, and let the parameter array of $\Phi = \Phi(\Gamma; x)$ be given as in (A.1). Suppose $w > 1$. Then Γ_Y is a Q-polynomial distance-regular graph precisely for Cases I, IA, II, IIA, IIB, IIC; or Case III with w^* even. If this is the case then the bilinear form $(\cdot|\cdot) : \boldsymbol{A}\hat{x} \times \boldsymbol{A}'\hat{x} \to \mathbb{C}$ defined by $(u|u') = u^\mathsf{T} u'$ is 0-balanced with respect to $\Phi,\ \Phi(\Gamma_Y; x)$.*

Proof. Write $W = \boldsymbol{A}\hat{x}$, $W' = \boldsymbol{A}'\hat{x}$. Note that $(E_i W | E_j' W') = 0$ whenever $\breve{E}_i E_j' = 0$. Hence it follows from (4.1) that $(E_i W | E_j' W') = 0$ if $i < j$ or $i > j + w^*$. Suppose Γ_Y is distance-regular. Then by these comments we find that $(\cdot|\cdot)$ is 0-balanced with respect to $\Phi,\ \Phi(\Gamma_Y; x)$. By virtue of (A.3), w^* must be even if $p(\Phi)$ is of Case III. Conversely, suppose $p(\Phi)$ (and w^*) satisfies one of the cases mentioned in (4.2). Then by [33, Theorem 7.3] there is a Leonard system $\Phi' = \left(\mathfrak{a}'; \mathfrak{a}^{*\prime}; \{\mathfrak{e}_i'\}_{i=0}^w; \{\mathfrak{e}_i^{*\prime}\}_{i=0}^w\right)$ with $W(\Phi') = W'$ such that $\mathfrak{e}_i' = E_i'|_{W'}$, $\mathfrak{e}_i^{*\prime} = E_i^{*\prime}|_{W'}$ where $E_i^{*\prime} = \text{diag}(\breve{A}_i \hat{x})$ $(0 \leqslant i \leqslant w)$. Note that $\breve{A}|_{W'} \in \langle \mathfrak{e}_0', \mathfrak{e}_1', \ldots, \mathfrak{e}_w' \rangle$, so that $\breve{A}|_{W'}$ is a polynomial in $\mathfrak{a}'$. Since $\breve{A}\mathfrak{e}_0^{*\prime} W' = \langle \breve{A}\hat{x} \rangle = \mathfrak{e}_1^{*\prime} W'$, it follows from (3.2) that $\breve{A}|_{W'} = \xi\mathfrak{a}' + \zeta\mathfrak{1}'$ for some $\xi, \zeta \in \mathbb{C}$ with $\xi \neq 0$, where $\mathfrak{1}'$ is the identity operator on W'. Hence $\langle E_i^{*\prime} \breve{A}^i \hat{x} \rangle = \mathfrak{e}_i^{*\prime}(\mathfrak{a}')^i \mathfrak{e}_0^{*\prime} W' = \mathfrak{e}_i^{*\prime} W' = \langle \breve{A}_i \hat{x} \rangle (\neq 0)$ for $0 \leqslant i \leqslant w$. In particular, Γ_Y is connected and thus distance-regular by (2.9). $\qquad\square$

By (4.2), connectivity and therefore distance-regularity of Γ_Y can be read off the parameters of Γ.

(4.3) Proposition. *Suppose Γ_Y is distance-regular. Then $\Phi(\Gamma_Y)$ is uniquely determined by $\Phi(\Gamma)$ up to isomorphism. Conversely, if $w \geqslant 3$ then $\Phi(\Gamma_Y)$ uniquely determines $\Phi(\Gamma)$ up to isomorphism.*

Proof. By (4.2), $\Phi(\Gamma_Y)$ is a 0-descendent of $\Phi(\Gamma)$. Hence the result follows from (3.13) together with the additional normalizations $b_0(\Psi) = \theta_0(\Psi)$, $b_0^*(\Psi) = \theta_0^*(\Psi)$, $c_1(\Psi) = c_1^*(\Psi) = 1$ for each $\Psi \in \{\Phi(\Gamma), \Phi(\Gamma_Y)\}$. $\qquad\square$

The following is another consequence of (4.1):

(4.4) Proposition. *Suppose Γ_Y is distance-regular. Then a nonempty subset of Y is a descendent of Γ_Y if and only if it is a descendent of Γ.*

Proof. Let $Z \subseteq Y$ have dual width $w^{*\prime}$ in Γ_Y. For $0 \leqslant i \leqslant w$, by (4.1) we find $\breve{E}_{i+w^*} \in \langle E_i', \ldots, E_w' \rangle$ and the coefficient of E_i' in $\breve{E}_{i+w^*}$ is nonzero. Since $\hat{Z}^\mathsf{T} E_i \hat{Z} = \hat{Z}^\mathsf{T} \breve{E}_i \hat{Z}$, it follows that Z has dual width $w^{*\prime} + w^*$ in Γ. $\qquad\square$

(4.5) Remark. Let $\mathscr{L}$ be the set of isomorphism classes of Q-polynomial distance-regular graphs with diameter at least three. For two isomorphism classes $[\Gamma], [\Delta] \in \mathscr{L}$, write $[\Delta] \preccurlyeq [\Gamma]$ if $[\Delta] = [\Gamma_Y]$ for some descendent Y of Γ. Then by (4.4) it follows that $\preccurlyeq$ is a partial order on $\mathscr{L}$. Determining all descendents of Γ amounts to describing the order ideal generated by $[\Gamma]$. Conversely, given $[\Gamma] \in \mathscr{L}$, it is a problem of some significance to determine the filter generated by $[\Gamma]$, i.e., $V_{[\Gamma]} = \{[\Delta] \in \mathscr{L} : [\Gamma] \preccurlyeq [\Delta]\}$.

Let $A^*(Y) = |X||Y|^{-1}\mathrm{diag}(E_1\hat{Y})$, $E_i^*(Y) = \mathrm{diag}(\hat{Y}_i)$ $(0 \leqslant i \leqslant w^*)$ where $Y_i = \{x \in X : \partial(x, Y) = i\}$. Let $\tilde{W} = \boldsymbol{A}\hat{Y} = \langle \hat{Y}_0, \hat{Y}_1, \ldots, \hat{Y}_{w^*} \rangle$. Note that $A^*(Y)\tilde{W} \subseteq \tilde{W}$. Following [22, Definition 3.7], we call Y *Leonard* (*with respect to θ_1*) if the matrix representing $A^*(Y)|_{\tilde{W}}$ with respect to the basis $\{E_i\hat{Y}\}_{i=0}^{w^*}$ for $\tilde{W}$ is irreducible[6] tridiagonal. Set $\mathfrak{b} = A|_{\tilde{W}}$, $\mathfrak{b}^* = A^*(Y)|_{\tilde{W}}$, $\mathfrak{f}_i = E_i|_{\tilde{W}}$, $\mathfrak{f}_i^* = E_i^*(Y)|_{\tilde{W}}$ $(0 \leqslant i \leqslant w^*)$. Then Y is Leonard if and only if $\Phi(\Gamma; Y) := \big(\mathfrak{b}; \mathfrak{b}^*; \{\mathfrak{f}_i\}_{i=0}^{w^*}; \{\mathfrak{f}_i^*\}_{i=0}^{w^*}\big)$ is a Leonard system. The following is dual to (4.2):

(4.6) Proposition. *Pick any $x \in Y$, and let the parameter array of $\Phi = \Phi(\Gamma; x)$ be given as in (A.1). Suppose $w^* > 1$. Then Y is Leonard (with respect to θ_1) precisely for Cases I, IA, II, IIA, IIB, IIC; or Case III with w even. If this is the case then the bilinear form $(\cdot|\cdot) : \boldsymbol{A}\hat{x} \times \boldsymbol{A}\hat{Y} \to \mathbb{C}$ defined by $(u|u') = u^\mathsf{T} u'$ is 0-balanced with respect to Φ^*, $\Phi(\Gamma; Y)^*$.*

Proof. Note that $E_i^*(x)E_j^*(Y) = 0$ whenever $i < j$ or $i > j + w$ (cf. (4.1)). Hence if Y is Leonard then $(\cdot|\cdot)$ is 0-balanced with respect to Φ^*, $\Phi(\Gamma; Y)^*$, so that by (A.3) it follows that w must be even if $p(\Phi)$ is of Case III.[7] Conversely, suppose $p(\Phi)$ (and w) satisfies one of the cases mentioned in (4.6). Then by [33, Theorem 7.3] there are operators $\mathfrak{c}$, $\mathfrak{c}^*$ on

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[6]A tridiagonal matrix is *irreducible* [38] if all the superdiagonal and subdiagonal entries are nonzero.

[7]The permutation $*$ (see footnote 5) leaves each of Cases I, IA, II, IIC, III invariant and swaps Cases IIA and IIB.

$\tilde{W}$ such that $\left(\mathfrak{c}; \mathfrak{c}^*; \{\mathfrak{f}_i\}_{i=0}^{w^*}; \{\mathfrak{f}_i^*\}_{i=0}^{w^*}\right)$ is a Leonard system. Note that $\mathfrak{b}^* \in \langle \mathfrak{f}_0^*, \mathfrak{f}_1^*, \ldots, \mathfrak{f}_{w^*}^* \rangle$, so that $\mathfrak{b}^*$ is a polynomial in $\mathfrak{c}^*$. Since $\mathfrak{b}^* \mathfrak{f}_0 \tilde{W} = \langle E_1 \hat{Y} \rangle = \mathfrak{f}_1 \tilde{W}$, it follows from (3.2) that $\mathfrak{b}^* = \xi^* \mathfrak{c}^* + \zeta^* \tilde{\mathfrak{i}}$ for some $\xi^*, \zeta^* \in \mathbb{C}$ with $\xi^* \neq 0$, where $\tilde{\mathfrak{i}}$ is the identity operator on $\tilde{W}$. Hence the matrix representing $\mathfrak{b}^*$ with respect to $\{E_i \hat{Y}\}_{i=0}^{w^*}$ is irreducible tridiagonal. In other words, Y is Leonard, as desired. $\qquad\square$

(4.7) **Remark.** Suppose Γ is a translation distance-regular graph [3, §11.1C] and Y is also a subgroup of the abelian group X. Then by [22, Proposition 3.3, Theorem 3.10], Y is Leonard if and only if the coset graph Γ/Y is Q-polynomial. Hence (4.6) strengthens [4, Theorem 4], which states that Γ/Y is Q-polynomial if it is primitive. Note that if Y is Leonard then $\Phi(\Gamma/Y; Y)$ (where Y is a *vertex* of Γ/Y) is affine isomorphic to $\Phi(\Gamma; Y)$.

It seems that (4.6) also motivates further analysis of the Terwilliger algebra *with respect to Y* in the sense of Suzuki [30]; this will be discussed elsewhere.

5 The bipartite case

(5.1) With reference to (2.12), in this section we further assume that Γ is bipartite and $d \geqslant 6$ (so that the halved graphs have diameter at least three).

With reference to (5.1), fix $x \in X$ and let $\Gamma_d^2 = \Gamma_d^2(x)$ be the graph with vertex set $\Gamma_d = \Gamma_d(x)$ and edge set $\{(y, z) \in \Gamma_d \times \Gamma_d : \partial(y, z) = 2\}$. Caughman [8, Theorems 9.2, 9.6, Corollary 4.4] showed that Γ_d^2 is distance-regular and Q-polynomial with diameter $\mathfrak{d}$, where $\mathfrak{d}$ equals half the width of Γ_d. In this section, we shall prove a result relating descendents of a halved graph of Γ to those of Γ_d^2; see (5.2) below.

Write $E_i^* = E_i^*(x)$ ($0 \leqslant i \leqslant d$) and $\boldsymbol{T} = \boldsymbol{T}(x)$. Let W be an irreducible $\boldsymbol{T}$-module with endpoint ϵ, dual endpoint ϵ^* and diameter δ (see (3.6)). By [7, Lemma 9.2, Theorem 9.4], W is thin, dual thin and $2\epsilon^* + \delta = d$. In particular, $0 \leqslant \epsilon^* \leqslant \lfloor d/2 \rfloor$ and $\epsilon \leqslant 2\epsilon^*$. Let U_{ij} be the sum of the irreducible $\boldsymbol{T}$-modules W in $\mathbb{C}^X$ with $\epsilon = i$ and $\epsilon^* = j$. By [7, Theorem 13.1], the (nonzero) U_{ij} are the homogeneous components of $\mathbb{C}^X$. Note that $E_d^* U_{ij} = 0$ unless $i = 2j$, so that $E_d^* \mathbb{C}^X = \sum_{j=0}^{\lfloor d/2 \rfloor} E_d^* U_{2j,j}$ (orthogonal direct sum). By [8, Theorem 9.2] $E_d^* \boldsymbol{A} E_d^*$ gives the Bose–Mesner algebra of Γ_d^2, and each of $E_d^* U_{2j,j}$ ($0 \leqslant j \leqslant \lfloor d/2 \rfloor$) is a (not necessarily maximal) eigenspace for $E_d^* \boldsymbol{A} E_d^*$.

We now compute the eigenvalue of the adjacency matrix $E_d^* A_2 E_d^*$ of Γ_d^2 on $E_d^* U_{2j,j}$. If Γ is the d-cube $H(d, 2)$ then Γ_d is a singleton and there is nothing to discuss. Suppose Γ is the folded cube $\bar{H}(2d, 2)$. Let $W \subseteq U_{2j,j}$ and let $\Phi = \Phi(W)$ be as in (3.6). By [36, Example 6.1], $a_i(\Phi) = 0$ ($0 \leqslant i \leqslant \delta$), $b_i(\Phi) = 2\delta - i$ ($0 \leqslant i \leqslant \delta - 1$), $c_i(\Phi) = i$ ($1 \leqslant i \leqslant \delta - 1$) and $c_\delta(\Phi) = 2\delta$, where $\delta = d - 2j$. Using $A^2 = c_2 A_2 + kI$ we find that $E_d^* A_2 E_d^*$ has eigenvalue $(d - 2j)^2 - 2j$ on $E_d^* W$ (and hence on $E_d^* U_{2j,j}$). Next suppose $\Gamma \neq H(d, 2), \bar{H}(2d, 2)$. Then by [7, pp. 89–91], $p(\Phi(\Gamma))$ satisfies Case I in (A.1) and in this case there are scalars $q, s^* \in \mathbb{C}$ (independent of j) such that $a_i(\Phi) = 0$ ($0 \leqslant i \leqslant \delta$),

$$b_i(\Phi) = \frac{h(q^\delta - q^i)(1 - s^* q^{4j+i+1})}{q^{\delta+j}(1 - s^* q^{4j+2i+1})}, \quad c_i(\Phi) = \frac{h(q^i - 1)(1 - s^* q^{4j+\delta+i+1})}{q^{\delta+j}(1 - s^* q^{4j+2i+1})}$$

for $1 \leqslant i \leqslant \delta - 1$, and $b_0(\Phi) = c_\delta(\Phi) = h(q^{-j} - q^{j-d})$, where

$$h = \frac{q^d(1 - s^*q^3)}{(q-1)(1 - s^*q^{d+2})}.$$

Likewise we find that the eigenvalue of $E_d^* A_2 E_d^*$ on $E_d^* U_{2j,j}$ is given by

$$\frac{1 - s^*q^5}{(q^2 - 1)(1 - s^*q^{d+2})(1 - s^*q^{d+3})(1 - s^*q^{2d-1})q}$$
$$\times \left((q^d - 1)(q^d - q)(1 - s^{*2}q^{2d+2}) + \frac{q^{2d}(1 - s^*q^3)(1 - q^{2j})(1 - s^*q^{2j})}{q^{2j}} \right).$$

(5.2) Proposition. *Referring to (5.1), suppose Γ_d^2 has diameter $\lfloor d/2 \rfloor$. If Y is a descendent of a halved graph of Γ with width w, then $Y \cap \Gamma_d$ is a descendent of Γ_d^2 with width w, provided that it is nonempty.*

Proof. Note that $Y \cap \Gamma_d$ has width at most w in Γ_d^2, and that the characteristic vector of $Y \cap \Gamma_d$ is $E_d^* \hat{Y}$. Let w^* be the dual width of Y in the halved graph. Then by [2, p. 328, Theorem 6.4] we find $\hat{Y} \in \sum_{j=0}^{w^*} (E_j + E_{d-j})\mathbb{C}^X$, so that $E_d^* \hat{Y} \in \sum_{j=0}^{w^*} E_d^* U_{2j,j}$. By assumption, $\Gamma \neq H(d, 2)$. Suppose $\Gamma = \bar{H}(2d, 2)$. Then Γ_d^2 is the folded Johnson graph $\bar{J}(2d, d)$. It follows (e.g., from [2, p. 301] or [36, Example 6.1]) that $(d - 2j)^2 - 2j$ is the jth eigenvalue of $\bar{J}(2d, d)$ in the Q-polynomial ordering ($0 \leqslant j \leqslant \lfloor d/2 \rfloor$). Likewise, if $\Gamma \neq H(d, 2), \bar{H}(2d, 2)$, then by the data in [8, p. 469], [2, pp. 264–265] we routinely find that the ordering $\{E_d^* U_{2j,j}\}_{j=0}^{\lfloor d/2 \rfloor}$ of the eigenspaces of Γ_d^2 also agrees with the Q-polynomial ordering of Γ_d^2. Hence it follows that $Y \cap \Gamma_d$ has dual width at most w^* in Γ_d^2. Since Γ_d^2 has diameter $\lfloor d/2 \rfloor = w + w^*$, we find that $Y \cap \Gamma_d$ is a descendent of Γ_d^2 and has width w, as desired. $\square$

6 Convexity and graphs with classical parameters

In this section, we shall prove our main results concerning convexity of descendents and classical parameters. Let Φ be the Leonard system from (3.1). By (3.10) we find

$$(6.1) \quad \frac{b_i(\Phi)}{c_1(\Phi)} = \frac{\varphi_{i+1} \eta_d^*(\theta_0^*) \tau_i^*(\theta_i^*)}{\phi_1 \eta_{d-1}^*(\theta_1^*) \tau_{i+1}^*(\theta_{i+1}^*)}, \quad \frac{c_i(\Phi)}{c_1(\Phi)} = \frac{\phi_i \eta_d^*(\theta_0^*) \eta_{d-i}^*(\theta_i^*)}{\phi_1 \eta_{d-1}^*(\theta_1^*) \eta_{d-i+1}^*(\theta_{i-1}^*)} \quad (0 \leqslant i \leqslant d).$$

Note that the values in (6.1) are invariant under affine transformation of Φ.[8] Moreover, if $\Phi = \Phi(\Gamma)$ then by (3.12) they coincide with $b_i(\Gamma)$ and $c_i(\Gamma)$, respectively, since $c_1(\Gamma) = 1$. The following is a refinement of [3, Theorem 8.4.1], and is verified using (2.5), (2.6), (6.1):

(6.2) Proposition. *With reference to (2.12), let the parameter array of $\Phi = \Phi(\Gamma)$ be given as in (A.1). Then Γ has classical parameters if and only if $p(\Phi)$ satisfies either Case I with $s^* = 0$; or Cases IA, IIA, IIC. If this is the case then $p(\Phi)$ and the classical parameters (d, q, α, β) are related as follows:*

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[8]Therefore, if $p(\Phi)$ satisfies, say, Case I in (A.1) then the resulting formulae involve q, r_1, r_2, s, s^* only, and are independent of $h, h^*, \theta_0, \theta_0^*$.

Case	q	α	β
I, $s^* = r_1 = 0$	q	$\dfrac{r_2(1-q)}{sq^d - r_2}$	$\dfrac{r_2 q - 1}{q(sq^d - r_2)}$
IA	q	$\dfrac{r(1-q)}{sq^d - r}$	$\dfrac{r}{sq^d - r}$
IIA	1	$\dfrac{1}{r - s - d}$	$\dfrac{-1 - r}{r - s - d}$
IIC	1	0	$\dfrac{-r}{r - ss^*}$

Recall that Y is called *convex* (or *geodetically closed*) if for any $x, y \in Y$ we have $\{z \in X : \partial(x, z) + \partial(z, y) = \partial(x, y)\} \subseteq Y$.

(6.3) Theorem. *With reference to (2.12), suppose $1 < w < d$. Then Y is convex precisely when Γ has classical parameters (with standard Q-polynomial ordering).*

Proof. Let $\Phi = \Phi(\Gamma)$ and let $p(\Phi)$ be given as in (A.1). First, we may assume that Γ_Y is distance-regular, or equivalently, by (4.2), that w^* is even if $p(\Phi)$ satisfies Case III. Indeed, if Y is convex then Γ_Y is connected and hence distance-regular by (2.9). On the other hand, if Γ has classical parameters then by (6.2), $p(\Phi)$ does not satisfy Case III in the first place. Let $\Phi' = \Phi(\Gamma_Y)$. Then, by (4.2), $p(\Phi')$ takes the form given in (A.3) with $\rho = 0$. Note that Y is convex if and only if $c_i(\Gamma) = c_i(\Gamma_Y)$ for all $1 \leqslant i \leqslant w$. Using (6.1) we find that $c_i(\Gamma)/c_i(\Gamma_Y)$ equals

$\dfrac{(1 - s^* q^{w+2})(1 - s^* q^{i+d+1})}{(1 - s^* q^{d+2})(1 - s^* q^{i+w+1})}$ for Case I,

$\dfrac{(s^* + w + 2)(s^* + i + d + 1)}{(s^* + d + 2)(s^* + i + w + 1)}$ for Cases II, IIB,

$\dfrac{-s^* + w + 2}{-s^* + d + 2}$ for Case III, d even, w even, i even,

$\dfrac{(-s^* + w + 2)(-s^* + i + d + 1)}{(-s^* + d + 2)(-s^* + i + w + 1)}$ for Case III, d even, w even, i odd,

$\dfrac{-s^* + i + d + 1}{-s^* + i + w + 1}$ for Case III, d odd, w odd, i even,

1 for Cases IA, IIA, IIC; or Case III, d odd, w odd, i odd.

Since $1 < w < d$ it follows that Y is convex precisely when $p(\Phi)$ satisfies one of the following: Case I, $s^* = 0$; or Cases IA, IIA, IIC. Hence the result follows from (6.2). $\square$

The following is another important consequence of (4.3), (6.2), (A.3):

(6.4) Theorem. *Given scalars q, α and β, if Γ has classical parameters (d, q, α, β) (with standard Q-polynomial ordering) then Γ_Y is distance-regular and has classical parameters (w, q, α, β). The converse also holds, provided $w \geqslant 3$.*

(6.5) **Remark.** In [31, Proposition 2], the author showed that $\iota(\Gamma_Y)$ is uniquely determined by w, and when Γ is of "semilattice-type" the convexity of Y was then derived from the existence of a specific example. We may remark that (6.3), (6.4) supersede these results and give an answer to the problem raised in [31, p. 907, Remark].

We end this section with a comment. Recall that Y is called *strongly closed* if for any $x, y \in Y$ we have $\{z \in X : \partial(x, z) + \partial(z, y) \leqslant \partial(x, y) + 1\} \subseteq Y$.

(6.6) *With reference to* (2.12), *suppose* $1 < w < d$. *Then* Y *is strongly closed precisely when* Γ *has classical parameters* $(d, q, 0, \beta)$ *(with standard Q-polynomial ordering).*

Proof. We may assume that Y is convex, or equivalently, by (6.3), that Γ has classical parameters (d, q, α, β). In particular, $c_i(\Gamma) = c_i(\Gamma_Y)$ $(1 \leqslant i \leqslant w)$. Note that Y is then strongly closed if and only if $a_i(\Gamma) = a_i(\Gamma_Y)$ for all $1 \leqslant i \leqslant w$. Since Γ_Y has classical parameters (w, q, α, β) by (6.4), it follows from (2.5) that this happens precisely when $\alpha = 0$, as desired. □

7 Quantum matroids and descendents

There are many results on distance-regular graphs with the property that any pair of vertices x, y is contained in a strongly closed subset with width $\partial(x, y)$; see e.g., [42, 19]. In view of the results of §6, in this section we assume Γ has classical parameters and look at the implications of similar existence conditions on descendents.

First we recall some facts concerning quantum matroids [37] and related distance-regular graphs. A finite nonempty poset[9] $\mathscr{P}$ is a *quantum matroid* if

(QM1) $\mathscr{P}$ is ranked;

(QM2) $\mathscr{P}$ is a (meet) semilattice;

(QM3) For all $x \in \mathscr{P}$, the interval $[0, x]$ is a modular atomic lattice;

(QM4) For all $x, y \in \mathscr{P}$ satisfying $\mathrm{rank}(x) < \mathrm{rank}(y)$, there is an atom $a \in \mathscr{P}$ such that $a \leqslant y$, $a \not\leqslant x$ and $x \vee a$ exists in $\mathscr{P}$.

We say $\mathscr{P}$ is *nontrivial* if $\mathscr{P}$ has rank $d \geqslant 2$ and is not a modular atomic lattice. Suppose $\mathscr{P}$ is nontrivial. Then $\mathscr{P}$ is called *q-line regular* if each rank 2 element covers exactly $q + 1$ elements; $\mathscr{P}$ is *β-dual-line regular* if each element with rank $d - 1$ is covered by exactly $\beta + 1$ elements; $\mathscr{P}$ is *α-zig-zag regular* if for all pairs (x, y) such that $\mathrm{rank}(x) = d - 1$, $\mathrm{rank}(y) = d$ and x covers $x \wedge y$, there are exactly $\alpha + 1$ pairs (x_1, y_1) such that y_1 covers both x and x_1, and y covers x_1. We say $\mathscr{P}$ is *regular* if $\mathscr{P}$ is line regular, dual-line regular and zig-zag regular. Suppose $\mathscr{P}$ is nontrivial and regular with parameters (d, q, α, β). Let $\mathrm{top}(\mathscr{P})$ be the top fiber of $\mathscr{P}$ and set $R = \{(x, y) \in \mathrm{top}(\mathscr{P}) \times \mathrm{top}(\mathscr{P}) : x, y \text{ cover } x \wedge y\}$. Then by [37, Theorem 38.2], $\Gamma = (\mathrm{top}(\mathscr{P}), R)$ is distance-regular. Moreover, it has classical parameters (d, q, α, β) provided that the diameter equals d.

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[9]See, e.g., [29, Chapter 3] for terminology from poset theory.

We now list five examples of nontrivial regular quantum matroids from [37]. In (7.1)–(7.5) below, the partial order on $\mathscr{P}$ will always be defined by inclusion and Y will denote a nontrivial descendent of Γ.

(7.1) **Example.** The truncated Boolean algebra $B(d,\nu)$ $(\nu > d)$. Let Ω be a set of size ν and $\mathscr{P} = \{x \subseteq \Omega : |x| \leqslant d\}$. $\mathscr{P}$ has parameters $(d,1,1,\nu-d)$ and $\mathrm{top}(\mathscr{P})$ induces the Johnson graph $J(\nu,d)$ [3, §9.1]. If $\nu \geqslant 2d$ then Y satisfies one of the following: (i) $Y = \{x \in \mathrm{top}(\mathscr{P}) : u \subseteq x\}$ for some $u \subseteq \Omega$ with $|u| = w^*$; (ii) $\nu = 2d$ and $Y = \{x \in \mathrm{top}(\mathscr{P}) : x \subseteq u\}$ for some $u \subseteq \Omega$ with $|u| = d + w$.

(7.2) **Example.** The Hamming matroid $H(d,\ell)$ $(\ell \geqslant 2)$. Let $\Omega_1, \Omega_2, \ldots, \Omega_d$ be pairwise disjoint sets of size ℓ, $\Omega = \bigcup_{i=1}^d \Omega_i$ and $\mathscr{P} = \{x \subseteq \Omega : |x \cap \Omega_i| \leqslant 1 \ (1 \leqslant i \leqslant d)\}$. $\mathscr{P}$ has parameters $(d,1,0,\ell-1)$ and $\mathrm{top}(\mathscr{P})$ induces the Hamming graph $H(d,\ell)$ [3, §9.2]. Y is of the form $\{x \in \mathrm{top}(\mathscr{P}) : u \subseteq x\}$ for some $u \in \mathscr{P}$ with $|u| = w^*$.

(7.3) **Example.** The truncated projective geometry $L_q(d,\nu)$ $(\nu > d)$. Let $\mathscr{P}$ be the set of subspaces x of $\mathbb{F}_q^\nu$ with $\dim x \leqslant d$. $\mathscr{P}$ has parameters (d,q,q,β) where $\beta + 1 = \begin{bmatrix} \nu-d+1 \\ 1 \end{bmatrix}_q$, and $\mathrm{top}(\mathscr{P})$ induces the Grassmann graph $J_q(\nu,d)$ [3, §9.3]. If $\nu \geqslant 2d$ then Y satisfies one of the following: (i) $Y = \{x \in \mathrm{top}(\mathscr{P}) : u \subseteq x\}$ for some subspace $u \subseteq \mathbb{F}_q^\nu$ with $\dim u = w^*$; (ii) $\nu = 2d$ and $Y = \{x \in \mathrm{top}(\mathscr{P}) : x \subseteq u\}$ for some subspace $u \subseteq \mathbb{F}_q^\nu$ with $\dim u = d + w$.

(7.4) **Example.** The attenuated space $A_q(d,d+e)$ $(e \geqslant 1)$. Fix a subspace E of $\mathbb{F}_q^{d+e}$ with $\dim E = e$, and let $\mathscr{P}$ be the set of subspaces x of $\mathbb{F}_q^{d+e}$ with $x \cap E = 0$. $\mathscr{P}$ has parameters $(d,q,q-1,q^e-1)$, and $\mathrm{top}(\mathscr{P})$ induces the bilinear forms graph $\mathrm{Bil}_q(d,e)$ [3, §9.5A]. If $d \leqslant e$ then Y satisfies one of the following: (i) $Y = \{x \in \mathrm{top}(\mathscr{P}) : u \subseteq x\}$ for some subspace $u \subseteq \mathbb{F}_q^{d+e}$ with $\dim u = w^*$ and $u \cap E = 0$; (ii) $d = e$ and $Y = \{x \in \mathrm{top}(\mathscr{P}) : x \subseteq u\}$ for some subspace $u \subseteq \mathbb{F}_q^{d+e}$ with $\dim u = d + w$ and $\dim u \cap E = w$.

(7.5) **Example.** The classical polar spaces. Let V be one of the following spaces over $\mathbb{F}_q$ equipped with a nondegenerate form:

Name	$\dim V$	Form	e
$[C_d(q)]$	$2d$	alternating	1
$[B_d(q)]$	$2d+1$	quadratic	1
$[D_d(q)]$	$2d$	quadratic (Witt index d)	0
$[^2D_{d+1}(q)]$	$2d+2$	quadratic (Witt index d)	2
$[^2A_{2d}(\ell)]$	$2d+1$	Hermitean $(q = \ell^2)$	$\frac{3}{2}$
$[^2A_{2d-1}(\ell)]$	$2d$	Hermitean $(q = \ell^2)$	$\frac{1}{2}$

Let $\mathscr{P}$ be the set of isotropic subspaces of V. $\mathscr{P}$ has parameters $(d,q,0,q^e)$ and $\mathrm{top}(\mathscr{P})$ induces the dual polar graph on V [3, §9.4]. Y is of the form $\{x \in \mathrm{top}(\mathscr{P}) : u \subseteq x\}$ for some $u \in \mathscr{P}$ with $\dim u = w^*$.

We recall the following isomorphisms: $J(\nu,d) \cong J(\nu,\nu-d)$, $J_q(\nu,d) \cong J_q(\nu,\nu-d)$, $\mathrm{Bil}_q(d,e) \cong \mathrm{Bil}_q(e,d)$. Note also that, in each of (7.1)–(7.5), the descendents with any fixed width form a single orbit under the full automorphism group of Γ.

(7.6) **Theorem**([37, Theorem 39.6]). *Every nontrivial regular quantum matroid with rank at least four is isomorphic to one of (7.1)–(7.5).*

Now we return to the general situation (2.12). Let $\mathscr{P}$ be a nonempty family of descendents of Γ. We say $\mathscr{P}$ *satisfies (UD)$_i$* if any two vertices $x, y \in X$ at distance i are contained in a unique descendent in $\mathscr{P}$, denoted $Y(x, y)$, with width i. We shall assume the following three conditions until (7.20):

(7.7) Γ has classical parameters (d, q, α, β);

(7.8) $\mathscr{P}$ satisfies (UD)$_i$ for all i;

(7.9) $Y_1 \cap Y_2 \in \mathscr{P}$ for all $Y_1, Y_2 \in \mathscr{P}$ such that $Y_1 \cap Y_2 \neq \emptyset$.

Referring to (7.7)–(7.9), define a partial order $\leqslant$ on $\mathscr{P}$ by reverse inclusion. Our goal is to show that $\mathscr{P}$ is a nontrivial regular quantum matroid. Note that X is the minimal element of $\mathscr{P}$ and the maximal elements of $\mathscr{P}$ are precisely the singletons. We shall freely use (2.8), (4.4), (6.3), (6.4). In particular, note that every $Y \in \mathscr{P}$ is convex, Γ_Y is distance-regular with classical parameters $(w(Y), q, \alpha, \beta)$, and $\mathscr{P}_Y := \{Z \in \mathscr{P} : Z \subseteq Y\}$ is a family of descendents of Γ_Y.

(7.10) *For $0 \leqslant i \leqslant j \leqslant d$, we have $|\{Y \in \mathscr{P} : x, y \in Y, \ w(Y) = j\}| = \begin{bmatrix} d-i \\ j-i \end{bmatrix}_q$ for any two vertices $x, y \in X$ with $\partial(x, y) = i$. In particular, q is a positive integer.*

Proof. Count in two ways the sequences $(z_1, \ldots, z_{j-i}, Y)$ such that $z_l \in \Gamma_{i+l}(x) \cap \Gamma(z_{l-1})$ $(1 \leqslant l \leqslant j - i)$ and $Y = Y(x, z_{j-i})$, where $z_0 = y$. $\qquad\square$

(7.11) $\mathscr{P}_Y$ *satisfies (UD)$_i$ in Γ_Y for all $Y \in \mathscr{P}$ and i.*

Proof. Let $x, y \in Y$ and set $Z = Y(x, y) \in \mathscr{P}$. Then $Y \cap Z$ belongs to $\mathscr{P}_Y$ and contains x and y, so that $Z = Y \cap Z \in \mathscr{P}_Y$. $\qquad\square$

(7.12) *For $0 \leqslant i \leqslant j \leqslant d$, we have $|\{Z \in \mathscr{P} : Y \subseteq Z, \ w(Z) = j\}| = \begin{bmatrix} d-i \\ j-i \end{bmatrix}_q$ for every $Y \in \mathscr{P}$ with $w(Y) = i$.*

Proof. Pick $x, y \in Y$ with $\partial(x, y) = i$ and let $Z \in \mathscr{P}$. Then by (7.11) we find $x, y \in Z$ if and only if $Y \subseteq Z$. Hence the result follows from (7.10). $\qquad\square$

(7.13) $\mathscr{P}$ *satisfies (QM1) and $\mathrm{rank}(Y) = w^*(Y)$ for every $Y \in \mathscr{P}$.*

Proof. Pick $Y, Z \in \mathscr{P}$ with $Y \subseteq Z$. Applying (7.12) to $(\Gamma_Z, \mathscr{P}_Z)$, we find Y covers Z precisely when $w(Z) = w(Y) + 1$, or equivalently, $w^*(Z) = w^*(Y) - 1$. Hence $\mathrm{rank}(Y)$ is well defined and equals $w^*(Y)$. $\qquad\square$

(7.14) $\mathscr{P}$ *satisfies (QM2). Moreover, $w(Y_1 \wedge Y_2) = w(Y_1 \cup Y_2)$ for any $Y_1, Y_2 \in \mathscr{P}$.*

Proof. Let $Y_1, Y_2 \in \mathscr{P}$ and suppose $Y_1 \not\subseteq Y_2$, $Y_2 \not\subseteq Y_1$. Since Y_1, Y_2 are completely regular, for every $x \in X$ and $i \in \{1, 2\}$ we have $w(\{x\} \cup Y_i) = \partial(x, Y_i) + w(Y_i)$. Hence there are $x_1 \in Y_1$, $x_2 \in Y_2$ such that $\partial(x_1, x_2) = w(Y_1 \cup Y_2)$. Set $Z = Y(x_1, x_2)$. Pick $y_1 \in Y_1$ with $\partial(x_2, y_1) = \partial(x_2, Y_1) = \partial(x_1, x_2) - w(Y_1)$. Then $\partial(x_1, y_1) = w(Y_1)$, so that $y_1 \in Z$ and $Y_1 = Y(x_1, y_1) \subseteq Z$ by (7.11). Likewise, $Y_2 \subseteq Z$. Hence Z is a lower bound for Y_1, Y_2. Any lower bound for Y_1, Y_2 contains x_1, x_2, and thus Z by (7.11), whence $Z = Y_1 \wedge Y_2$. □

(7.15) $\mathscr{P}$ *satisfies (QM3).*

Proof. Let $Y \in \mathscr{P}$. Then the interval $[X, Y]$ is a lattice since $\mathscr{P}$ satisfies (QM2) by (7.14). By (7.12), every $Z \in [X, Y]$ with $w^*(Z) \geqslant 2$ covers $\left[{w^*(Z) \atop 1} \right]_q \geqslant 2$ elements in $\mathscr{P}$, so that $[X, Y]$ is atomic. Let $Y_1, Y_2 \in [X, Y]$ be distinct and suppose $w^*(Y_1) = w^*(Y_2)$. Set $i = w(Y_1)$, $j = w^*(Y_1)$ for brevity. We claim $w^*(Y_1 \wedge Y_2) = j - 1$ if and only if $w^*(Y_1 \cap Y_2) = j + 1$, where we recall $Y_1 \vee Y_2 = Y_1 \cap Y_2$. First suppose $w^*(Y_1 \cap Y_2) = j + 1$. Then $w(Y_1 \cap Y_2) = i - 1$ and by (4.4), (2.8) we find $Y_1 \cap Y_2$ has covering radius one in each of Γ_{Y_1}, Γ_{Y_2}, so that $w(Y_1 \cup Y_2) \leqslant w(Y_1 \cap Y_2) + 2 = i + 1$. Since $w(Y_1 \wedge Y_2) > i$, we find $w(Y_1 \wedge Y_2) = i + 1$ by (7.14) and thus $w^*(Y_1 \wedge Y_2) = j - 1$. Next suppose $w^*(Y_1 \wedge Y_2) = j - 1$. Then by (7.12) and since $\left[{j \atop 1} \right]_q > \left[{j-1 \atop 1} \right]_q$, there is $C \in \mathscr{P}$ such that $w^*(C) = 1$, $Y_1 \subseteq C$, $Y_1 \wedge Y_2 \not\subseteq C$. We have $Y_1 = C \cap (Y_1 \wedge Y_2)$ and hence $Y_1 \cap Y_2 = C \cap Y_2$. Since $\hat{C} \circ \hat{Y_2} \in \sum_{l=0}^{j+1} E_l \mathbb{C}^X$ by virtue of (2.3), we find $w^*(Y_1 \cap Y_2) \leqslant j + 1$. By (2.8) and since $w(Y_1 \cap Y_2) < i$, it follows that $w^*(Y_1 \cap Y_2) = j + 1$. The claim now follows, and therefore $[X, Y]$ is modular. □

(7.16) $\mathscr{P}$ *satisfies (QM4).*

Proof. Let $Y_1, Y_2 \in \mathscr{P}$ and suppose $w^*(Y_1) < w^*(Y_2)$. First assume $Y_1 \cap Y_2 \neq \emptyset$. Then by (7.12) and since $\left[{w^*(Y_2) \atop 1} \right]_q > \left[{w^*(Y_1) \atop 1} \right]_q$, there is $C \in \mathscr{P}$ such that $w^*(C) = 1$, $Y_2 \subseteq C$, $Y_1 \not\subseteq C$. Since $Y_1 \cap C \neq \emptyset$ we find $Y_1 \vee C = Y_1 \cap C$. Next assume $Y_1 \cap Y_2 = \emptyset$, and pick $x_1, y_1 \in Y_1$ and $x_2 \in Y_2$ such that $\partial(x_1, x_2) = w(Y_1 \cup Y_2)$ and $\partial(x_2, y_1) = \partial(x_2, Y_1) = \partial(x_1, x_2) - w(Y_1)$ as in the proof of (7.14). Set $Z_1 = Y_1 \wedge Y_2$, $Z_2 = \{y_1\} \wedge Y_2$. Then $Z_2 \subseteq Z_1$, and since $w(\{y_1\} \cup Y_2) = \partial(y_1, Y_2) + w(Y_2) \leqslant \partial(y_1, x_2) + w(Y_2) = \partial(x_1, x_2) - w(Y_1) + w(Y_2) < w(Y_1 \cup Y_2)$, it follows from (7.14) that $w(Z_1) > w(Z_2)$, i.e., $w^*(Z_1) < w^*(Z_2)$. Again there is $C \in \mathscr{P}$ such that $w^*(C) = 1$, $Z_2 \subseteq C$, $Z_1 \not\subseteq C$. Note that $Y_2 \subseteq C$ and $Y_1 \not\subseteq C$. Since $y_1 \in C$, we find $Y_1 \cap C \neq \emptyset$ and $Y_1 \vee C = Y_1 \cap C$, as desired. □

(7.17) $\mathscr{P}$ *is q-line regular, β-dual-line regular and α-zig-zag regular.*

Proof. By (7.12), $\mathscr{P}$ is q-line regular. Pick any $Y \in \mathscr{P}$ with $w(Y) = 1$. Then $|Y| = \beta + 1$ by (6.4), (2.5), so that $\mathscr{P}$ is β-dual-line regular. Let $x \in X$ and suppose $w(\{x\} \wedge Y) = 2$, i.e., $\partial(x, Y) = 1$. We count the pairs $(y, Z) \in X \times \mathscr{P}$ such that $w(Z) = 1$, $y \in Y$, $y \in Z$, $x \in Z$. Note that x, y must be adjacent and $Z = Y(x, y)$. Hence the number of such pairs is $|\Gamma(x) \cap Y|$. By (6.4), (2.5), the strongly regular graph $\Delta = \Gamma_{\{x\} \wedge Y}$ satisfies $k(\Delta) = \beta(q + 1)$, $a_1(\Delta) = \beta + \alpha q - 1$, $c_2(\Delta) = (\alpha + 1)(q + 1)$, and hence ([3, Theorem 1.3.1]) has smallest eigenvalue $\theta_2(\Delta) = -(q + 1)$, so that Y attains the Hoffman bound $1 - k(\Delta)/\theta_2(\Delta) = \beta + 1$. By [3, Proposition 1.3.2], $|\Gamma(x) \cap Y| = -c_2(\Delta)/\theta_2(\Delta) = \alpha + 1$ and therefore $\mathscr{P}$ is α-zig-zag regular. □

To summarize:

(7.18) **Proposition.** *Suppose* Γ *and* $\mathscr{P}$ *satisfy* (7.7)–(7.9)*. Then* $\mathscr{P}$ *is a nontrivial regular quantum matroid with parameters* (d, q, α, β)*.*

Hence it follows from (7.6) that

(7.19) **Theorem.** *Suppose* Γ *and* $\mathscr{P}$ *satisfy* (7.7)–(7.9)*, and suppose further* $d \geqslant 4$*. Then* Γ *is either a Johnson, Hamming, Grassmann, bilinear forms, or dual polar graph.*

One may wish to modify the conditions (7.7)–(7.9) so that we still obtain a characterization similar to (7.19). For example, it seems natural to assume that $\mathscr{P}$ is the set of *all* descendents of Γ. We now show that in this case (7.9) is redundant.

(7.20) **Proposition.** *Suppose* $\mathscr{P}$ *is the set of descendents of* Γ*. Then* (7.7)*,* (7.8) *together imply* (7.9)*.*

Proof. Note that (7.10) holds without change. We next show (7.11). Let i be given. By induction, assume that $\mathscr{P}_Z$ satisfies (UD)$_l$ in Γ_Z for all $Z \in \mathscr{P}$ and $l > i$. Let $Y \in \mathscr{P}$ and pick any $x, y \in Y$ with $\partial(x, y) = i$. Since $\mathscr{P}_Y$ satisfies (UD)$_{i+1}$ in Γ_Y, for any $z \in \Gamma_{i+1}(x) \cap \Gamma(y) \cap Y$ we have $x, y \in Y(x, z) \subseteq Y$, so that by replacing Y with $Y(x, z)$ we may assume $w(Y) = i + 1$. Since $\begin{bmatrix} d-i \\ 1 \end{bmatrix}_q > \begin{bmatrix} d-i-1 \\ 1 \end{bmatrix}_q$, it follows from (7.10) that there is $C \in \mathscr{P}$ with $x, y \in C$, $Y \nsubseteq C$, $w^*(C) = 1$. Set $Z = Y \cap C$. Suppose $w(Z) = i + 1$ and pick $u, v \in Z$ with $\partial(u, v) = i + 1$. Then $Y = Y(u, v)$. But since $u, v \in C$ and $\mathscr{P}_C$ satisfies (UD)$_{i+1}$, we would have $Y \subseteq C$, a contradiction. Hence $w(Z) = i$. Moreover, $\hat{Z} = \hat{Y} \circ \hat{C} \in \sum_{l=0}^{d-i} E_l \mathbb{C}^X$ by virtue of (2.3), whence $w^*(Z) \leqslant d - i$. By (2.8) it follows that $Z \in \mathscr{P}$ and therefore $Y(x, y) = Z \subseteq Y$. Hence $\mathscr{P}_Y$ satisfies (UD)$_i$ in Γ_Y. We have now shown (7.11). Finally, we show (7.9). Let $Y_1, Y_2 \in \mathscr{P}$ and suppose $N := Y_1 \cap Y_2 \neq \emptyset$. Pick $x, y \in N$ such that $\partial(x, y) = w(N)$. Then $Y(x, y) \subseteq N$ by (7.11). Since $Y(x, y)$ is completely regular and has width $w(N)$, for every $z \in N$ we have $w(N) \geqslant w(\{z\} \cup Y(x, y)) = \partial(z, Y(x, y)) + w(N)$, so that $z \in Y(x, y)$ and thus $N = Y(x, y) \in \mathscr{P}$, as desired. $\square$

(7.21) **Remark.** It should be noted that $J(2d, d)$, $J_q(2d, d)$, $\mathrm{Bil}_q(d, d)$ are the only examples among the graphs listed in (7.19) which do *not* possess the property that "the set of all descendents satisfies (UD)$_i$ for all i." This property seems particularly strong, so that it is a reasonable guess that (7.7) would also be redundant.

8 Classifications

In §7, we focused on the 5 families of distance-regular graphs associated with short regular semilattices (or nontrivial regular quantum matroids). In this section, we shall extend the classification of descendents to all of the 15 known infinite families with classical parameters and with unbounded diameter. We shall freely use (2.8), (6.3), (6.4).

(8.1) Throughout this section, Y shall always denote a nontrivial descendent of Γ with width w. Descriptions of some of the graphs below involve $n \in \{2d, 2d + 1\}$, in which cases we use the following notation:

$$
m = \begin{cases} 2d - 1 & \text{if } n = 2d, \\ 2d + 1 & \text{if } n = 2d + 1. \end{cases}
$$

Doob graphs

Let d_1, d_2 be positive integers, and let $\Gamma = \mathrm{Doob}(d_1, d_2) := \Gamma^1 \times \Gamma^2 \times \cdots \times \Gamma^{d_1+d_2}$ be a Doob graph [3, §9.2B], where $\Gamma^1, \ldots, \Gamma^{d_1}$ are copies of the Shrikhande graph S on 16 vertices and $\Gamma^{d_1+1}, \ldots, \Gamma^{d_1+d_2}$ are copies of the complete graph K_4 on 4 vertices. Γ has classical parameters $(d, 1, 0, 3)$, where $d = 2d_1 + d_2$. In particular, $\iota(\Gamma) = \iota(H(d, 4))$.

Γ_Y has classical parameters $(w, 1, 0, 3)$, so that $|Y| = 4^w$. Observe that the convex subsets of X are precisely the direct products $Y^1 \times Y^2 \times \cdots \times Y^{d_1+d_2}$ where $Y^1, \ldots, Y^{d_1}$ are convex subsets of VS and $Y^{d_1+1}, \ldots, Y^{d_1+d_2}$ are nonempty subsets of VK_4 (cf. [23, Proposition 5.11]). Since S has clique number three, it follows that

(8.2) **Theorem.** *Every descendent of* $\mathrm{Doob}(d_1, d_2)$ *is of the form* $Y = Y^1 \times Y^2 \times \cdots \times Y^{d_1+d_2}$, *where either* $Y^i = V\Gamma^i$ *or* $|Y^i| = 1$, *for each* i $(1 \leqslant i \leqslant d_1 + d_2)$.

Halved cubes

Let $\Gamma = \frac{1}{2}H(n, 2)$ $(n \in \{2d, 2d + 1\})$ be a halved cube [3, §9.2D] with vertex set $X = X_n^\varepsilon$, where $\varepsilon \in \mathbb{F}_2$ and X_n^0 (resp. X_n^1) is the set of even- (resp. odd-)weight vectors of $\mathbb{F}_2^n$. Γ has classical parameters $(d, 1, 2, m)$.

For every $x \in \mathbb{F}_2^n$ let $\mathrm{wt}(x)$ be its (Hamming) weight. First suppose $w > 1$. Fix $x, y \in Y$ with $\partial(x, y) = w$. For simplicity of notation, we assume $x = (x_1, t)$, $y = (y_1, t)$, where $x_1, y_1 \in X_{2w}^{\varepsilon_1}$, $t \in X_{n-2w}^{\varepsilon_2}$ $(\varepsilon_1, \varepsilon_2 \in \mathbb{F}_2$, $\varepsilon_1 + \varepsilon_2 = \varepsilon)$ and $\mathrm{wt}(x_1 - y_1) = 2w$. Note that $X_{2w}^{\varepsilon_1} \times \{t\} \subseteq Y$ by convexity. Let $z = (z_1, z_2) \in Y \backslash (X_{2w}^{\varepsilon_1} \times \{t\})$. Then it follows that $\mathrm{wt}(z_2 - t) = 1$. Moreover, for every $u = (u_1, u_2) \in Y \backslash (X_{2w}^{\varepsilon_1} \times \{t\})$ we have $u_2 = z_2$, for otherwise by convexity and since $w > 1$ there would be a vector $v = (v_1, v_2) \in Y$ such that $\mathrm{wt}(v_2 - t) = 2$. It follows that Γ_Y has valency at most $\binom{2w+1}{2} = w(2w + 1)$; but this is smaller than the expected valency wm except when $n = 2d$ and $w = d - 1$, in which case we must have $Y = (X_{2w}^{\varepsilon_1} \times \{t\}) \cup (X_{2w}^{1+\varepsilon_1} \times \{z_2\})$.

Next suppose $w = 1$. Then $|Y| = m + 1$. On the other hand, $\frac{1}{2}H(n, 2)$ has maximal clique sizes 4 and n [16, Theorem 14], from which it follows that $n = 2d$.

(8.3) **Theorem.** *Let* Y *be a nontrivial descendent of* $\frac{1}{2}H(n, 2)$. *Then* $n = 2d$ *and one of the following holds: (i)* $w = 1$ *and* $Y = \{x \in X : \mathrm{wt}(x - z) = 1\}$ *for some* $z \in X_n^{1+\varepsilon} = \mathbb{F}_2^n \backslash X$; *(ii)* $w = d - 1$ *and there are* $a \in \mathbb{F}_2$ *and* $i \in \{1, 2, \ldots, n\}$ *such that* $Y = \{x = (\xi_1, \xi_2, \ldots, \xi_n) \in X : \xi_i = a\}$.

Hemmeter graphs

Let q be an *odd* prime power and Σ the dual polar graph on $[C_{d-1}(q)]$. The Hemmeter graph $\Gamma = \mathrm{Hem}_d(q)$ [3, §9.4C] is the extended bipartite double of Σ, so that Γ has vertex set $X = X^+ \cup X^-$, where $X^\pm = \{x^\pm : x \in V\Sigma\}$ are two copies of $V\Sigma$. Γ has classical parameters $(d, q, 0, 1)$, which coincide with those of the dual polar graph on $[D_d(q)]$.

If $w = 1$ then Y is an edge. Suppose $w > 1$. Let $\mathring{Y} = \{x \in V\Sigma : x^+ \in Y \text{ or } x^- \in Y\}$. Then $\mathring{Y}$ is a convex subset of $V\Sigma$ with width $\mathring{w} \in \{w, w-1\}$. By [23, Proposition 5.19] and [16, Lemma 10], there is an isotropic subspace u of $[C_{d-1}(q)]$ with $\dim u = d - 1 - \mathring{w}$ such that $\mathring{Y} = I(u) := \{x \in V\Sigma : u \subseteq x\}$ if $\mathring{w} > 1$, and $\mathring{Y} \subseteq I(u)$ if $\mathring{w} = 1$. Note that $|Y| = 2\prod_{i=1}^{w-1}(q^i + 1)$ and $|I(u)| = \prod_{i=1}^{\mathring{w}}(q^i + 1)$ (cf. [3, Lemma 9.4.1]). If $\mathring{w} = w(> 1)$ then $|Y| < |I(u)| = |\mathring{Y}|$, a contradiction. Hence $\mathring{w} = w - 1$. It follows that

(8.4) Theorem. *Let Y be a nontrivial descendent of $\mathrm{Hem}_d(q)$. If $w = 1$ then Y is an edge. If $w > 1$ then there is an isotropic subspace u of $[C_{d-1}(q)]$ with $\dim u = w^*$ such that $Y = \{x^+, x^- : x \in V\Sigma, u \subseteq x\}$.*

Hermitean forms graphs

In this subsection and the next, we realize Γ as "affine subspaces" of dual polar graphs. See [3, §9.5E] for the details.

Let ℓ be a prime power and let Δ be the dual polar graph on $[^2A_{2d-1}(\ell)]$. Fix a vertex z of Δ. The subgraph $\Gamma = \mathrm{Her}(d, \ell)$ of Δ induced on $X := \Delta_d(z)$ is the Hermitean forms graph [3, §9.5C]. Γ has classical parameters $(d, -\ell, -\ell-1, -(-\ell)^d - 1)$.

By [23, Proposition 5.30], every noncomplete convex subset of X is either of the form $I(u) = \{x \in X : u \subseteq x\}$ where u is an isotropic subspace of $[^2A_{2d-1}(\ell)]$ with $\dim u \leqslant d - 2$ and $z \cap u = 0$, or isomorphic to the 4-cycle $K_{2,2}$. The latter case occurs only when $\ell = 2$. Note that $I(u)$ induces $\mathrm{Her}(d - \dim u, \ell)$. By [16, Theorem 21], the maximal cliques of Γ are the $I(u)$ with $\dim u = d - 1$. Comparing the classical parameters it follows that

(8.5) Theorem. $\mathrm{Her}(d, \ell)$ *has no nontrivial descendent.*

Alternating forms graphs

Let ℓ be a prime power and let Δ be the dual polar graph on $[D_n(\ell)]$, where $n \in \{2d, 2d+1\}$. Fix a vertex z of Δ. The subgraph $\Gamma = \mathrm{Alt}(n, \ell)$ of the distance-2 graph of Δ induced on $X := \Delta_n(z)$ is the alternating forms graph [3, §9.5B], i.e., $\Gamma = \Delta_n^2(z)$ with the notation of §5. Γ has classical parameters $(d, \ell^2, \ell^2 - 1, \ell^m - 1)$.

By [23, Proposition 5.26], every noncomplete convex subset of X is of the form $I(u) = \{x \in X : u \subseteq x\}$ where u is an isotropic subspace of $[D_n(\ell)]$ with $\dim u \leqslant n - 4$ and $z \cap u = 0$. Note that $I(u)$ induces $\mathrm{Alt}(n - \dim u, \ell)$. By [16, Lemma 19], Γ has clique number ℓ^{n-1} and the maximum cliques are of the form $C(x, v) = \{x\} \cup \{y \in \Gamma(x) : x \cap y \subseteq v\}$ where $x \in X$ and v is a subspace of x with $\dim v = n - 1$. It follows that

(8.6) Theorem. *Let Y be a nontrivial descendent of* $\mathrm{Alt}(n, \ell)$. *Then $n = 2d$ and Y takes one of the following forms: (i) $w = 1$ and $Y = C(x, v)$; (ii) $w = d - 1$ and $Y = I(u)$ with $\dim u = 1$.*

Quadratic forms graphs

Let ℓ be a prime power and let X be the set of quadratic forms on $V = \mathbb{F}_\ell^{n-1}$ over $\mathbb{F}_\ell$, where $n \in \{2d, 2d+1\}$. For $x \in X$ let $\mathrm{Rad}\, x = x^{-1}(0) \cap \mathrm{Rad}\, B_x$ and $\mathrm{rk}\, x = \dim(V/\mathrm{Rad}\, x)$, where B_x is the symmetric bilinear form associated with x and $\mathrm{Rad}\, B_x$ denotes its radical. Let $\Gamma = \mathrm{Quad}(n-1, \ell)$ be the quadratic forms graph on V [3, §9.6]: $x, y \in X$ are adjacent if $\mathrm{rk}(x - y) = 1$ or 2. Γ has classical parameters $(d, \ell^2, \ell^2 - 1, \ell^m - 1)$. We remark that if ℓ is odd then Γ is isomorphic to the subgraph of the distance 1-or-2 graph of the dual polar graph Σ on $[C_{n-1}(\ell)]$ induced on $\Sigma_{n-1}(z)$ with $z \in V\Sigma$, or equivalently, Γ is isomorphic to $\Delta_n^2(z)$, where $\Delta = \mathrm{Hem}_n(\ell)$ and $z \in V\Delta$; see [9].

By [23, Proposition 5.36], [26, Theorem 1.2], every noncomplete convex subset of X is of the form[10] $I(x, u) = \{y \in X : \mathrm{Rad}(x - y) \supseteq u\}$ where $x \in X$ and u is a subspace of V with $\dim u \leqslant n - 4$. Note that $I(x, u)$ induces $\mathrm{Quad}(n - 1 - \dim u, \ell)$. By [17, Theorems 14, 24, 26], Γ has clique number ℓ^{n-1} and there are two types of maximum cliques: A *type 1* clique is defined to be $C(x) = \{y \in X : \mathrm{rk}(x - y) \leqslant 1\}$ where $x \in X$. If q is odd then a *type 2* clique is defined to be $C(x, u) = \{y \in X : B_y|_{u \times u} = B_x|_{u \times u}\}$ where $x \in X$ and u is a subspace of V with $\dim u = n - 2$. If q is even then it is defined to be $C(x, u, \gamma, a) = \{y \in X : B_y|_{u \times u} = B_x|_{u \times u}, \ (x - y)(\xi) = a((B_x - B_y)(\gamma, \xi))^2 \text{ for all } \xi \in u\}$ where $x \in X$, u a subspace of V with $\dim u = n - 2$, $\gamma \in V \setminus u$ and $a \in \mathbb{F}_\ell$. See also [6, 18]. Cliques of types 1 and 2 are called *grand cliques*. It follows that

(8.7) Theorem. *Let Y be a nontrivial descendent of* $\mathrm{Quad}(n - 1, \ell)$. *Then $n = 2d$ and Y takes one of the following forms: (i) $w = 1$ and Y is a grand clique; (ii) $w = d - 1$ and $Y = I(x, u)$ with $\dim u = 1$.*

Unitary dual polar graphs with second Q-polynomial ordering

Let ℓ be a prime power and let $\Gamma = U(2d, \ell)$ be the dual polar graph on $[{}^2A_{2d-1}(\ell)]$. Besides the ordinary $(d, \ell^2, 0, \ell)$, Γ has another set of classical parameters $(d, -\ell, \alpha, \beta)$, where $\alpha + 1 = \frac{1+\ell^2}{1-\ell}$ and $\beta + 1 = \frac{1-(-\ell)^{d+1}}{1-\ell}$ [3, Corollary 6.2.2]. Here we consider the standard Q-polynomial ordering with respect to the latter classical parameters; it is $\{E_0, E_d, E_1, E_{d-1}, \dots\}$ in terms of the ordinary ordering $\{E_i\}_{i=0}^d$.

By [23, Proposition 5.19], every noncomplete convex subset of X is of the form $I(u) = \{x \in X : u \subseteq x\}$ for some isotropic subspace u of $[{}^2A_{2d-1}(\ell)]$ with $\dim u = w^*$. By [16, Lemma 10], the maximal cliques of Γ are the $I(u)$ with $\dim u = d - 1$. It follows that

(8.8) Theorem. $U(2d, \ell)$ *(with second Q-polynomial ordering) has no nontrivial descendent.*

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[10]If ℓ is odd, then in terms of the above identification $X = \Sigma_{n-1}(z)$, every noncomplete convex subgraph of Γ is rewritten as $I(u) = \{y \in X : u \subseteq y\}$ for some isotropic subspace of $[C_{n-1}(\ell)]$ with $\dim u \leqslant n - 4$ and $z \cap u = 0$.

Half dual polar graphs

Let ℓ be a prime power and let Δ be the bipartite dual polar graph on $[D_n(\ell)]$ with $V\Delta = X^+ \cup X^-$, where $n \in \{2d, 2d+1\}$ and X^+, X^- are bipartite halves of Δ. The path-length distance for Δ is denoted ∂_Δ. Let $\Gamma = D_{n,n}(\ell)$ be a halved graph of Δ with vertex set $X = X^\varepsilon$ where $\varepsilon \in \{+, -\}$, whence $2\partial = \partial_\Delta|_{X \times X}$ [3, §9.4C]. Γ has classical parameters $(d, \ell^2, \alpha, \beta)$, where $\alpha + 1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_\ell$ and $\beta + 1 = \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_\ell$. Pepe, Storme and Vanhove [28, §5] recently classified the descendents of Γ when $n = 2d$ and $w = d - 1$, partly based on the Erdős–Ko–Rado theorem for Grassmann graphs (cf. [31, Theorem 3]). Our approach below uses (8.6) instead.

First suppose $w > 1$. Fix $x \in Y$ and pick any $z \in \Delta_{n-2w}(x)$ such that $\Delta_n(z) \cap Y \neq \emptyset$. By (5.2), (8.6) we find $n = 2d$, $w = d - 1$ and there is an isotropic subspace u of $[D_{2d}(\ell)]$ with $\dim u = 1$ and $z \cap u = 0$ such that $Y_0 := \Delta_{2d}(z) \cap Y = \{y \in \Delta_{2d}(z) : u \subseteq y\}$. Note that $z \in \Gamma(x) \backslash Y$ and $Y_0 = \Gamma_d(z) \cap Y \subseteq (\Gamma_Y)_{d-1}(x)$. For convenience, let Ξ be the dual polar graph on the residual polar space of u, so that $V\Xi = \{y \in V\Delta : u \subseteq y\}$.

(8.9) $x \in V\Xi$.

Proof. Suppose $u \not\subseteq x$. Let $x^\dagger = u + (x \cap u^\perp)$, $z^\dagger = u + (z \cap u^\perp) \in V\Xi$. Then $\partial_\Delta(x, y) = \partial_\Delta(x^\dagger, y) + 1$ for any $y \in V\Xi$, and similarly for $z^\dagger$. Since $Y_0 \subseteq \Xi_{2d-3}(x^\dagger)$ we find $x^\dagger \in \Delta_3(z)$, i.e., $x^\dagger \in \Xi_2(z^\dagger)$. But then $\Gamma_d(x) \cap Y_0 = \Xi_{2d-1}(x^\dagger) \cap \Xi_{2d-1}(z^\dagger) \neq \emptyset$, a contradiction. $\square$

(8.10) $(\Gamma_Y)_{d-1}(x) \subseteq V\Xi$.

Proof. Let $z_1 \in \Gamma(x) \backslash Y$. Since Y is completely regular in Γ, $Y_1 := \Gamma_d(z_1) \cap Y \neq \emptyset$. Let u_1 be the isotropic subspace of $[D_{2d}(\ell)]$ with $\dim u_1 = 1$, $z_1 \cap u_1 = 0$ and $Y_1 = \{y \in \Gamma_d(z_1) : u_1 \subseteq y\}$. Note that $|Y_0| = |Y_1| = \ell^{(d-1)(2d-1)}$ and $|(\Gamma_Y)_{d-1}(x)| = \prod_{i=1}^{d-1}(b_{i-1}(\Gamma_Y)/c_i(\Gamma_Y)) = \ell^{(d-1)(2d-3)} \begin{bmatrix} 2d-1 \\ 1 \end{bmatrix}_\ell$. Suppose $u \neq u_1$. Then $u + u_1 \subseteq x$ by (8.9), so that by looking at the residual polar space of $u + u_1$ we find $|Y_0 \cap Y_1| \leqslant |\mathrm{Alt}(2d-2, \ell)| = \ell^{(d-1)(2d-3)}$. If $\ell > 2$ then $|Y_0 \cap Y_1| \geqslant |Y_0| + |Y_1| - |(\Gamma_Y)_{d-1}(x)| > \ell^{(d-1)(2d-3)}$, a contradiction. Hence $\ell = 2$, $Y_0 \cup Y_1 = (\Gamma_Y)_{d-1}(x)$ and $Y_0 \cap Y_1 = \{y \in (\Gamma_Y)_{d-1}(x) : u + u_1 \subseteq y\}$, from which it follows that $Y_0 = \{y \in (\Gamma_Y)_{d-1}(x) : u \subseteq y\}$, $Y_1 = \{y \in (\Gamma_Y)_{d-1}(x) : u_1 \subseteq y\}$. Pick $v \in Y_0 \cap Y_1$. Since $a_{d-1}(\Gamma) > a_{d-1}(\Gamma_Y)$ there is $z_2 \in \Gamma(x) \cap \Gamma_{d-1}(v) \backslash Y$. Let u_2 be the isotropic subspace of $[D_{2d}(2)]$ with $\dim u_2 = 1$, $z_2 \cap u_2 = 0$ and $Y_2 := \Gamma_d(z_2) \cap Y = \{y \in \Gamma_d(z_2) : u_2 \subseteq y\}$. Since $v \notin Y_2$ we find $Y_2 \neq Y_0, Y_1$, so that $u_2 \neq u, u_1$, and thus $|Y_0 \cap Y_2|, |Y_1 \cap Y_2| \leqslant 2^{(d-1)(2d-3)}$ by the above argument. But $\max\{|Y_0 \cap Y_2|, |Y_1 \cap Y_2|\} \geqslant \frac{1}{2}|Y_2| > 2^{(d-1)(2d-3)}$, a contradiction. Hence $u = u_1$. Since $(\Gamma_Y)_{d-1}(x) = \bigcup_{z_1 \in \Gamma(x) \backslash Y} \Gamma_d(z_1) \cap Y$, the proof is complete. $\square$

It follows from (8.9), (8.10) that $Y \subseteq V\Xi \cap X$, so that Γ_Y is a subgraph of a halved graph of Ξ. Since $\iota(\Gamma_Y) = \iota(D_{2d-1,2d-1}(\ell))$ we conclude $Y = V\Xi \cap X$.

Next suppose $w = 1$. Then $|Y| = \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_\ell$. By [16, Lemma 12] or [5, Theorem 3.5], Γ has clique number $\begin{bmatrix} 2d \\ 1 \end{bmatrix}_\ell$ and the maximum cliques are of the form $\Delta(z)$ where $z \in V\Delta \backslash X$.

(8.11) **Theorem.** *Let Y be a nontrivial descendent of $D_{n,n}(\ell)$. Then $n = 2d$ and one of the following holds: (i) $w = 1$ and $Y = \Delta(z)$ for some $z \in V\Delta \backslash X$; (ii) $w = d - 1$ and $Y = \{x \in X : u \subseteq x\}$ for some isotropic subspace u of $[D_{2d}(\ell)]$ with $\dim u = 1$.*

Ustimenko graphs

Let ℓ be an *odd* prime power and let Σ be the dual polar graph on $[C_{n-1}(\ell)]$ with vertex set $X = V\Sigma$, where $n \in \{2d, 2d+1\}$. The path-length distance for Σ is denoted ∂_Σ. The Ustimenko graph $\Gamma = \mathrm{Ust}_{n-1}(\ell)$ is the distance 1-or-2 graph of Σ, whence $\partial = \lceil \frac{1}{2}\partial_\Sigma \rceil$ [3, §9.4C]. Γ has classical parameters $(d, \ell^2, \alpha, \beta)$, where $\alpha + 1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_\ell$ and $\beta + 1 = \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_\ell$. Note that Γ may also be viewed as a halved graph of $\Delta = \mathrm{Hem}_n(\ell)$. Pepe et al. [28, §7] classified the descendents of Γ when $n = 2d$ and $w = d - 1$ by a different approach from the following.

First suppose $w > 1$. Fix $x \in Y$ and pick any $z \in \Delta_{n-2w}(x)$ such that $\Delta_n(z) \cap Y \neq \emptyset$. By (5.2), (8.7) we find $n = 2d$, $w = d-1$ and there is an isotropic subspace u of $[C_{2d-1}(\ell)]$ with $\dim u = 1$ and $z \cap u = 0$ such that $Y_0 := \Delta_{2d}(z) \cap Y = \{y \in \Delta_{2d}(z) : u \subseteq y\}$.[11] Note that $z \in \Gamma(x)\backslash Y$ and $Y_0 = \Gamma_d(z) \cap Y = \Sigma_{2d-1}(z) \cap Y \subseteq (\Gamma_Y)_{d-1}(x)$. Let Ξ be the dual polar graph on the residual polar space of u, so that $V\Xi = \{y \in V\Sigma : u \subseteq y\}$.

(8.12) $x \in V\Xi$.

Proof. Suppose $u \not\subseteq x$. Let $x^\dagger = u + (x \cap u^\perp)$, $z^\dagger = u + (z \cap u^\perp) \in V\Xi$. Then $\Gamma_d(x) \cap Y_0 = \Xi_{2d-2}(x^\dagger) \cap \Xi_{2d-2}(z^\dagger) \neq \emptyset$, a contradiction. $\qquad\square$

(8.13) $(\Gamma_Y)_{d-1}(x) \subseteq V\Xi$.

Proof. Let $z_1 \in \Gamma(x)\backslash Y$. Since Y is completely regular in Γ, $Y_1 := \Gamma_d(z_1) \cap Y \neq \emptyset$. Let u_1 be the isotropic subspace of $[C_{2d-1}(\ell)]$ with $\dim u_1 = 1$, $z_1 \cap u_1 = 0$ and $Y_1 = \{y \in \Gamma_d(z_1) : u_1 \subseteq y\}$. Note that $|Y_0| = |Y_1| = \ell^{(d-1)(2d-1)}$ and $|(\Gamma_Y)_{d-1}(x)| = \ell^{(d-1)(2d-3)} \begin{bmatrix} 2d-1 \\ 1 \end{bmatrix}_\ell$. Suppose $u \neq u_1$. Then $u + u_1 \subseteq x$ by (8.12), so that $|Y_0 \cap Y_1| \leqslant |\mathrm{Quad}(2d-3, \ell)| = \ell^{(d-1)(2d-3)}$. But $|Y_0 \cap Y_1| \geqslant |Y_0| + |Y_1| - |(\Gamma_Y)_{d-1}(x)| > \ell^{(d-1)(2d-3)}$, a contradiction. Hence $u = u_1$. Since $(\Gamma_Y)_{d-1}(x) = \bigcup_{z_1 \in \Gamma(x)\backslash Y} \Gamma_d(z_1) \cap Y$, the proof is complete. $\qquad\square$

(8.14) $Y \subseteq V\Xi$.

Proof. Suppose there is $y \in Y\backslash V\Xi$. Pick $v \in (\Gamma_Y)_{d-1}(x)$ such that $\partial(x,y) + \partial(y,v) = d-1$. Set $y^\dagger = u + (y \cap u^\perp) \in V\Xi$. Then by (8.12), (8.13) it follows that $\partial_\Sigma(x,v) \leqslant \partial_\Sigma(x, y^\dagger) + \partial_\Sigma(y^\dagger, v) = \partial_\Sigma(x, y) + \partial_\Sigma(y, v) - 2 \leqslant 2d - 4$, a contradiction. $\qquad\square$

By (8.14) and since $\iota(\Gamma_Y) = \iota(\mathrm{Ust}_{2d-2}(\ell))$ we conclude $Y = V\Xi$.

Next suppose $w = 1$. Then $|Y| = \begin{bmatrix} m+1 \\ 1 \end{bmatrix}_\ell$. By [5, Theorem 3.7], Γ has clique number $\begin{bmatrix} 2d \\ 1 \end{bmatrix}_\ell$ and the maximum cliques are of the form $\Sigma(x) \cup \{x\}$ where $x \in X$.

(8.15) **Theorem.** *Let Y be a nontrivial descendent of $\mathrm{Ust}_{n-1}(\ell)$. Then $n = 2d$ and one of the following holds: (i) $w = 1$ and $Y = \Sigma(x) \cup \{x\}$ for some $x \in X$; (ii) $w = d-1$ and $Y = \{x \in X : u \subseteq x\}$ for some isotropic subspace u of $[C_{2d-1}(\ell)]$ with $\dim u = 1$.*

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[11]See footnote 10.

Twisted Grassmann graphs

Let q be a prime power and fix a hyperplane H of $\mathbb{F}_q^{2d+1}$. Let X_1 be the set of $(d+1)$-dimensional subspaces of $\mathbb{F}_q^{2d+1}$ not contained in H, and X_2 the set of $(d-1)$-dimensional subspaces of H. The twisted Grassmann graph $\Gamma = \tilde{J}_q(2d+1, d)$ [12, 14, 1, 27] has vertex set $X = X_1 \cup X_2$, and $x, y \in X$ are adjacent if $\dim x + \dim y - 2\dim x \cap y = 2$. Γ has classical parameters (d, q, q, β), where $\beta + 1 = \begin{bmatrix} d+2 \\ 1 \end{bmatrix}_q$. Note that X_2 is a descendent of Γ and induces $J_q(2d, d-1)$. We observe

(8.16) $2\partial(x, y) = \dim x + \dim y - 2\dim x \cap y \quad (x, y \in X)$.

(8.17) *For $x, y, z \in X$, we have $\partial(x, z) + \partial(z, y) = \partial(x, y)$ if and only if $x \cap y \subseteq z = (x \cap z) + (y \cap z)$.*

Proof. Observe $\dim x \cap z + \dim y \cap z \leqslant \dim z + \dim x \cap y \cap z \leqslant \dim z + \dim x \cap y$, and equality holds if and only if $x \cap y \subseteq z = (x \cap z) + (y \cap z)$. Hence the result follows from (8.16). $\qquad\square$

(8.18) *Let Z be a nonempty subset of X_1 such that $\{z \in X_1 : x \cap y \subseteq z = (x \cap z) + (y \cap z)\} \subseteq Z$ for all $x, y \in Z$. Then at least one of the following holds: (i) there is a subspace u of $\mathbb{F}_q^{2d+1}$ with $\dim u = d - w(Z) + 1$ such that $u \subseteq z$ for all $z \in Z$; (ii) there is a subspace v of $\mathbb{F}_q^{2d+1}$ with $\dim v = d + w(Z) + 1$ and not contained in H such that $z \subseteq v$ for all $z \in Z$.*

Proof. Fix $x, y \in Z$ with $\partial(x, y) = w(Z)$ and recall $\dim x \cap y = d - w(Z) + 1$ by (8.16). Let $z \in X_1$. We claim that if $x \cap y \not\subseteq z \not\subseteq x + y$ then $z \notin Z$. Suppose $z \in Z$. Let $\gamma \in (x \cap y) \backslash z$ and $\sigma \in z \backslash ((x + y) \cup H)$. Let E be a complementary subspace of $\langle \gamma \rangle$ in x such that $x \cap z \subseteq E$. Set $z^\dagger = E + \langle \sigma \rangle \in X_1$. Since $x \cap z \subseteq z^\dagger = (x \cap z^\dagger) + (z \cap z^\dagger)$ we find $z^\dagger \in Z$. But then $\gamma \notin z^\dagger \cap y (= E \cap y) \subseteq x \cap y$ implies $\dim z^\dagger \cap y < d - w(Z) + 1$ and thus $\partial(z^\dagger, y) > w(Z)$ by (8.16), a contradiction. Hence the claim follows. It follows that every $z \in Z$ satisfies $x \cap y \subseteq z$ or $z \subseteq x + y$ (or both). Next we claim that there is no pair (z_1, z_2) of elements of Z such that $x \cap y \not\subseteq z_1 \subseteq x + y$ and $x \cap y \subseteq z_2 \not\subseteq x + y$. Suppose such a pair (z_1, z_2) exists. Let $\zeta \in z_2 \backslash ((x + y) \cup H)$. Let E be a hyperplane of z_1 such that $z_1 \cap z_2 \subseteq E$. Set $z^\ddagger = E + \langle \zeta \rangle \in X_1$. Since $z_1 \cap z_2 \subseteq z^\ddagger = (z_1 \cap z^\ddagger) + (z_2 \cap z^\ddagger)$ we find $z^\ddagger \in Z$. But this is absurd since $x \cap y \not\subseteq z^\ddagger \not\subseteq x + y$. Hence the claim follows. Setting $u = x \cap y$ and $v = x + y$, we find that at least one of (i), (ii) holds. $\qquad\square$

(8.19) *Let Z be a nonempty convex subset of X. If there are vertices $x \in Z \cap X_1$, $y \in Z \cap X_2$ with $\partial(x, y) = w(Z)$, then for each $i = 1, 2$ at least one of the following holds: (i) there is a subspace u of H with $\dim u = d - w(Z)$ such that $u \subseteq z$ for all $z \in Z \cap X_i$; (ii) there is a subspace v of $\mathbb{F}_q^{2d+1}$ with $\dim v = d + w(Z)$ and not contained in H such that $z \subseteq v$ for all $z \in Z \cap X_i$.*

Proof. Similar to the proof of (8.18), by virtue of (8.17). $\qquad\square$

(8.20) **Theorem.** *Let Y be a nontrivial descendent of $\tilde{J}_q(2d+1, d)$. Then $Y = \{x \in X_2 : u \subseteq x\}$ for some subspace u of H with $\dim u = d - w - 1$.*

Proof. Note that $\iota(\Gamma_Y) = \iota(J_q(d + w + 1, w))$, whence $|Y| = \begin{bmatrix} d+w+1 \\ w \end{bmatrix}_q$. First suppose $Y \subseteq X_1$. Then in view of (8.17), if (8.18)(i) holds then $|Y| \leqslant \begin{bmatrix} d+w \\ w \end{bmatrix}_q < \begin{bmatrix} d+w+1 \\ w \end{bmatrix}_q$, and if (8.18)(ii) holds then $|Y| \leqslant \begin{bmatrix} d+w+1 \\ d+1 \end{bmatrix}_q - \begin{bmatrix} d+w \\ d+1 \end{bmatrix}_q < \begin{bmatrix} d+w+1 \\ w \end{bmatrix}_q$, a contradiction. Hence $Y \nsubseteq X_1$. Next suppose $Y \cap X_1, Y \cap X_2$ are nonempty. Let $z \in Y \cap X_1$, $y \in Y \cap X_2$, and pick any $x \in Y$ with $\partial(x, y) = w$, $\partial(x, z) = w - \partial(z, y)$. Since X_2 is convex we find $x \in Y \cap X_1$. Let $w_1 := w(Y \cap X_1) \leqslant w$. By (8.17), $Y \cap X_1$ satisfies the assumptions of (8.18). If $Y \cap X_1$ satisfies (8.18)(i) then $|Y \cap X_1| \leqslant \begin{bmatrix} d+w_1 \\ w_1 \end{bmatrix}_q \leqslant \begin{bmatrix} d+w \\ w \end{bmatrix}_q$. If $Y \cap X_1$ satisfies (8.19)(ii) then $|Y \cap X_1| \leqslant \begin{bmatrix} d+w \\ d+1 \end{bmatrix}_q \leqslant \begin{bmatrix} d+w \\ w \end{bmatrix}_q$. If $Y \cap X_1$ satisfies both (8.18)(ii) and (8.19)(i) then $|Y \cap X_1| \leqslant \begin{bmatrix} w+w_1+1 \\ w+1 \end{bmatrix}_q \leqslant \begin{bmatrix} d+w \\ w \end{bmatrix}_q$ since $w < d$. Hence we always have $|Y \cap X_1| \leqslant \begin{bmatrix} d+w \\ w \end{bmatrix}_q$. On the other hand, if $Y \cap X_2$ satisfies (8.19)(i) then $|Y \cap X_2| \leqslant \begin{bmatrix} d+w \\ w-1 \end{bmatrix}_q$, and if $Y \cap X_2$ satisfies (8.19)(ii) then $|Y \cap X_2| \leqslant \begin{bmatrix} d+w-1 \\ w \end{bmatrix}_q$. Since $q^{d+1} \begin{bmatrix} d+w \\ w-1 \end{bmatrix}_q > \begin{bmatrix} d+w-1 \\ w \end{bmatrix}_q$, it follows that $|Y| = |Y \cap X_1| + |Y \cap X_2| < \begin{bmatrix} d+w \\ w \end{bmatrix}_q + q^{d+1} \begin{bmatrix} d+w \\ w-1 \end{bmatrix}_q = \begin{bmatrix} d+w+1 \\ w \end{bmatrix}_q$, a contradiction. Hence $Y \subseteq X_2$, and by (4.4), (7.3), Y must be of the form as in (8.20). $\square$

Summary and remarks

Let $\mathscr{P}$ be the set of descendents of Γ and $w(\mathscr{P}) = \{w(Y) : Y \in \mathscr{P}\}$. In the table below, we list $w(\mathscr{P})$ for each of the 15 families of graphs with classical parameters.

Γ	#	$w(\mathscr{P}) \backslash \{0, d\}$
$J(\nu, d)$ $(\nu \geqslant 2d)$	(7.1)	$\{1, 2, \ldots, d-1\}$
$H(d, \ell)$ $(\ell \geqslant 2)$	(7.2)	$\{1, 2, \ldots, d-1\}$
$J_q(\nu, d)$ $(\nu \geqslant 2d)$	(7.3)	$\{1, 2, \ldots, d-1\}$
$\mathrm{Bil}_q(d, e)$ $(e \geqslant d)$	(7.4)	$\{1, 2, \ldots, d-1\}$
Dual polar graph	(7.5)	$\{1, 2, \ldots, d-1\}$
$\mathrm{Doob}(d_1, d_2)$ $(d = 2d_1 + d_2)$	(8.2)	$\{1, 2, \ldots, d-1\}$
$\mathrm{Hem}_d(q)$ $(q : \mathrm{odd})$	(8.4)	$\{1, 2, \ldots, d-1\}$
$\tilde{J}_q(2d+1, d)$	(8.20)	$\{1, 2, \ldots, d-1\}$
$\frac{1}{2}H(n, 2)$	(8.3)	$\{1, d-1\}$ $(n = 2d)$, $\emptyset$ $(n = 2d+1)$
$\mathrm{Her}(d, \ell)$	(8.5)	$\emptyset$
$\mathrm{Alt}(n, \ell)$	(8.6)	$\{1, d-1\}$ $(n = 2d)$, $\emptyset$ $(n = 2d+1)$
$\mathrm{Quad}(n-1, \ell)$	(8.7)	$\{1, d-1\}$ $(n = 2d)$, $\emptyset$ $(n = 2d+1)$
$U(2d, \ell)$	(8.8)	$\emptyset$
$D_{n,n}(\ell)$	(8.11)	$\{1, d-1\}$ $(n = 2d)$, $\emptyset$ $(n = 2d+1)$
$\mathrm{Ust}_{n-1}(\ell)$ $(\ell : \mathrm{odd})$	(8.15)	$\{1, d-1\}$ $(n = 2d)$, $\emptyset$ $(n = 2d+1)$

Note that the 5 families of the first group in the table are associated with regular semilattices, and that the 3 families of the second group have the classical parameters

of graphs belonging to the first group. It should be remarked however that, for every i $(0 < i < d)$, the graphs in the second group possess pairs of vertices at distance i which are not contained in any descendent with width i, with the exception of $(\Gamma, i) = (\mathrm{Hem}_d(q), 1)$.

In (4.5) we posed the problem of determining the filter $V_{[\Gamma]}$ of the poset $\mathscr{L}$ generated by the isomorphism class $[\Gamma]$. We end this section with describing $V_{[\Gamma]}$ for some examples where Γ has classical parameters (d, q, α, β) with $q = 1$ or $q < -1$. The distance-regular graphs with classical parameters $(d, 1, \alpha, \beta)$ $(d \geqslant 3)$ are known: the Johnson graphs, Hamming graphs, Doob graphs, halved cubes and the Gosset graph; see e.g., [3, Theorem 6.1.1]. By virtue of (6.4), (7.1), (7.2), (8.2) and (8.3), we have the following table:

Γ	$V_{[\Gamma]}$
$J(\nu, d)$ $(\nu \geqslant 2d)$	$\{[J(\nu + i, d + i)] : 0 \leqslant i \leqslant \nu - 2d\}$
$H(d, \ell)$ $(\ell \neq 4)$	$\{[H(e, \ell)] : e \geqslant d\}$
$H(d, 4)$	$\{[H(e, 4)] : e \geqslant d\} \cup \{[\mathrm{Doob}(d_1, d_2)] : d_2 \geqslant d\}$
$\mathrm{Doob}(d_1, d_2)$	$\{[\mathrm{Doob}(e_1, e_2)] : e_1 \geqslant d_1, \ e_2 \geqslant d_1\}$
$\frac{1}{2}H(2d, 2)$	$\{[\frac{1}{2}H(2d, 2)]\}$
$\frac{1}{2}H(2d + 1, 2)$	$\{[\frac{1}{2}H(2d + 1, 2)], [\frac{1}{2}H(2d + 2, 2)]\}$

Weng [43] showed that if a distance-regular graph Δ having classical parameters (d, q, α, β) satisfies $d \geqslant 4$, $q < -1$, $a_1 \neq 0$, $c_2 > 1$ then either (i) $\Delta = \mathrm{Her}(d, \ell)$ $(q = -\ell)$; (ii) $\Delta = U(2d, \ell)$ $(q = -\ell)$; or (iii) $\alpha = (q - 1)/2$, $\beta = -(1 + q^d)/2$ and $-q$ is a power of an odd prime. Hence, by (6.4), (8.5), (8.8) it follows that $[\mathrm{Her}(d, \ell)]$, $[U(2d, \ell)]$ are maximal elements in $\mathscr{L}$ for all $d \geqslant 3$, so that the filter generated by any of them is a singleton. It would be interesting if the poset $\mathscr{L}$ is of some use, e.g., in the classification of distance-regular graphs Γ with $\iota(\Gamma) = \iota(J_q(2d + 1, d)) = \iota(\tilde{J}_q(2d + 1, d))$.

A   The list of parameter arrays

We display the parameter arrays of Leonard systems. The data in (A.1) is taken from [40], but the presentation is changed so as to be consistent with the notation in [2, 34, 35, 36].

(A.1) **Theorem**([40, Theorem 5.16]). *Let Φ be the Leonard system from (3.1) and let $p(\Phi) = \left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right)$ be as in (3.7). Then at least one of the following cases I, IA, II, IIA, IIB, IIC, III hold:*

(I) $p(\Phi) = p(\mathrm{I}; q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$ *where* $r_1 r_2 = ss^* q^{d+1}$,

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i},$$
$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^* q^{i+1})q^{-i}$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1 q^i)(1-r_2 q^i),$$

$$\phi_i = \begin{cases} hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^* & \text{if } s^* \neq 0, \\ hh^* q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s - r_1 q^{i-d-1} - r_2 q^{i-d-1}) & \text{if } s^* = 0 \end{cases}$$

for $1 \leqslant i \leqslant d$.

(IA) $p(\Phi) = p(\text{IA}; q, h^*, r, s, \theta_0, \theta_0^*, d)$ *where*

$$\theta_i = \theta_0 - sq(1-q^i),$$
$$\theta_i^* = \theta_0^* + h^*(1-q^i)q^{-i}$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = -rh^* q^{1-i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = h^* q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s - rq^{i-d-1})$$

for $1 \leqslant i \leqslant d$.

(II) $p(\Phi) = p(\text{II}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$ *where* $r_1 + r_2 = s + s^* + d + 1$,

$$\theta_i = \theta_0 + hi(i+1+s),$$
$$\theta_i^* = \theta_0^* + h^* i(i+1+s^*)$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = hh^* i(i-d-1)(i+r_1)(i+r_2),$$
$$\phi_i = hh^* i(i-d-1)(i+s^*-r_1)(i+s^*-r_2)$$

for $1 \leqslant i \leqslant d$.

(IIA) $p(\Phi) = p(\text{IIA}; h, r, s, s^*, \theta_0, \theta_0^*, d)$ *where*

$$\theta_i = \theta_0 + hi(i+1+s),$$
$$\theta_i^* = \theta_0^* + s^* i$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = hs^* i(i-d-1)(i+r),$$
$$\phi_i = hs^* i(i-d-1)(i+r-s-d-1)$$

for $1 \leqslant i \leqslant d$.

(IIB) $p(\Phi) = p(\text{IIB}; h^*, r, s, s^*, \theta_0, \theta_0^*, d)$ *where*

$$\theta_i = \theta_0 + si,$$
$$\theta_i^* = \theta_0^* + h^*i(i + 1 + s^*)$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = h^*si(i - d - 1)(i + r),$$
$$\phi_i = -h^*si(i - d - 1)(i + s^* - r)$$

for $1 \leqslant i \leqslant d$.

(IIC) $p(\Phi) = p(\text{IIC}; r, s, s^*, \theta_0, \theta_0^*, d)$ *where*

$$\theta_i = \theta_0 + si,$$
$$\theta_i^* = \theta_0^* + s^*i$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = ri(i - d - 1),$$
$$\phi_i = (r - ss^*)i(i - d - 1)$$

for $1 \leqslant i \leqslant d$.

(III) $p(\Phi) = p(\text{III}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$ *where* $r_1 + r_2 = -s - s^* + d + 1$,

$$\theta_i = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i),$$
$$\theta_i^* = \theta_0^* + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i)$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = \begin{cases} -4hh^*i(i + r_1) & \textit{if } i \textit{ even}, d \textit{ even}, \\ -4hh^*(i - d - 1)(i + r_2) & \textit{if } i \textit{ odd}, d \textit{ even}, \\ -4hh^*i(i - d - 1) & \textit{if } i \textit{ even}, d \textit{ odd}, \\ -4hh^*(i + r_1)(i + r_2) & \textit{if } i \textit{ odd}, d \textit{ odd}, \end{cases}$$

$$\phi_i = \begin{cases} 4hh^*i(i - s^* - r_1) & \textit{if } i \textit{ even}, d \textit{ even}, \\ 4hh^*(i - d - 1)(i - s^* - r_2) & \textit{if } i \textit{ odd}, d \textit{ even}, \\ -4hh^*i(i - d - 1) & \textit{if } i \textit{ even}, d \textit{ odd}, \\ -4hh^*(i - s^* - r_1)(i - s^* - r_2) & \textit{if } i \textit{ odd}, d \textit{ odd} \end{cases}$$

for $1 \leqslant i \leqslant d$.

We call an array $p(\cdot; \cdots)$ in (A.1) *feasible* if it is the parameter array of an actual Leonard system. See [40, Examples 5.3–5.15] for the explicit feasibility conditions of these arrays. The following is essentially[12] given (without proof) in [34, Lemma 2.4] and can be read off the proof of [40, Theorem 5.16]:

(A.2) **Proposition.** *Referring to* (A.1), *(i)–(iv) hold below:*

(i) *The following arrays are equal:*

$$p(\mathrm{I}; q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d),$$
$$p(\mathrm{I}; q, h, h^*, r_2, r_1, s, s^*, \theta_0, \theta_0^*, d).$$

(ii) *If $ss^* \neq 0$ then the following arrays are equal:*

$$p(\mathrm{I}; q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d),$$
$$p(\mathrm{I}; q^{-1}, hsq, h^*s^*q, r_1^{-1}, r_2^{-1}, s^{-1}, s^{*-1}, \theta_0, \theta_0^*, d).$$

(iii) *The following arrays are equal:*

$$p(\mathrm{II}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d),$$
$$p(\mathrm{II}; h, h^*, r_2, r_1, s, s^*, \theta_0, \theta_0^*, d).$$

(iv) *If d is odd then the following arrays are equal:*

$$p(\mathrm{III}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d),$$
$$p(\mathrm{III}; h, h^*, r_2, r_1, s, s^*, \theta_0, \theta_0^*, d).$$

Moreover, there is no equality among the feasible arrays $p(\cdot; \cdots)$ with $d \geqslant 3$, other than (i)–(iv) above (and combinations of (i), (ii)).

Finally, we describe the parameter arrays of the ρ-descendents of Φ:

(A.3) **Theorem**([33, Theorem 6.9]). *Let Φ be the Leonard system* (3.1) *and let $p(\Phi)$ be given as in* (A.1). *Let Φ' be a Leonard system with diameter $d' \leqslant d$. Given an integer ρ $(0 \leqslant \rho \leqslant d - d')$, Φ' is a ρ-descendent of Φ precisely when $p(\Phi')$ takes the following form:*

Case I:

$$p(\Phi') = p(\mathrm{I}; q, h', h^{*\prime}, r_1 q^\rho, r_2 q^\rho, sq^{d-d'}, s^* q^{2\rho}, \theta_0', \theta_0^{*\prime}, d').$$

Case IA:

$$p(\Phi') = p(\mathrm{IA}; q, h^{*\prime}, r', s', \theta_0', \theta_0^{*\prime}, d') \quad where \quad s'/r' = q^{d-d'-\rho}s/r.$$

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[12]The scalars φ_i, ϕ_i first appeared in [38]. Hence we reproduce [34, Lemma 2.4] in terms of $p(\Phi)$ for accuracy purposes.

Case II:

$$p(\Phi') = p(\text{II}; h', h^{*\prime}, r_1 + \rho, r_2 + \rho, s + d - d', s^* + 2\rho, \theta_0', \theta_0^{*\prime}, d').$$

Case IIA:

$$p(\Phi') = p(\text{IIA}; h', r + \rho, s + d - d', s^{*\prime}, \theta_0', \theta_0^{*\prime}, d').$$

Case IIB:

$$p(\Phi') = p(\text{IIB}; h^{*\prime}, r + \rho, s', s^* + 2\rho, \theta_0', \theta_0^{*\prime}, d').$$

Case IIC:

$$p(\Phi') = p(\text{IIC}; r', s', s^{*\prime}, \theta_0', \theta_0^{*\prime}, d') \quad where \quad s's^{*\prime}/r' = ss^*/r.$$

Case III, d even, d′ even, ρ even; or Case III, d odd, d′ odd, ρ even:

$$p(\Phi') = p(\text{III}; h', h^{*\prime}, r_1 + \rho, r_2 + \rho, s - d + d', s^* - 2\rho, \theta_0', \theta_0^{*\prime}, d').$$

Case III, d even, d′ even, ρ odd:

$$p(\Phi') = p(\text{III}; h', h^{*\prime}, r_2 + \rho, r_1 + \rho, s - d + d', s^* - 2\rho, \theta_0', \theta_0^{*\prime}, d').$$

Case III, d even, d′ = 1:

$$p(\Phi') = p(\text{IIC}; r', s', s^{*\prime}, \theta_0', \theta_0^{*\prime}, 1) \quad where \quad s's^{*\prime}/r' = 1 - \phi_{\rho+1}/\varphi_{\rho+1}.$$

Case III, d even, d′ odd ⩾ 3; or Case III, d odd, either d′ even or ρ odd: Does not occur.

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