A short approach to Catalan numbers modulo 2^r

Guoce Xin[1] and Jing-Feng Xu[2*]

[1] Department of Mathematics
Capital Normal University
Beijing 100048, PR China
guoce.xin@gmail.com

[2] China Institute for Actuarial Science
Central University of Finance and Economics
Beijing 100081, PR China
xujf_cufe@126.com

Submitted: Jul 3, 2011; Accepted: Aug 28, 2011; Published: Sep 9, 2011
Mathematics Subject Classifications: 05A10, 11B50

Abstract

We notice that two combinatorial interpretations of the well-known Catalan numbers $C_n = (2n)!/n!(n+1)!$ naturally give rise to a recursion for C_n. This recursion is ideal for the study of the congruences of C_n modulo 2^r, which attracted a lot of interest recently. We present short proofs of some known results, and improve Liu and Yeh's recent classification of C_n modulo 2^r. The equivalence $C_n \equiv_{2^r} C_{\bar{n}}$ is further reduced to $C_n \equiv_{2^r} C_{\tilde{n}}$ for simpler $\tilde{n}$. Moreover, by using connections between weighted Dyck paths and Motzkin paths, we find new classes of combinatorial sequences whose 2-adic order is equal to that of C_n, which is one less than the sum of the digits of the binary expansion of $n+1$.

1 Introduction

In this paper, we always denote by p a prime number and by r a positive integer. There have been many results on the congruence for combinatorial numbers modulo a prime power p^r. For integer $q \geq 2$ let $n = n_d q^d + n_{d-1} q^{d-1} + \cdots + n_0$ be the base q expansion of n. We denote by $[n]_q = \langle n_d n_{d-1} \cdots n_0 \rangle_q$ the corresponding q-ary digits. The elegant result of Lucas [13] states that $\binom{n}{k} \equiv_p \prod_i \binom{n_i}{k_i}$ if n_i and k_i come from $[n]_p$ and $[k]_p$, where $\equiv_p$ denote the congruence equivalence modulo p. The modulo p^r case is considered in [2], but is much more complicated.

*The authors would like to thank the referee for valuable suggestions improving this exposition. The first named author was partially supported by the Natural Science Foundation of China and by the SRF for ROCS, SEM.

For a positive integer n, the *p-adic order* $\omega_p(n)$ defined by $\omega_p(n) = \max\{t \in \mathbb{N} : p^t | n\}$ is very important when studying the p-adic property of n. In words, $p^{\omega_p(n)}$ is the largest power of p dividing n. In some literature, $\omega_p(n)$ is also called the p-adic valuation. The study of congruences of combinatorial numbers usually starts with their p-adic order. The p-adic order of $\binom{n}{k}$ was first studied by Kummer [9]. We also denote by $\delta_q(n)$ the sum of the digits in the base q expansion $[n]_q$. In particular, if $q = 2$ then we will omit the subscript q.

The well-known Catalan numbers C_n defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!},$$

are one of the most important sequences in combinatorics. Richard Stanley has collected more than 200 interpretations of C_n. See [16]. Recently, the congruences of C_n attracted a lot of interests of study. Alter and Kubota [1] considered $\omega_p(C_n)$, while the following result dates back to Kummer (see Dickson's book [5]):

Theorem 1. *For all nonnegative integers n we have*

$$\omega(C_n) = s(n) := \delta(n+1) - 1.$$

Recently Deutsch and Sagan [3] gave a combinatorial proof of Theorem 1 using group actions. Thereafter Postnikov and Sagan [14] studied the 2-adic order of some weighted Catalan numbers. The congruence class of C_n modulo 8 was first obtained in [6] as an important step for setting the conjecture that the well-known Motzkin numbers are nonzero when modulo 8. Their technique rely on the factorial representation of C_n, and was further developed by Liu and Yeh [11] to classify the congruence of C_n modulo 64. In the same paper, the congruence of C_n modulo 2^r is settled by reducing to only finite cases for given r. These results were also considered in [7] in a more general setting.

Our approach to the Catalan numbers modulo 2^r is short and only relies on the following recursion.

Theorem 2. *The Catalan numbers C_n is recursively determined by $C_0 = 1$ and*

$$C_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 2^{n-2i} C_i. \tag{1}$$

To prove this result is easy. The shortest proof might be by Zeilberger's creative telescoping method [19] since this is in the scope of hypergeometric sum identities. We discover this recursion by the connection between Catalan numbers and 2-Motzkin numbers. See Section 4.

The recursion in Theorem 2 is ideal for the study of C_n modulo 2^r, and is the starting point of this paper. In Section 2, we give short proofs of some known results on the Catalan congruences. In Section 3, we improve Liu and Yeh's reduction result for C_n modulo 2^r. In Section 4, we study Postnikov and Sagan's question on what kind of weighted Catalan number has the same 2-adic order as C_n. We construct several classes of such weighted Catalan numbers by using the Dyck path and Motzkin path model. Finally in Section 5, we discuss some possible directions for generalization.

2 Short derivation of some known results

Our derivation relies on the following lemma and Theorem 2.

Lemma 3. *For positive integers n, we have $s(n) \le s(n-i)+i$ for $0 \le i \le n$.*

Proof. By definition, it is clear that $\delta(n+1) \le \delta(n)+1$, which is equivalent to $s(n) \ge s(n-1)+1$. Thus repeated application of this inequality gives, for $i > 1$, $s(n) \ge s(n-1)+1 \ge s(n-2)+2 \ge \cdots \ge s(n-i)+i$. □

We begin with a simple proof of Theorem 1 not relying on the factorial representation. Deutsch and Sagan [3] gave a beautiful proof of it by using the complete binary tree representation of C_n and group actions, but the idea is not useful to compute the congruences.

Let us rewrite the recursion in Theorem 2 more explicitly as follows.

$$C_{2m+1} = C_m + \sum_{i\ge 1} \binom{2m}{2i} 2^{2i} C_{m-i}, \text{ for } m \ge 0, \tag{2}$$

$$C_{2m} = \sum_{i\ge 1} \binom{2m-1}{2i-1} 2^{2i-1} C_{m-i}, \text{ for } m \ge 1. \tag{3}$$

These two equations and Lemma 3 will be used frequently.

Proof of Theorem 1. We prove the result by induction on n. The theorem clearly holds for $n = 0$. Assume the theorem holds for all smaller n.

For odd $n = 2m+1$, we use (2). By Lemma 3 and the induction hypothesis, we have, for $i \ge 1$,

$$\omega(2^{2i} C_{m-i}) = 2i + s(m-i) \ge i + s(m) > s(m) = \omega(C_m).$$

Thus $\omega(C_{2m+1}) = \omega(C_m) = s(m) = \delta(m+1) - 1 = \delta(2m+2) - 1 = s(2m+1)$.

For even $n = 2m$, we use (3). By the induction hypothesis, $\omega(2(2m-1)C_{m-1}) = s(m-1)+1$, together with Lemma 3, we have, for $i \ge 2$,

$$\omega(2^{2i-1} C_{m-i}) = 2i - 1 + s(m-i) \ge s(m-1) + i > s(m-1) + 1.$$

Thus $\omega(C_{2m}) = \omega(2(2m-1)C_{m-1}) = s(m-1)+1 = \delta(m) = \delta(2m+1) - 1 = s(2m)$. This completes the proof. □

From Theorem 1, it is easy to derive the following congruence result. See, e.g., [1].

Theorem 4. *The Catalan number C_n is odd if and only if $n = 2^a - 1$:*

$$C_n \equiv_2 \chi(n = 2^a - 1, a \ge 0) = \begin{cases} 1 & \text{if } n = 2^a - 1, a \ge 0; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The modulo 4 and 8 congruences were computed in [6] by developing a general approach for dealing with factorials.

Theorem 5. *The Catalan numbers are not congruent to 3 when modulo 4 and*

$$C_n \equiv_4 \begin{cases} 1 & \text{if } n = 2^a - 1; a \geq 0 \\ 2 & \text{if } n = 2^b + 2^a - 1, b > a \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 6. *The Catalan numbers are not congruent to 3,7 when modulo 8 and*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0, 1 \\ 5 & \text{if } n = 2^a - 1, a \geq 2; \\ 2 & \text{if } n = 2^{a+1} + 2^a - 1, a \geq 0; \\ 6 & \text{if } n = 2^b + 2^a - 1, b - 2 \geq a \geq 0; \\ 4 & \text{if } n = 2^c + 2^b + 2^a - 1, c > b > a \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Our approach by recursion is much shorter.

Proof of Theorems 4-6. Modulo 2, the recursions (2) and (3) reduce to

$$C_{2m+1} \equiv_2 C_m \quad \text{ and } C_{2m} \equiv_2 0, m \geq 1.$$

We can then deduce that if $n = 2^{a+1}\alpha + 2^{a-1} + a^{a-2} + \cdots + 1 = (2\alpha + 1)2^a - 1$, then $C_n \equiv_2 C_{2\alpha}$, which is 1 if $\alpha = 0$ and 0 otherwise. Theorem 4 then follows.

For $r = 2$, the recursions reduce to

$$C_{2m+1} \equiv_4 C_m \quad \text{ and } C_{2m} \equiv_4 2C_{m-1}.$$

Similar to the $r = 1$ case, if $n = 2^a(2\alpha + 1) - 1, a \geq 1$, then $C_n \equiv_4 C_{2\alpha}$. If $\alpha = 0$, then $n = 2^a - 1$, and we have $C_n \equiv_4 1$; otherwise $C_{2\alpha} \equiv_4 2C_{\alpha-1} \equiv_4 2\chi(\alpha = 2^{a'}, a' \geq 0)$ by Theorem 4. Summarizing the above gives Theorem 5.

The $r = 3$ case is a bit more complicated. Our recursions reduce to

$$C_{2m+1} \equiv_8 C_m + 4m(2m-1)C_{m-1} \equiv_8 C_m + 4mC_{m-1}, \tag{7}$$

$$C_{2m} \equiv_8 2(2m-1)C_{m-1}. \tag{8}$$

Equation (7) can be simplified further:

$$\begin{aligned} C_{2m+1} &\equiv_8 C_m + 4m\chi(m = 2^a, a \geq 0) \\ &\equiv_8 C_m + 4 \cdot 2^a \chi(m = 2^a, a \geq 0) \\ &\equiv_8 C_m + 4\chi(m = 1). \end{aligned}$$

Let $n = 2^a(2\alpha + 1) - 1$. If $\alpha = 0$, then $n = 2^a - 1$ and we have $C_0 = C_1 = 1 \pmod 8$, $C_{2^a-1} \equiv_8 5$ for $a \geq 2$; If $\alpha \geq 1$, then

$$C_n \equiv_8 C_{2\alpha} \equiv_8 2(2\alpha - 1)C_{\alpha-1}.$$

On the other hand, by (5) and (8) we obtain:

$$C_{2\alpha} \equiv_8 2(2\alpha-1)C_{\alpha-1} \equiv_4 \begin{cases} 2 & \text{if } \alpha = 1; \\ 6 & \text{if } \alpha = 2^b, b \geq 1; \\ 4 & \text{if } \alpha = 2^c + 2^b, c > b \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6 is then obtained by collecting the above information. □

From the above computation, we see that we can recursively solve for $C_n \pmod{2^r}$ even for symbolic n. However the recursion becomes complicated and we will have to splits into too many cases when r becomes large. This approach is not convincing as can be seen in the next section.

3 Improvements on Catalan numbers modulo 2^r

The following result was conjectured by Liu and Yeh [11] and was soon proved by Lin [10]. Around the same time, Luca and Young [12] discovered and proved this result independently for a different purpose. Both proofs rely on the factorial form of C_n.

Theorem 7. *Let $k \geq 2$, The odd congruences $C_{2^a-1} \pmod{2^k}, a \geq 0$ remain constant for $a \geq k-1$ and are distinct for $a = 1, 2, \ldots, k-1$.*

We are able to generalize this result as follows.

Theorem 8. *Let $r \geq 1$ and $\alpha \in \mathbb{N}$ with $\delta(\alpha) < r$. Then if $b \geq r-1-\delta(\alpha)$, we have $C_{2^b(2\alpha+1)-1} \equiv C_{2^{r-1-\delta(\alpha)}(2\alpha+1)-1} \pmod{2^r}$. Moreover, the congruence classes $C_{2^b(2\alpha+1)-1} \pmod{2^r}, b = 1, 2, \ldots, r-1-\delta(\alpha)$ are all distinct.*

Proof. We prove the result by induction on r. The cases $r = 1, 2$ are easily checked by Theorems 4 and 5. For the second part, by using the induction hypothesis and the fact that $x \equiv y \pmod{2^r}$ implies that $x \equiv y \pmod{2^{r-1}}$, we only need to show that $C_{2^{r-1-\delta(\alpha)}(2\alpha+1)-1} \not\equiv C_{2^{r-2-\delta(\alpha)}(2\alpha+1)-1} \pmod{2^r}$.

Now write $n = 2^b(2\alpha+1)-1$ and $n' = 2^{b-1}(2\alpha+1)-1$. We will show that $C_n \equiv C_{n'} \pmod{2^r}$ for $b \geq r-\delta(\alpha) > 0$. Then iterating application of this equality yields the first part of the theorem. For general $b > 0$, take $m = n'$ in (2) and investigate each term of the right-hand-side. By Theorem 1, we have, for $i \geq 1$,

$$\omega(2^{2i}C_{m-i}) = 2i + s(m-i) \geq s(m-1) + i + 1 = \delta(m) + i = \delta(\alpha) + b - 1 + i.$$

Thus most terms vanish when modulo 2^r and we are left with: i) $C_n \equiv C_{n'} \pmod{2^r}$ if $b \geq r-\delta(\alpha)$, as desired; ii) $C_n \equiv C_{n'} + m(2m-1)2^2C_{m-1} \equiv C_{n'} + 2^{r-1} \pmod{2^r}$ if $b = r-\delta(\alpha)-1 \geq 2$ (there is nothing to show when $r-\delta(\alpha)-1 = 1$), since $\omega(2^2C_{m-1}) = 2 + s(m-1) = \delta(\alpha) + b = r-1$. □

Suppose $[n]_2 = \langle 10^{a_s}10^{a_{s-1}}\cdots 10^{a_1}01^{a_0}\rangle_2$ where $a_i \geq 0$ and 0^{a_i} means the concatenation of a_i 0's. The extra 0 after 0^{a_1} guarantees that $s(n) = s$ for all $a_1 \geq 0$. Liu and Yeh [11] reduce the congruence of C_n to that of $C_{\bar{n}}$ when modulo 2^r, where $[\bar{n}]_2$ is obtained from $[n]_2$ by change a_i to $\min(a_i, r-1)$ for all i. We improve this reduction to $\tilde{n}$ defined by

$$[\tilde{n}]_2 = \langle 10^{\min(a_s,r-s-1)}10^{\min(a_{s-1},r-s-1)}\cdots 10^{\min(a_1,r-s-1)}01^{\min(a_0,r-s-1)}\rangle_2.$$

In words, $[\tilde{n}]_2$ is obtained from $[n]_2$ by changing any run of 1 or 0 of length greater than $r - s(n) - 1$ to length $r - s(n) - 1$, provided the rightmost 0 is not counted into the runs.

Theorem 9. *For positive integers n and r, we have*

$$C_n \equiv_{2^r} \begin{cases} C_{\tilde{n}}, & \text{if } s(n) \leq r-1; \\ 0, & \text{if } s(n) \geq r. \end{cases} \tag{9}$$

Before giving the proof, we clarify the statement by some examples.

1) The $r = 1$ case. If $s(n) = 0$ then $[n]_2 = \langle 1^{a_0}\rangle_2$ with $a_0 \geq 0$ and the theorem asserts $C_n \equiv_2 = C_0 = 1$. This is consistent with Theorem 4.

2) The $r = 2$ case. If $s(n) = 1$ then $[n]_2 = \langle 10^{a_1}01^{a_0}\rangle_2$ with $a_i \geq 0$, and the theorem asserts $C_n \equiv_4 C_2 = 2$; if $s(n) = 0$ then for $[n]_2 = \langle 1^{a_0}\rangle_2$ with $a_0 \geq 2-0-1 = 1$ we have $C_n \equiv_4 C_1 = 1$, which together with $C_0 \equiv_4 1$ classify this case.

3) The $r = 3$ case. If $s(n) = 2$ then $[n]_2 = \langle 10^{a_2}10^{a_1}01^{a_0}\rangle_2$ with $a_i \geq 0$, and the theorem asserts $C_n \equiv_8 C_6 = 396 \equiv_8 4$ (also followed by $s(n) = 2 = r-1$); if $s(n) = 0$ then $C_0 = C_1 = 1$ for $a_0 = 0, 1$ and $C_n \equiv_8 C_3 \equiv_8 5$ for $a_0 \geq 2$; finally the $s(n) = 1$ case is listed in the following table.

$[n]_2 = \langle 10^{a_1}01^{a_0}\rangle_2$	$a_0 = 0$	$a_0 \geq 1$
$a_1 = 0$	$C_2 \equiv_8 2$	$C_n \equiv_8 C_5 \equiv_8 2$
$a_1 \geq 1$	$C_n \equiv_8 C_4 \equiv_8 6$	$C_n \equiv_8 C_9 \equiv_8 6$

In summary, the above computation is consistent with Theorem 6.

Proof of Theorem 9. We prove the result by induction on r. The theorem holds for $r = 1, 2, 3$ as it has been verified above. Now assume the theorem holds for smaller r. The reduction of a_0 to $\min(a_0, r - s(n) - 1)$ is confirmed by Theorem 8.

By Theorem 1, it is clear that $C_n \equiv_{2^r} 0$ if $s(n) \geq r$ and $C_n \equiv_{2^r} 2^{r-1}$ if $s(n) = r-1$. Now let us assume $s(n) \leq r-2$. The reduction for a_i with $i \geq 1$ follows by iterative application of the following Lemma 10. ☐

Lemma 10. *Suppose $r \geq 1$. Fix $\alpha, e \in \mathbb{N}$ with $2^{a-1} \leq e < 2^a$. For $n = (2\alpha+1)2^{a+b}+e$ set $n' = (2\alpha+1)2^{a+b-1}+e$ if $b \geq 2$. Then $C_n \equiv C_{n'} \pmod{2^r}$ if $b \geq r - s(n) + \chi(e = 2^a - 1)$.*

Proof. We prove the result by induction on r and then on a. The lemma has been verified for $r = 1, 2$, so assume $r \geq 3$.

For the initial case $a = 0$, e has to be 0 and $\chi(e = 2^a - 1) = 1$. Write $n = 2^b(2\alpha + 1)$ and $n' = m = 2^{b-1}(2\alpha + 1) = 2m'$ and use (3). Then $s(n) = s(n') = \delta(\alpha) + 1$. By Theorem 1, we have, for $i \geq 2$,

$$\omega(2^{2i-1}C_{m-i}) = 2i - 1 + s(m-i) \geq i + 1 + s(m-2) = i + \delta(m-1) = i + \delta(\alpha) + b - 1.$$

Thus for $b+\delta(\alpha)\geq r$, or equivalently $b\geq r+1-s(n)$, we are left with

$$C_n \equiv 2(2m-1)C_{m-1} \equiv -2C_{m-1} \pmod{2^r},$$
$$C_{n'} \equiv 2(2m'-1)C_{m'-1} \equiv -2C_{m'-1} \pmod{2^r},$$

where we used the fact that $\omega(4m'C_{m'-1}) = b+\delta(\alpha)\geq r$ and similarly for m. Now $C_{m-1}\equiv C_{m'-1} \pmod{2^{r-1}}$ by Theorem 8, since $m-1$ ends with a run of 1 of length $b-1\geq r-1-\delta(\alpha)$. It then follows that $C_n\equiv C_{n'} \pmod{2^r}$, as desired.

Now for $a\geq 1$, take $m=2^{a+b-1}(2\alpha+1)+f$ and $m'=2^{a+b-2}(2\alpha+1)+f$, where $e=2f+1$ or $e=2f$ according to $n=2m+1$ or $n=2m$ and similarly for n' and m'. The key observation is that $s(m-f-2)$ is large:

$$s(m-f-2)=\delta(2^{a+b-1}(2\alpha+1)-1)-1=\delta(\alpha)+a+b-2.$$

For the case $n=2m$ even, we also need the equality

$$s(m'-1)=s(m-1)=\delta(m)-1=\delta(2m+1)-2=s(2m)-1=s(n)-1.$$

In (3), by using Theorem 1 we obtain, for $i\geq f+2$,

$$\omega(2^{2i-1}C_{m-i})=2i-1+s(m-i)\geq s(m-f-2)+i+f+1\geq \delta(\alpha)+a+b+1+2f.$$

Thus for $b+s(n)\geq r$ we have $b+a+\delta(\alpha)\geq b+s(n)\geq r$, and hence

$$C_n \equiv \binom{2m-1}{2f+1}2^{2f+1}C_{2^{a+b-1}(2\alpha+1)-1)}+\sum_{i=1}^{f}\binom{2m}{2i-1}2^{2i-1}C_{m-i} \pmod{2^r},$$
$$C_{n'} \equiv \binom{2m'-1}{2f+1}2^{2f+1}C_{2^{a+b-2}(2\alpha+1)-1)}+\sum_{i=1}^{f}\binom{2m'}{2i-1}2^{2i-1}C_{m'-i} \pmod{2^r},$$

where we write separately the term corresponds to $i=f+1$. To see that $C_n\equiv C_{n'}$ $\pmod{2^r}$ we need to check the following three facts: i) $C_{2^{a+b-1}(2\alpha+1)-1}\equiv C_{2^{a+b-2}(2\alpha+1)-1}$ $\pmod{2^{r-1}}$ by Theorem 8; ii) $C_{m-i}\equiv C_{m'-i} \pmod{2^{r-2i+1}}$ for $1\leq i\leq f$. Since $e-i$ can not be $2^{a-1}-1$ and

$$b+s(m-i)\geq b+s(m-1)-(i-1)=b+s(n)-i\geq r-i\geq r-2i+1,$$

the condition for the lemma holds with respect to $a-1$ and $f-i$, and hence the induction hypothesis applies. iii) For $1\leq i\leq f+1$ we have $\binom{2m-1}{2i-1}2^{2i-1}C_{m'-i}\equiv\binom{2m'-1}{2i-1}2^{2i-1}C_{m'-i}$ $\pmod{2^r}$. Firstly we have

$$\omega(2^{2i-1}C_{m'-i})=2i-1+s(m'-i)\geq i+s(m'-1)=s(n)+i-1\geq s(n).$$

Secondly, $\omega(2m-2m')=\omega(2^{a+b-1}(2\alpha+1))=a+b-1\geq r-s(n)$, and hence $\binom{2m-1}{2i-1}\equiv\binom{2m'-1}{2i-1} \pmod{2^{r-s(n)}}$.

The case $n = 2m + 1$ odd is similar. We need the fact $s(n) = s(m) = s(m')$. Looking at (2), we have similarly $\omega(2^{2i}C_{m-i}) \geq \delta(\alpha) + a + b + 2 + 2f$ for $i \geq f + 2$. Thus if $b + s(n) \geq r + \chi(e = 2^a - 1)$, then we are left with

$$C_n \equiv \binom{2m}{2f+2} 2^{2f+2} C_{2^{a+b-1}(2\alpha+1)-1)} + \sum_{i=0}^{f} \binom{2m}{2i} 2^{2i} C_{m-i} \pmod{2^r},$$

$$C_{n'} \equiv \binom{2m'}{2f+2} 2^{2f+2} C_{2^{a+b-2}(2\alpha+1)-1)} + \sum_{i=0}^{f} \binom{2m'}{2i} 2^{2i} C_{m'-i} \pmod{2^r}.$$

To see that $C_n \equiv C_{n'} \pmod{2^r}$ we need four facts: i) $C_{2^{a+b-1}(2\alpha+1)-1)} \equiv C_{2^{a+b-2}(2\alpha+1)-1)} \pmod{2^{r-2}}$, also followed by Theorem 8; ii) $C_{m-i} \equiv C_{m'-i} \pmod{2^r - 2i}$ for $1 \leq i \leq f$. This can be similarly checked by first observing that

$$b + s(m - i) \geq b + s(m) - i = b + s(n) - i \geq r - i \geq r - 2i$$

and then applying the induction hypothesis. iii) For $1 \leq i \leq f + 1$ we have $\binom{2m}{2i} 2^{2i} C_{m'-i} \equiv \binom{2m'}{2i} 2^{2i} C_{m'-i} \pmod{2^r}$. Similarly, we obtain

$$\omega(2^{2i}C_{m'-i}) = 2i + s(m' - i) \geq i + s(m') = s(n) + i \geq s(n),$$

and $\omega(2m-2m') = \omega(2^{a+b-1}(2\alpha+1)) = a+b-1 \geq r-s(n)$ which implies that $\binom{2m}{2i} \equiv \binom{2m'}{2i} \pmod{2^{r-s(n)}}$; iv) $C_m \equiv C_{m'} \pmod{2^r}$ for the $i = 0$ term. We also apply the induction hypothesis by verifying that $b + s(m) = b + s(n) \geq r + \chi(e = 2^a - 1) = r + \chi(f = 2^{a-1} - 1)$. □

4 Generalizations from the view of recursion

A natural question is to find more combinatorial sequences D_n satisfying the property $\omega(D_n) = s(n)$. By generalizing the idea in the proof of Theorem 1, we construct classes of weighted Catalan number having this property.

A Motzkin path of length n is a $\mathbb{Z}^2$ lattice path from $(0, 0)$ to $(n, 0)$ with steps $(1, 1), (1, 0), (1, -1)$, called *up*, *level*, or *down* step respectively, and never go below level 0, i.e., the horizontal axis. Given two sequences $h = (h_0, h_1, h_2, \dots)$ and $u = (u_0, u_1, u_2, \dots)$, an up step start at level i has weight u_i, a level step at level i has weight h_i, and any down step has weight 1. The weight of a Motzkin path is defined to be the product of the weights of all its steps, and the weighted Motzkin number $M_n^{h,u}$ is defined to be the sum of the weights of all Motzkin paths of length n. When h is constant, say $h_i = \alpha$ for all i, we simply write α for h, and similarly for u. The ordinary Motzkin number M_n is the special case $M_n^{1,1}$.

It is clear that $M_{2n+1}^{0,u} = 0$. Denote by $C_n^u = M_{2n}^{0,u}$ and call it the weighted Catalan number, since $C_n^1 = C_n$. Motzkin paths without level steps are also called Dyck paths. It is clear that $C_n^\beta = \beta^n C_n$. It is worth mentioning that both generating functions $\sum_{n\geq 0} C_n^u x^n$ and $\sum_{n\geq 0} M_n^{h,u} x^n$ have continued fraction representations, one called the S-fraction and the other called the J-fraction. See [17]. For constant h, say $h_i = \alpha$, we have the following identity

$$M_n^{\alpha,u} = \sum_{i\geq 0} \binom{n}{2i} \alpha^{n-2i} C_i^u, \tag{10}$$

by classifying the appropriate Motzkin paths according to the number of level steps.

The following result is useful in constructing sequences with 2-adic order $s(n)$. Since it is only a special case of the more general Proposition 17, we omit the proof.

Proposition 11. *For any sequence E_n with $\omega(E_n) = s(n)$ and odd numbers α, β, if*

$$D_{2n+1} = \beta^n E_n + \sum_{1\leq i\leq n} \binom{2n}{2i} (2\alpha)^{2i} \beta^{n-i} E_{n-i}, \ \textit{for } n \geq 0, \tag{11}$$

$$D_{2n} = \sum_{1\leq i\leq n} \binom{2n-1}{2i-1} (2\alpha)^{2i-1} \beta^{n-i} E_{n-i}, \ \textit{for } n \geq 1, \tag{12}$$

then $\omega(D_n) = s(n)$ for all n. In particular, if D_n is recursively defined as above with $E_n = D_n$ and $\omega(D_0) = 0$, then $\omega(D_n) = s(n)$.

Corollary 12. *For any integer α, β and sequence u such that $\omega(C_n^u) = s(n)$, we have $\omega(M_n^{4\alpha+2,u}) = s(n+1)$. In particular, $\omega(M_n^{4\alpha+2,2\beta+1}) = s(n+1)$.*

By comparing Theorem 2 with (10), we see that $C_{n+1} = M_n^{2,1}$. Next we establish a connection between weighted Catalan numbers and weighted Motzkin numbers.

Theorem 13. *For any sequence $u = (u_0, u_1, u_2, \dots)$ we have*

$$C_{n+1}^u = u_0 M_n^{(u_0+u_1, u_2+u_3, \dots), (u_1 u_2, u_3 u_4, \dots)}.$$

Proof. For a given Dyck path P from $(0, 0)$ to $(2n + 2, 0)$, by removing the first up step (with weight u_0) and the final down step, we obtain a path from $(1, 1)$ to $(2n + 1, 1)$ that never go below level 0. Now combine every consecutive two steps together to form a Motzkin path M of length n: a) an up-up step in P start at level $2i + 1$ with weight $u_{2i+1}u_{2i+2}$ becomes an up step at level i for $i \geq 0$; b) a down-down step in P start at level $2i + 1$ with weight 1 becomes a down step start at level i for $i \geq 1$ (we can not have a down-down step in P start at level 1); c) an up-down step or a down-up step start at level $2i + 1$ with combined weights $u_{2i+1} + u_{2i}$ becomes a level step at level i for $i \geq 0$. It is easy to see that this is a weight preserving bijection. □

Remark 14. *Though we are not able to find a reference, Theorem 13 is probably known, since the proof is short and weighted Motzkin and Catalan numbers are well studied objects. Indeed, the ordinary case $C_{n+1} = M_n^{2,1}$ was established by Deutsch and Shapiro [4], where $M_n^{2,1}$ is called the 2-Motzkin numbers.*

Postnikov and Sagan [14] established a sufficient condition for $\omega(C_n^u) = s(n)$, in particular for $C_n^{(1^2,3^2,5^2,\ldots)}$. We derive sufficient conditions of different kind. Denote by $h = (\dot{h}_0, h_1, \ldots, \dot{h}_{d-1})$ if h is periodic with $h_{d+i} = h_i$. We have

Theorem 15. *For odd integers α, β with $\alpha + \beta \equiv_4 2$, we have $\omega(C_n^{(\dot{\alpha},\dot{\beta})}) = s(n)$.*

Proof. By Theorem 13, $C_n^{(\dot{\alpha},\dot{\beta})} = \alpha M_{n-1}^{\alpha+\beta,\alpha\beta}$. Now apply Corollary 12. □

Theorem 16. *For odd integers u_0, u_2, and $\alpha \equiv_4 2$, we have $\omega(C_n^{(\dot{u}_0,\alpha-u_0,u_2,\alpha-\dot{u}_2)}) = s(n)$.*

Proof. Denote by $u_1 = \alpha - u_0, u_3 = \alpha - u_2$. We have $C_n^{(\dot{u}_0,u_1,u_2,\dot{u}_3)} = u_0 M_{n-1}^{\alpha,(\dot{u_1 u_2},\dot{u_3 u_0})}$ by using Theorem 13. By Theorem 15 and the condition that

$$u_2u_1 + u_3u_0 = u_2(\alpha - u_0) + u_0(\alpha - u_2) \equiv_4 2(u_2 + u_0 - u_0u_2) = -2u_0u_2 \equiv_4 2,$$

we conclude that $\omega(C_n^{\alpha,(\dot{u_1 u_2},\dot{u_3 u_0})}) = s(n)$. Now apply Corollary 12. □

A concrete example would be $\omega(C_n^{(\dot{1},5,3,\dot{3})}) = s(n)$. If we allow negative numbers, the simplest example would be $\omega(C_n^{(\dot{1},1,-1,\dot{3})}) = s(n)$. We can obtain more classes of sequences, but they will be more complicated. For instance, if we want $\omega(C_n^{(\dot{u}_0,u_1,\ldots,\dot{u}_7)}) = s(n)$, we need u_i being odd, $u_{2i} + u_{2i+1} = \alpha \equiv_4 2$, and $u_1u_2 + u_3u_4 = u_5u_6 + u_7u_0 \equiv_4 2$.

We conclude by the following very general result, in the same vein of Theorem 1.

Proposition 17. *Fix a prime p. For any sequence E_n satisfying $\omega_p(E_n) = s(n)$, if*

$$D_{2n+1} = a_{2n+1,0}E_n + \sum_{1\le i\le n} a_{2n+1,i}p^{i+1}E_{n-i}, \text{ for } n \ge 0, \tag{13}$$

$$D_{2n} = pa_{2n,1}E_{n-1} + \sum_{2\le i\le n} a_{2n,i}p^{i+1}E_{n-i}, \text{ for } n \ge 1, \tag{14}$$

where $\omega_p(a_{2n+1,0}) = 0$ and $\omega_p(a_{2n,1}) = 0$, then $\omega_p(D_n) = s(n)$ for all n. In particular, if $E_n = D_n$ and D_n is recursively defined as above with $\omega_p(D_0) = 0$, then $\omega_p(D_n) = s(n)$.

Proof. For the odd case, we check that, for $i \ge 1$,

$$\omega_p(a_{2n+1,i}p^{i+1}E_{n-i}) \ge i + 1 + s(n-i) \ge s(n) + 1 > s(n) = \omega_p(a_{2n+1,0}E_n).$$

It follows that $\omega_p(D_{2n+1}) = s(n) = s(2n+1)$.

Similarly for the even case, we check that, for $i \ge 2$,

$$\omega_p(a_{2n,i}p^{i+1}E_{n-i}) \ge i + 1 + s(n-i) \ge 2 + s(n-1) > s(n-1) + 1 = \omega_p(pa_{2n,1}E_{n-1}).$$

It follows that $\omega_p(D_{2n}) = s(n-1) + 1 = \delta(n) = \delta(2n+1) - 1 = s(2n)$. □

However, it is hard to specialize p and $a_{n,i}$ to produce nice combinatorial sequences.

5 Further discussion and future work

Deutsch and Sagan [3] give a combinatorial proof of $\omega(C_n) = s(n)$ by using the complete binary tree representation of C_n and group actions. They conjectured that the closely related Motzkin numbers

$$M_n = \sum_{i\geq 0} \binom{n}{2i} C_i$$

is nonzero when modulo 8. This conjecture was settled by Eu, Liu and Yeh [6], where the congruences of the Catalan numbers in Theorems 4-6 play important roles.

In [18], the first named author will find a similar, but not very nice, recursion as in Theorem 2 that not only gives rise to a simple proof of the Eu-Liu-Yeh theorem but also helps classifying the congruence classes of M_n modulo 2^r.

The recursion in Theorem 2 is very helpful in computing the congruences. It is natural to ask if we have similar recursion for prime $p > 2$. A possible starting point would be to consider generalized Catalan numbers, which is denoted

$$G_n^{(k)} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

This counts the number of many combinatorial objects such as complete k-ary trees, and the $k = 2$ case reduces to the ordinary Catalan numbers. See, e.g., [16]. The following result was obtained independently by Stănică [15] and by Konvalinka [8].

Proposition 18. *Let p be a prime and $k \geq 1$. We have*

$$\omega_p(G_n^{(p^k)}) = \frac{\delta_p((p^k-1)n+1)-1}{p-1}.$$

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