Colorful Paths in Vertex Coloring of Graphs

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Abstract

A colorful path in a graph G is a path with $\chi(G)$ vertices whose colors are different. A v-colorful path is such a path, starting from v. Let $G \neq C_7$ be a connected graph with maximum degree $\Delta(G)$. We show that there exists a $(\Delta(G)+1)$-coloring of G with a v-colorful path for every $v \in V(G)$. We also prove that this result is true if one replaces $(\Delta(G)+1)$ colors with $2\chi(G)$ colors. If $\chi(G) = \omega(G)$, then the result still holds for $\chi(G)$ colors. For every graph G, we show that there exists a $\chi(G)$-coloring of G with a rainbow path of length $\lfloor \chi(G)/2 \rfloor$ starting from each $v \in V(G)$.

Keywords: Vertex-coloring, Colorful path, Rainbow path

1 Introduction

Throughout this paper all graphs are simple. Let G be a graph and $V(G)$ be the vertex set of G. In a connected graph G, for any two vertices $u, v \in V(G)$ let $d_G(u, v)$ denote the

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length of the shortest path between u and v in G. We denote the *DFS tree* in G rooted at v by $T(G, v)$ (which is defined in [2, p.139]). For every $u \in V(G)$, each vertex on the path between u and v in $T(G, v)$ is called an *ancestor* of u. By Theorem 6.6 of [2], in every DFS tree if w and w' are adjacent, then one of them is ancestor of another.

In a graph G, a *k-coloring* of G is a function $c : V(G) \to \{0, \ldots, k-1\}$ such that $c(u) \neq c(v)$ for every adjacent vertices $u, v \in V(G)$. The *chromatic number* of G denoted by $\chi(G)$, is the smallest k for which G has a k-coloring. For simplicity we denote a $\chi(G)$-coloring of G by χ-coloring. For a coloring of graph G, we say path P of G is a *rainbow path* if all vertices of P have different colors. A *v-rainbow path* is a rainbow path starting from the vertex v. A *v-colorful path* is a rainbow path starting from the vertex v with $\chi(G)$ vertices. The colorful paths and rainbow paths have been studied by several authors, see [4], [5] and [6].

For each $u \in V(G)$, let $N(u)$ be the set of all vertices adjacent to u. We denote a cycle of order n by C_n. Also we denotes the size of the maximum clique in G by $\omega(G)$. A *good cycle* in a graph G is a cycle of order ℓ in which $\ell \geq \chi(G)$ and $\ell = 0$ *or* $\ell = 1\,(mod\ \chi(G))$.

2 The Existence of $(\Delta(G)+1)$-Colorings with Colorful Paths

Let G be a graph. We recall that a path in G is said to represent all $\chi(G)$ colors if all the colors $0, \ldots, \chi(G) - 1$ appear on this path. The following problem was posed in [6].

Problem. Let G be a connected graph. Does there always exist a proper vertex coloring of G with $\chi(G)$ colors such that every vertex of G is on a path with $\chi(G)$ vertices which represents all $\chi(G)$ colors?

The following conjecture was proposed in [1].

Conjecture. *Let $G \neq C_7$ be a connected graph. Then there exists a $\chi(G)$-coloring of G such that for every $v \in V(G)$, there exists a v-colorful path.*

In [1] it is shown that the local version of conjecture is true, that is for an arbitrary $v \in V(G)$, there exists a χ-coloring of G with a v-colorful path. We start with the following theorem.

Theorem 1 *Let $G \neq C_7$ be a connected graph. If G contains a good cycle, then there is a $(\Delta(G) + 1)$-coloring of G with a v-colorful path for every $v \in V(G)$.*

Proof. For complete graphs the assertion is trivial. Fig.1 shows a proper 3-coloring for odd cycles except C_7, with a v-colorful path for every $v \in V(G)$. Thus assume that G is neither an odd cycle nor a complete graph.

Assume that C is a good cycle of the minimum order k in G, with vertices $v_0, v_1, \ldots,$ v_{k-1}, such that $k = 0$ *or* $k = 1\,(mod\ \chi(G))$. For every i, $0 \leq i \leq k-1$, we color the

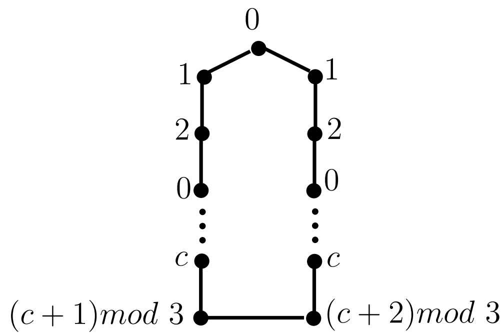

Figure 1: Coloring of odd cycles not isomorphic to C_7

vertex v_i by $i \mod \chi(G)$ using the colors $0, \ldots, \chi(G)-1$. In the case $k = 1\,(mod\ \chi(G))$, we color v_{k-1} by the color $\chi(G)$ and call v_{k-1} by v^*. Note that because of the minimality of the order of C, there is no edge between two vertices of the same color and for each i, $0 \leq i \leq k-1$, there is a v_i-colorful path on C. As a consequence of Brooks' Theorem (Theorem 14.4 of [2]), in the coloring of C we use at most $\Delta(G)+1$ colors. For each i, $0 \leq i \leq k-1$, let *father of* v_i (for abbreviation $F(v_i)$) be $v_{((i+1)\,mod\ k)}$.

Now, we provide an algorithm to color the remaining vertices of G with $\Delta(G)+1$ colors such that there is a v-colorful path for each $v \in V(G)$. For simplicity, define $Next(t)$ the color $(t+1)\,mod\ (\Delta(G)+1)$, for every t, $0 \leq t \leq \Delta(G)$.

In each step of the algorithm, let u be one of the vertices with no color, but adjacent to some colored vertices. Let $c(N(u))$ be the set of all colors appeared in the neighbors of u. Since $|c(N(u))| \leq \Delta(G)$, we can choose an available color t such that $t \notin c(N(u))$ but $Next(t) \in c(N(u))$.

Let $F(u)$ be one of the vertices in $N(u)$ whose color is $Next(t)$. Assign the color t to u and continue the algorithm until all vertices are colored.

Obviously the algorithm produces a proper coloring c. Now, we show that there is a u-colorful path. Consider the following sequence of the vertices $Q(u) : q_1, \ldots, q_{\chi(G)}$ such that $q_1 = u$ and for every i, $1 < i \leq \chi(G)$: $q_i = F(q_{i-1})$. We prove that $Q(u)$ is a u-colorful path. We claim that the colors of $q_1, \ldots, q_{\chi(G)}$ are distinct.
The proof is by contradiction. It can be easily checked that the following holds:

$$c(q_{i+1}) = \begin{cases} c(q_i)+1\ (mod\ (\Delta(G)+1)) & \text{if } q_i \notin C \\ c(q_i)+1\ (mod\ \chi(G)) & \text{if } q_i, q_{i+1} \in C\backslash\{v^*\} \\ c(q_i)+1\ (mod\ (\chi(G)+1)) & \text{if } q_i = v^* \text{ or } q_{i+1} = v^*. \end{cases}$$

Assume that for some $a \neq b$, $c(q_a) = c(q_b)$. It is clear that for some i, $a \leq i < b$, $c(q_i) = 0$. Let $M = \max\{\, i \,|\, i < b, c(q_i) = 0\,\}$. The colors of the vertices $q_M, q_{M+1}, \ldots, q_b$ are $0, 1, \ldots, c(q_b)$, respectively. Since the number of vertices of $Q(u)$ is $\chi(G)$, we have $0 < c(q_b) < \chi(G)$.

Now, let $m = \min\{\, i \,|\, a < i, c(q_i) = 0 \,\}$. Since $c(q_a) \neq 0$, we have $m \leq M$. The number of vertices in the sequence $q_M, \ldots, q_b$ is exactly $c(q_b) + 1$. Since $c(q_m) = 0$, $c(q_{m-1}) \in \{\chi(G)-1, \chi(G), \Delta(G)\}$. So the number of vertices in the sequence $q_a, \ldots, q_{m-1}$ is at least $\chi(G) - c(q_a)$. Therefore the number of vertices of $Q(u)$ should be at least $\chi(G) + 1$, a contradiction. The claim is proved. $\square$

Before stating our main results, we need to prove another theorem.

Lemma 1 *Let G be a connected graph with no cycle of order $\chi(G)$. For a given vertex v, there exists $u \in V(G)$ such that $2\chi(G) - 2 \leq d_{T(G,v)}(u, v)$.*

Proof. Let $T = T(G, v)$. If for every $w \in V(G)$, $2\chi(G) - 2 > d_T(w, v)$, then we show one can properly color the vertices of G using $\chi(G) - 1$ colors. To see this we define a coloring c as follows. For every $w \in V(G)$, let $c(w) = d_T(w, v)\,(mod\ (\chi(G) - 1))$. Assume that $w_1, w_2 \in V(G)$ are adjacent and $c(w_1) = c(w_2)$. Since T is a DFS tree, with no loss of generality we can suppose that w_1 is an ancestor of w_2. Thus $d_T(w_1, w_2) = 0\,(mod\ (\chi(G)-1))$. If $d_T(w_1, w_2) \neq \chi(G)-1$, then $d_T(w_2, v) \geq 2\chi(G)-2$; a contradiction. Hence $d_T(w_1, w_2) = \chi(G)-1$. Since w_1 and w_2 are adjacent we find a cycle of order $\chi(G)$; a contradiction. $\square$

The following theorem proves the assertion of Theorem 1 for the graphs with no good cycle.

Theorem 2 *Let $G \neq C_7$ be a connected graph. If G has no good cycle, then there is a $(\Delta(G) + 1)$-coloring of G with a v-colorful path for every $v \in V(G)$.*

Proof. As we see in the proof of Theorem 1, the assertion holds for odd cycles except C_7. Thus assume that G is not an odd cycle. Let v be an arbitrary vertex of G and $T = T(G, v)$. By Lemma 1, there exists a vertex u such that $2\chi(G) - 2 \leq d_T(u, v)$. Let $P : v = p_0, p_1, \ldots, p_k = u$ be the path between v and u in T. Let Q represent the set of vertices of G whose ancestors(including the vertex itself) are not in the set $\{p_{\chi(G)-1}, p_{\chi(G)}, \ldots, p_k\}$. Define $S = Q \backslash P$ (See Fig.2).

For each $w \in V(G) \backslash S$, color w with $d_T(w, v) \mod \chi(G)$. Since there are no good cycles in G, therefore the coloring of $V(G) \backslash S$ is proper. For each $w \in Q \backslash S$, there is a w-colorful path in $V(G) \backslash S$ going downward in T through P, by passing from each vertex to its child in P. For each $w \in V(G) \backslash Q$, there is a w-colorful path in $V(G) \backslash S$ going upward in T by passing from each vertex to its parent.

So for each $w \in V(G) \backslash S$, there is a w-colorful path. All uncolored vertices are contained in S. We color them in such a way that for each $w \in S$ there exists a vertex $w' \in N(w)$, where $c(w') = Next(c(w))$. Recall that for a color t, $Next(t) = (t+1)\,mod\ (\Delta(G) + 1)$. We denote w' by $F(w)$. Since T is a DFS tree there are no edges between S and $V(G) \backslash Q$. Therefore $F(w) \in Q$. Such coloring can be obtained using the algorithm discussed in the proof of Theorem 1. Now, we show that for each $w \in S$, there exists a w-colorful path.

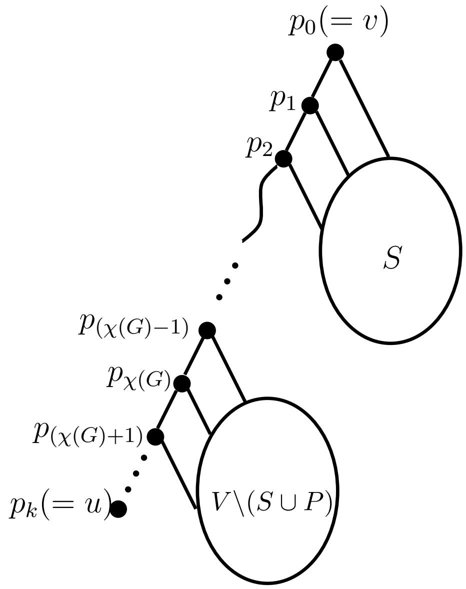

Figure 2: The DFS tree T, rooted at v. This figure illustrates only the edges of T.

For every i, $0 \leq i \leq k-1$, let $F(p_i) = p_{i+1}$. Consider the sequence of the vertices $Q(w) : q_0(= w), \ldots, q_{\chi(G)-1}$, where $F(q_i) = q_{i+1}$, for every i, $0 \leq i < \chi(G) - 1$. Note that for each i, $0 \leq i < \chi(G) - 1$, $c(q_{i+1})$ is either $Next(c(q_i))$ or $c(q_i) + 1$ $(mod\ \chi(G))$. Hence there are no vertices with the same color in $Q(w)$. Therefore $Q(w)$ is a w-colorful path. □

The following theorem shows that for every graph G the conjecture is true for $\Delta(G)+1$ colors instead of $\chi(G)$ colors. In [1] it was proved that the conjecture is true for $\chi(G) + \Delta(G) - 1$ colors. The following theorem is an improvement of this result.

Theorem 3 *Let $G \neq C_7$ be a connected graph. Then there is a $(\Delta(G) + 1)$-coloring of G with a v-colorful path, for every $v \in V(G)$.*

Proof. If $G \neq C_7$ contains a good cycle, then by Theorem 1 there is a $(\Delta(G)+1)$-coloring of G with a v-colorful path, for every $v \in V(G)$. Thus, we may assume that G does not have a good cycle. In this case, Theorem 2 shows that there is a $(\Delta(G) + 1)$-coloring of G with the same properties. □

3 The Existence of $(2\chi(G))$-Colorings with Colorful Paths

Let c be a χ-coloring of a given graph G. Let G_c be a directed graph with the same vertex set of G which has a directed edge from u to v if and only if (i) u and v are adjacent in G; and (ii) $c(v) = c(u) + 1$ $(mod\ \chi(G))$.

Lemma 2 *Let c be a χ-coloring of a connected graph G. For a given subgraph H of G, there exists a χ-coloring c', such that for every $v \in V(H)$, $c'(v) = c(v)$ and for every $u \in V(G)$, there is a directed path from u to at least one of the vertices of $V(H)$ in $G_{c'}$.*

Proof. For an arbitrary χ-coloring of G like c, a vertex u in G_c is called *nice* if there exists a directed path from u to a vertex of H. Assuming that we have a χ-coloring c, we give a polynomial-time algorithm to obtain the coloring c' from c, such that all the vertices are nice. Let $c' = c$ and let $S \in V(G)$ be the set of all vertices of G which are not nice in c'. We will decrease $|S|$, by modifying c' in each iteration of the algorithm. After at most $|V|$ iterations, all the vertices would be nice.

In each iteration, we do as follows:
Let c'_i, for i, $1 \leq i < \chi(G)$, be the coloring of G such that:

$$c'_i(v) = \begin{cases} c'(v) & \text{if } v \notin S \\ c'(v) + i \ (mod\ \chi(G)) & \text{if } v \in S. \end{cases}$$

Since G is connected, at least one of these colorings is not proper. Assume that t is the smallest natural number for which c'_t is not proper. By the definition of S, there is no directed edge from S to $V(G)\backslash S$ in $G_{c'}$. Hence c'_1 is proper. Now, consider the proper coloring c'_{t-1}. Since c'_t is not proper, there are two adjacent vertices $u \in S$ and $v \notin S$ such that $c'_{t-1}(u) + 1 = c'_{t-1}(v)\,(mod\ \chi(G))$. Therefore u is also a nice vertex in $G_{c'_{t-1}}$. Now, let c' be c'_{t-1} and continue with the next iteration (note that the vertices of $G\backslash S$ remain nice in c' and u becomes a nice vertex).
After at most $|V|$ iterations the algorithm will find a coloring c' such that all vertices are nice, and each iteration can be implemented in $O(|V| + |E|)$ time (by considering the edges between S and $G\backslash S$). □

We denote the χ-coloring c', given in the proof of Lemma 2, by $\mathcal{C}(G, H, c)$. Next theorem shows that for every graph G the conjecture holds if one replaces $\chi(G)$ colors with $2\chi(G)$ colors.

Theorem 4 *Let G be a connected graph. Then there exists a $2\chi(G)$-coloring of G with a v-colorful path for every $v \in V(G)$.*

Proof. Let $H = C$ when there is a cycle C of order $\chi(G)$ or $\chi(G) + 1$, otherwise let H be the path with $2\chi(G) - 1$ vertices according to Lemma 1. In either case, choose an arbitrary vertex of H and call it by v^*. Let c be a χ-coloring of G and set $c' = \mathcal{C}(G, v^*, c)$. Now we recolor vertices of H with at most $\chi(G)$ new colors $\chi(G), \ldots, 2\chi(G) - 1$ such that:

- If H is a cycle, then color vertices of $H \backslash v^*$ with one of the colors $\chi(G), \ldots, 2\chi(G) - 1$. Color v^* as the same as its color in $\mathcal{C}(G, v^*, C)$.

- If $H : p_0, \ldots, p_{2\chi(G)-2}$ is a path, then color p_i with $\chi(G) + (i\ mod\ \chi(G))$.

We first claim that c' is a proper coloring. This is trivial in the first case. In the case H is a path P, if there are two adjacent vertices $u, v \in V(G)$ with the same color in c', then $u, v \in V(P)$, because $V(G)\backslash H$ is properly colored with the colors $0, \ldots, \chi(G)-1$ and H is colored with the colors $\chi(G), \ldots, 2\chi(G)-1$. Let $p_i = u$ and $p_j = v$. With no loss of generality suppose that $i < j$. Note that in the coloring of P, we should have $i = j\ (mod\ \chi(G))$. So the vertices $p_i, \ldots, p_j$ form a cycle of order $\chi(G)+1$, a contradiction.

Now, we show that for each $v \in V(G)$, there is a v-colorful path in c'.

Case 1. H is a cycle with the vertices $D : v_0, \ldots, v_k$, where $k = \chi(G)-1$ or $\chi(G)$. Let v be an arbitrary vertex of G. If $v \in D$, then it is clear that there is a v-colorful path in D. If $v \notin D$, then by Lemma 2, there exists a directed path starting from v and ending to v^* in G_f, where $f = \mathcal{C}(G, v^*, c)$. Call this path by $Q : q_0(= v), \ldots, q_k(= v^*)$. If $k \geq \chi(G)-1$, then $q_0, \ldots, q_{\chi(G)-1}$ is a v-colorful path. So assume that $k < \chi(G)-1$. Let i be the smallest index such that $q_i \in D$. Consider the q_i-colorful path in D and call it by $Q' : q'_0(= q_i), \ldots, q'_{\chi(G)-1}$. We claim that $Q'' : q_0, \ldots, q_i, q'_1, \ldots, q'_{\chi(G)-i-1}$ is a v-colorful path. The vertices of D are differently colored with the colors $c(v^*), \chi(G), \ldots, 2\chi(G)-1$. Since $k < \chi(G)-1$, there are no vertices colored with $c(v^*)$ in $\{q_0, \ldots, q_i\}$. Therefore Q'' is a v-colorful path.

Case 2. H is a path P. Let v be an arbitrary vertex of G. If $v \in V(P)$, then according to the length of P, there is a v-colorful path in P. If $v \notin V(P)$, then by Lemma 2, there is a directed path starting from v and ending to v^* in G_f, where $f = \mathcal{C}(G, v^*, c)$. Call this path by $Q : q_0(= v), \ldots, q_k(= v^*)$. Let i be the smallest index such that $q_i \in V(P)$. If $i \geq \chi(G)-1$, then $q_0, \ldots, q_{\chi(G)-1}$ is a v-colorful path. If $i < \chi(G)-1$, then consider the q_i-colorful path in P and call it by $Q' : q'_0(= q_i), \ldots, q'_{\chi(G)-1}$. Then $q_0, \ldots, q_i, q'_1, \ldots, q'_{\chi(G)-i-1}$ is a v-colorful path and the proof is complete. $\square$

4 Long Rainbow Paths in $\chi(G)$-Colorings

The following theorem shows that for every graph G with $\chi(G) = \omega(G)$, the conjecture is true.

Theorem 5 *Let G be a graph with $\omega(G) = \chi(G)$. Then there exists a $\chi(G)$-coloring of G with a v-colorful path for every $v \in V(G)$.*

Proof. Assume that $M = \{v_1, \ldots, v_{\chi(G)}\}$ is a maximum clique in G. We claim that the assertion holds for the coloring $f = \mathcal{C}(G, M, c)$, where c is an arbitrary coloring of G. By Lemma 2, for every $v \in V(G)$, there exists a directed path in G_f, starting from v and ending in M. Call this path by $P : p_1, \ldots, p_k$. Let $M' = \{u_1, \ldots, u_{\chi(G)-k}\}$ be a subset of M such that for every j, $1 \leq j \leq \chi(G)-k$, $c(u_j) \notin \{c(p_1), \ldots, c(p_k)\}$. Clearly, $p_1, \ldots, p_k, u_1, \ldots, u_{\chi(G)-k}$ is a v-colorful path. $\square$

In the previous theorems, we proved the existence of v-colorful paths (rainbow paths of length $\chi(G)$), for every $v \in V(G)$, using a set of colors with different sizes. We close this paper by showing that there are some χ-colorings of G in which there exist long v-rainbow paths, for every $v \in V(G)$.

Theorem 6 *Let G be a connected graph. Then there is a $\chi(G)$-coloring of G in which for every $v \in V(G)$, there exists a v-rainbow path of length $\lfloor \frac{\chi(G)}{2} \rfloor$.*

Proof. Let c be a χ-coloring of G. As a consequence of Proposition 5 in [3], there is a path $P : p_0, \ldots, p_{\chi(G)-1}$ such that

$$c(p_i) = \begin{cases} i & \text{if } 0 \le i \le m \\ \chi(G) + m - i & \text{if } m+1 \le i \le \chi(G) - 1, \end{cases}$$

where $m = \lfloor \frac{\chi(G)-1}{2} \rfloor$. Let $c' = \mathcal{C}(G, P, c)$. By Lemma 2, for every $v \in V(G)$, there is a path $Q(v) : v = q_1, \ldots, q_k = p_s$, where $c'(q_{i+1}) = c'(q_i) + 1 \,(mod\ \chi(G))$ for $1 \le i < k$. With no loss of generality, assume that $q_k \in V(P)$ and $q_i \notin V(P)$ for each i, $1 \le i \le k-1$. Let $Q'(v) : q'_1, \ldots, q'_{k+\lfloor \frac{\chi(G)}{2} \rfloor}$ be the path of length $k + \lfloor \frac{\chi(G)}{2} \rfloor - 1$ such that

$$q'_i = \begin{cases} q_i & \text{if } 1 \le i \le k \\ p_{s+(i-k)} & \text{if } k+1 \le i \le k + \lfloor \frac{\chi(G)}{2} \rfloor \text{ and } s \le m \\ p_{s-(i-k)} & \text{if } k+1 \le i \le k + \lfloor \frac{\chi(G)}{2} \rfloor \text{ and } m < s. \end{cases}$$

We claim that the first $\lfloor \frac{\chi(G)}{2} \rfloor + 1$ vertices of $Q'(v)$ form a v-rainbow path. We prove this in the case $s \le m$. The other case($s > m$) is similar.
Let t be the integer that $q'_t = p_m$. If $t \ge \lfloor \frac{\chi(G)}{2} \rfloor + 1$, then it is clear that there is a v-rainbow path of length $\lfloor \frac{\chi(G)}{2} \rfloor$. Thus assume that $t \le \lfloor \frac{\chi(G)}{2} \rfloor$. We have

- $c'(q'_{i+1}) = c'(q'_i) + 1$, for i, $1 \le i < t$; and
- $c'(q'_i) = c'(q'_{i+1}) + 1$, for i, $t+1 \le i \le \lfloor \frac{\chi(G)}{2} \rfloor$.

Therefore, $c'(q'_i) \in \{0, \ldots, m\}$ for i, $1 \le i \le t$, and $c'(q'_i) \in \{m+1, \ldots, \chi(G)-1\}$ for i, $t+1 \le i \le \lfloor \frac{\chi(G)}{2} \rfloor + 1$. Hence the color of the vertices of $q'_1, \ldots, q'_{\lfloor \frac{\chi(G)}{2} \rfloor + 1}$ are distinct and this path is a v-rainbow path. $\square$

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