Promotion and evacuation on standard Young tableaux of rectangle and staircase shape

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Abstract

(Dual-)promotion and (dual-)evacuation are bijections on $SYT(\lambda)$ for any partition λ. Let c^r denote the rectangular partition $(c, \dots, c)$ of height r, and let sc_k $(k > 2)$ denote the staircase partition $(k, k-1, \dots, 1)$. We demonstrate a promotion- and evacuation-preserving embedding of $SYT(sc_k)$ into $SYT(k^{k+1})$. We hope that this result, together with results by Rhoades on rectangular tableaux, can help to demonstrate the cyclic sieving phenomenon of promotion action on $SYT(sc_k)$.

1 Introduction

Promotion and evacuation (denoted here by ∂ and ϵ, respectively – see Definitions 2.1 and 2.7) are closely related permutations on the set of standard Young tableaux $SYT(\lambda)$ for any given shape λ. Schützenberger studied them in [14, 15, 16] as bijections on $SYT(\lambda)$, and later as permutations on the linear extensions of any finite poset. Edelman and Greene [3], and Haiman [6] described some of their important properties; in particular, they showed that the order of promotion on $SYT(sc_k)$ is $k(k+1)$, where $sc_k = (k, k-1, \cdots, 1)$

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is the staircase tableau. In 2008, Stanley gave a terrific survey [20] of previous results on promotion and evacuation.

In the paper, we report the construction of an embedding

$$\iota : SYT(sc_k) \hookrightarrow SYT(k^{(k+1)})$$

that preserves promotion and evacuation:

$$\begin{aligned} \iota \circ \partial &= \partial \circ \iota, \text{ and} \\ \iota \circ \epsilon &= \epsilon \circ \iota. \end{aligned}$$

This result arises from our on-going project aimed at understanding the promotion cycle structure on rectangle-shaped standard Young tableaux $SYT(r^c)$ and staircase standard Young tableaux $SYT(sc_k)$. The eventual (not yet achieved) goal of this project is the demonstration of the **cyclic sieving phenomenon** (CSP) of promotion action on $SYT(sc_k)$.

Let X be a finite set and let $C = \langle a \rangle$ be a cyclic group of order N acting on X. Let $X(q) \in \mathbb{Z}[q]$ be a polynomial with integer coefficients. We say that the triple $(X, C, X(q))$ exhibits the CSP if for any integer k, we have

$$X(\zeta^k) = \#\{x \in X \mid a^k \cdot x = x\}, \tag{1.1}$$

where $\zeta = e^{2\pi i/N}$ is a primitive N-th root of unity.

V. Reiner, D. Stanton, and D. White first formalized the notion of the CSP in [10]. Before them, Stembridge considered the "$q = -1$" phenomenon [23], which is the special case of the CSP with $N = 2$ (where $\zeta = e^{2\pi i/2} = -1$).

Important instances of the CSP arise from the actions of promotion and evacuation on standard Young tableaux. For example, Stembridge [23] showed that $(SYT(\lambda), \langle\epsilon\rangle, X(q))$ exhibits the CSP, where λ is any partition shape and $X(q)$ is the generating function of the comajor index.

More recently, B. Rhoades [11] showed representation-theoretically that, for an arbitrary rectangular partition (c^r), $(SYT(c^r), \langle\partial\rangle, X(q))$ exhibits the CSP, where ∂ is promotion on $SYT(c^r)$ and $X(q)$ is the generating function of $\overline{\mathrm{maj}}$, a statistic on standard Young tableaux that is closely related to the major index. [1]

Via the embedding $\iota : SYT(sc_k) \hookrightarrow SYT(k^{(k+1)})$, we are able to extend Rhoades' definition of "extended descent" from rectangular tableaux to staircase tableaux. This further enables us to demonstrate facts about the promotion cycle structure on staircase tableaux. For example, we will be able to show that full-cycle(s) always exist (Corollary 4.18), and half-cycles never exist (Corollary 4.19).

This paper is organized in the following way: In Section 2, we define the terminology and notation, and review several basic results that are used in later sections.

[1] More precisely, $\overline{\mathrm{maj}} = \mathrm{maj} - \mathrm{b}(\lambda)$, where $\mathrm{b}(\lambda)$ is a quantity that depends only on the shape λ, see [19, 7.21.5] for details.

In Section 3, we construct the embedding ι and prove our main results about ι, Theorems 3.6 and 3.7, which state that promotion and evacuation are preserved under the embedding.

In Section 4, we extend Rhoades' construction of "extended descent" on rectangular tableaux to that on staircase tableaux by using ι; our main results in this section are Theorems 4.11, 4.12, 4.13 and 4.14, which state that the extended descent data nicely records the actions of (dual-)promotion and (dual-)evacuation on both rectangular and staircase tableaux.

In Section 5, we explain how the embedding ι arose and pose open questions.

2 Definitions and Preliminaries

This section is a review of those notions, notations and facts about Young tableaux that are directly used in the following sections. We assume the reader's basic knowledge of tableaux theory, including partitions, standard Young tableaux, Knuth equivalence, reading word of a tableau, jeu-de-taquin, and the RSK algorithm. All of our tableaux and directional references (e.g., north, west, etc.) will refer to tableaux in "English" notation. For more on these topics, see [19] or [5].

2.1 Basic definitions

Definition 2.1. Given $T \in SYT(\lambda)$ for any (skew) shape $\lambda \vdash n$, the **promotion** action on T, denoted by $\partial(T)$, is given as follows:

Find in T the outside corner that contains the number n, and remove it to create an empty box. Apply jeu-de-taquin repeatedly to move the empty box northwest until the empty box is an inside corner of λ. (We call this process **sliding**, the sequence of positions that the empty box moves along in this process is called **sliding path**). Place 0 in the empty box. Now add one to each entry of the current filling of λ so that we again have a standard Young tableau. This new tableau is $\partial(T)$, the promotion of T.

In the case that sliding is used to define promotion (there are other equivalent descriptions), we will refer to the sliding path as the **promotion path**.

Remark 2.2. Edelman and Greene ([3]) call ∂ defined above "elementary promotion." They call ∂^n the "promotion operator."

Example 2.3. Promotion on standard tableaux.

$$
\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&8\\\hline 3&7&13\\\hline 9&10&15\\\hline 11&14\\\cline{1-2} 12\\\cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&8\\\hline 3&7&13\\\hline 9&10&\\\hline 11&14\\\cline{1-2} 12\\\cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&8\\\hline 3&7&\\\hline 9&10&13\\\hline 11&14\\\cline{1-2} 12\\\cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6&\\\hline 3&7&8\\\hline 9&10&13\\\hline 11&14\\\cline{1-2} 12\\\cline{1-1}\end{array}
\to
\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&&6\\\hline 3&7&8\\\hline 9&10&13\\\hline 11&14\\\cline{1-2} 12\\\cline{1-1}\end{array}
$$

$$\rightarrow \begin{array}{ccc} 1 & & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 8 \\ 9 & 10 & 13 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} & 1 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 8 \\ 9 & 10 & 13 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 0 & 1 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 8 \\ 9 & 10 & 13 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 1 & 2 & 6 \\ 3 & 5 & 7 \\ 4 & 8 & 9 \\ 10 & 11 & 14 \\ 12 & 15 & \\ 13 & & \end{array}$$

If we label the boxes by (i, j), with i being the row index from top to bottom and j being the column index from left to right, and the northwest corner being labelled $(1, 1)$, then the promotion path of the above example is $[(4, 3), (3, 3), (2, 3), (2, 2), (1, 2), (1, 1)]$.

Promotion ∂ has a dual operation, called **dual-promotion**, denoted by ∂^* and defined as follows:

Definition 2.4. Find in T the inside corner that contains 1, and remove it to create an empty box. Apply jeu-de-taquin repeatedly to move the empty box southeast until it is an outside corner of λ. (We call this process **dual-sliding**, and the sequence of positions that the empty box moves along in this process is called the **dual-sliding path**). Place the number $n + 1$ in this outside corner. Now subtract one from each entry so that we again have a standard Young tableau. This new tableau is $\partial^*(T)$, the dual-promotion of T.

In the case that dual-sliding is used to define promotion, we will refer to the dual-sliding path as the **dual-promotion path**.

Example 2.5. Dual-promotion on standard tableaux.

$$\begin{array}{ccc} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 13 \\ 9 & 10 & 15 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 13 \\ 9 & 10 & 15 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 2 & 4 & 5 \\ & 6 & 8 \\ 3 & 7 & 13 \\ 9 & 10 & 15 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 2 & 4 & 5 \\ 3 & 6 & 8 \\ & 7 & 13 \\ 9 & 10 & 15 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 7 & & 13 \\ 9 & 10 & 15 \\ 11 & 14 & \\ 12 & & \end{array}$$

$$\rightarrow \begin{array}{ccc} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 7 & 10 & 13 \\ 9 & & 15 \\ 11 & 14 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 7 & 10 & 13 \\ 9 & 14 & 15 \\ 11 & & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 7 & 10 & 13 \\ 9 & 14 & 15 \\ 11 & 16 & \\ 12 & & \end{array} \rightarrow \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 7 \\ 6 & 9 & 12 \\ 8 & 13 & 14 \\ 10 & 15 & \\ 11 & & \end{array}$$

The dual-promotion path of the above example is $[(1, 1), (2, 1), (3, 1), (3, 2), (4, 2), (5, 2)]$.

Remark 2.6. It is easy to see that $\partial^* = \partial^{-1}$; thus, they are both bijections on $SYT(\lambda)$. Moreover, the the promotion path of T is the reverse of the dual-promotion path of $\partial(T)$.

Definition 2.7. Given $T \in SYT(\lambda)$ for any $\lambda \vdash n$, the **evacuation** action on T, denoted by $\epsilon(T)$, is described in the following algorithm:

Let $T_0 = T$ and $\lambda_0 = \lambda$, and let U be an "empty" tableau of shape λ. We will fill in the entries of U to get $\epsilon(T)$.

1. Apply sliding to T_k. The last box of the sliding path is an inside corner of λ_k; call this box (i_k, j_k). Fill in the number $k+1$ in the (i_k, j_k) box of U.

2. Remove (i_k, j_k) from λ_k to get λ_{k+1}, and remove the corresponding box and entry from T_k to get T_{k+1}.

3. Repeat steps (1) and (2) until $\lambda_n = \emptyset$ and U is completely filled. Then define $\epsilon(T) = U$.

Example 2.8. The following is a "slow motion" demonstration of the above process, where the T_k and U have been condensed. Bold entries indicate the current fillings of U.

$$T = \begin{ytableau} 1 & 3 & 8 \\ 2 & 4 \\ 5 & 9 \\ 6 & 10 \\ 7 \end{ytableau} \to \begin{ytableau} 1 & 3 & 8 \\ 2 & 4 \\ 5 & 9 \\ 6 & \ \\ 7 \end{ytableau} \to \begin{ytableau} 1 & 3 & 8 \\ 2 & 4 \\ 5 & \ \\ 6 & 9 \\ 7 \end{ytableau} \to \begin{ytableau} 1 & 3 & 8 \\ 2 & 4 \\ \ & 5 \\ 6 & 9 \\ 7 \end{ytableau} \to \begin{ytableau} 1 & 3 & 8 \\ \ & 4 \\ 2 & 5 \\ 6 & 9 \\ 7 \end{ytableau} \to \begin{ytableau} \ & 3 & 8 \\ 1 & 4 \\ 2 & 5 \\ 6 & 9 \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & 3 & 8 \\ 1 & 4 \\ 2 & 5 \\ 6 & 9 \\ 7 \end{ytableau}$$

$$\to \begin{ytableau} \mathbf{1} & 3 & 8 \\ 1 & 4 \\ 2 & 5 \\ 6 & \ \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & 3 & 8 \\ 1 & 4 \\ 2 & 5 \\ \ & 6 \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & 3 & 8 \\ 1 & 4 \\ \ & 5 \\ 2 & 6 \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & 3 & 8 \\ \ & 4 \\ 1 & 5 \\ 2 & 6 \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & 3 & 8 \\ \mathbf{2} & 4 \\ 1 & 5 \\ 2 & 6 \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & 3 & \ \\ \mathbf{2} & 4 \\ 1 & 5 \\ 2 & 6 \\ 7 \end{ytableau} \to \begin{ytableau} \mathbf{1} & \ & 3 \\ \mathbf{2} & 4 \\ 1 & 5 \\ 2 & 6 \\ 7 \end{ytableau}$$

$$\to \begin{ytableau} \mathbf{1} & \mathbf{3} & 3 \\ \mathbf{2} & 4 \\ 1 & 5 \\ 2 & 6 \\ 7 \end{ytableau} \to \cdots \to \begin{ytableau} \mathbf{1} & \mathbf{3} & \mathbf{8} \\ \mathbf{2} & \mathbf{5} \\ \mathbf{4} & \mathbf{6} \\ \mathbf{7} & \mathbf{10} \\ \mathbf{9} \end{ytableau} = \epsilon(T)$$

Remark 2.9. The above definition of evacuation follows the convention of Edelman and Greene in [3]. Stanley's "evacuation" [19, A1.2.8] would be our "dual-evacuation" defined below.

Definition 2.10. Given $T \in SYT(\lambda)$ for any $\lambda \vdash n$, the **dual-evacuation** of T, denoted by $\epsilon^*(T)$, is described in the following algorithm:

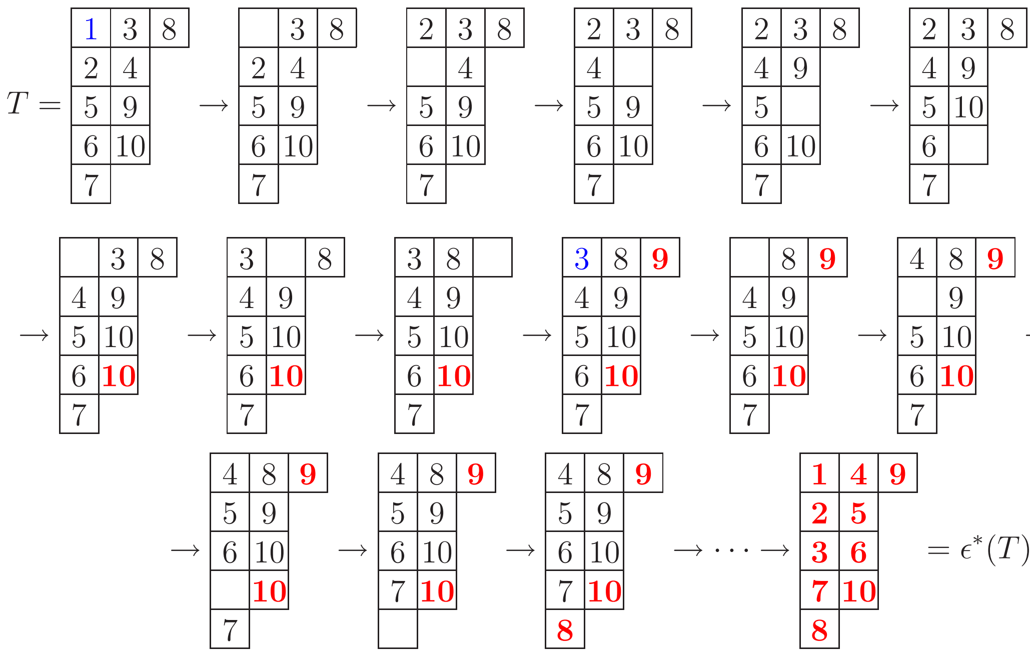
Let $T_0 = T$ and $\lambda_0 = \lambda$, and let U be an "empty" tableau of shape λ. We will fill in the entries of U to get $\epsilon^*(T)$.

1. Apply dual-sliding to T_k. The last box of the dual-sliding path is an outside corner of λ_k; call this box (i_k, j_k). Fill in the number $n-k$ in the (i_k, j_k) box of U.

2. Remove (i_k, j_k) from λ_k to get λ_{k+1}, and remove the corresponding box and entry from T_k to get T_{k+1}.

3. Repeat steps (1) and (2) until $\lambda_n = \emptyset$ and U is completely filled. Then define $\epsilon^*(T) = U$.

Example 2.11. The following is a "slow motion" demonstration of the above process, where the T_k and U have been condensed. Bold entries indicate the current fillings of U.



Remark 2.12. There is an equivalent definition of ϵ^* via the RSK algorithm [19, A1.2.10]. (Recall that Stanley's "evacuation" is our "dual-evacuation".) For a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ (in one-line notation), let $w^\sharp \in \mathfrak{S}_n$ be given by

$$w^\sharp = (n+1-w_n)\cdots(n+1-w_2)(n+1-w_1).$$

For example, in the case $w = 3547126$, $w^\sharp = 2671435$. The operation $w \to w^\sharp$ is equivalent to composing by the longest element in $\mathfrak{S}_n$. Then if w corresponds to (P, Q) under RSK, $w^\sharp$ corresponds to $(\epsilon^*(P), \epsilon^*(Q))$ under RSK. We are not aware of any RSK definition of ϵ for general shape λ.

Definition 2.13. For $T \in SYT(\lambda)$, i is a **descent** of T if $i+1$ appears strictly south of i in T. The **descent set** of T, denoted by $\mathrm{Des}(T)$, is the set of all descents of T.

Example 2.14. In the case that $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 9 \\ \hline 5 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$, $\mathrm{Des}(T) = \{3, 4, 6, 7\}$.

Remark 2.15. Descent statistics were originally defined on permutations. For $\pi \in \mathfrak{S}_n$, i is a **right descent** of π if $\pi(i) > \pi(i+1)$, and i is a **left descent** of π if i is to the right of $i+1$ in the one-line notation of π.

It is straightforward to check that left descents are preserved by Knuth equivalence. Therefore the descent set of any tableau T is the set of left descents of any reading word of T.

2.2 Basic facts

We list those basic facts of (dual-)promotion and (dual-)evacuation that we will assume. If not specified otherwise, the following facts are about $SYT(\lambda)$ for general $\lambda \vdash n$.

Fact 2.16. ϵ and ϵ^* are involutions.

Fact 2.17. $\epsilon \circ \partial = \partial^* \circ \epsilon$ and $\epsilon^* \circ \partial = \partial^* \circ \epsilon^*$.

Fact 2.18. $\epsilon \circ \epsilon^* = \partial^n$.

The above results are due to Schützenberger [14, 15]. Alternative proofs are given by Haiman in [6].

Fact 2.19. For any $R \in SYT(c^r)$, let $n = |c^r| = r \cdot c$. Then $\partial^n(R) = R$.

The above result is often attributed to Schützenberger.

Fact 2.20. On rectangular tableaux, $\epsilon = \epsilon^*$.

The above result is an easy consequence of Fact 2.18 and Fact 2.19.

Fact 2.21. For any $S \in SYT(sc_k)$, let $n = |sc_k| = k(k+1)/2$. Then $\partial^{2n}(S) = S$ and $\partial^n(S) = S^t$, where S^t is the transpose of S.

The above result is due to Edelman and Greene [3].

Fact 2.22. For any $S \in SYT(sc_k)$, $\epsilon^*(S) = \epsilon(S)^t$.

The above result is an easy consequence of Fact 2.16, Fact 2.18, and Fact 2.21.

3 The embedding of $SYT(sc_k)$ into $SYT(k^{(k+1)})$

In this section we describe the embedding $\iota : SYT(sc_k) \to SYT(k^{(k+1)})$.

Definition 3.1. Given $S \in SYT(sc_k)$, let $N = k(k+1)$. Construct $R = \iota(S)$ as follows:

- $R[i,j] = S[i,j]$ for $i + j \le k+1$ (northwest (upper) staircase portion).
- $R[i,j] = N+1-\epsilon(T)[k+2-i, k+1-j]$ for $i+j > k+1$ (southeast (lower) staircase portion).

This amounts to the following visualization:

Example 3.2. Let $S = \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4\\\cline{1-1}\end{array}$; then $\epsilon(S) = \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1}\end{array}$. Rotating $\epsilon(S)$ by π, we get $\begin{array}{ccc}\cline{3-3} &&\multicolumn{1}{|c|}{3}\\\cline{2-3} &\multicolumn{1}{|c|}{6}&\multicolumn{1}{c|}{2}\\\hline \multicolumn{1}{|c|}{5}&4&\multicolumn{1}{c|}{1}\\\hline\end{array}$. Now we take the complement of each filling by $N+1 = 13$ and get $S' = \begin{array}{ccc}\cline{3-3} &&\multicolumn{1}{|c|}{10}\\\cline{2-3} &\multicolumn{1}{|c|}{7}&\multicolumn{1}{c|}{11}\\\hline \multicolumn{1}{|c|}{8}&9&\multicolumn{1}{c|}{12}\\\hline\end{array}$.

There is an obvious way to put S and S' together to create a standard tableau of shape 3^4, which is $\iota(t) = \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5&10\\\hline 4&7&11\\\hline 8&9&12\\\hline\end{array}$.

Remark 3.3. Recall that $\epsilon^*(S) = \epsilon(S)^t$. Thus we could have computed $\epsilon^*(S) = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$, and flipped it along the staircase diagonal to get $\begin{array}{ccc}\cline{3-3} &&\multicolumn{1}{|c|}{3}\\\cline{2-3} &\multicolumn{1}{|c|}{6}&\multicolumn{1}{c|}{2}\\\hline \multicolumn{1}{|c|}{5}&4&\multicolumn{1}{c|}{1}\\\hline\end{array}$, which is the same as rotating $\epsilon(S)$ by π. This point of view manifests the fact that $n \in \mathrm{Des}(\iota(S))$ (Definition 4.1) if and only if the corner of n in $\epsilon^*(S)$ is southwest of the corner of n in S.

It is also an arbitrary choice to embed $SYT(sc_k)$ into $SYT(k^{(k+1)})$ instead of into $SYT((k+1)^k)$. For example, we could have put together the above S and S' to form

$$\begin{array}{|c|c|c|c|}\hline 1&2&6&10\\\hline 3&5&7&11\\\hline 4&8&9&12\\\hline\end{array}.$$

Our arguments below apply to either choice with little modification.

From the construction of ι, we see that $\iota(S)$ contains the upper staircase portion, which is just S, and the lower staircase portion, which is essentially $\epsilon(S)$. Therefore, we can just identify $\iota(S)$ with the pair $(S, \epsilon(S))$. We would like to understand how the promotion action on $\iota(S)$ factors through this identification. It is clear from the construction that promotion on $\iota(S)$, when restricted to the lower staircase portion, corresponds to dual-promotion on $\epsilon(S)$. If the promotion path in $\iota(S)$ passes through the box containing $n = k(k+1)/2$ (the largest number in the upper staircase portion of $\iota(S)$), then we know that promotion on $\iota(S)$, when restricted to the upper staircase portion, corresponds to promotion on S. The following arguments show that this is indeed the case.

Lemma 3.4. Let $T \in STY(\lambda)$, and $n = |\lambda|$. If the number n is in box (i, j) of T (clearly, it must be an outside corner), then the dual-promotion path of $\epsilon^*(T)$ ends on box (i, j) of $\epsilon^*(T)$.

Proof. It follows from the definition of dual-evacuation using dual-sliding that the position of n in $\epsilon^* \circ \partial(T)$ is the same as the position of n in T (because the sliding in the action of promotion and the first application of dual-sliding in the definition of dual-evacuation will "cancel out" with respect to the position of n). By the fact that $\epsilon^*(T) = \partial \circ \epsilon^* \circ \partial(T)$ (Fact 2.17) and the fact that the dual-promotion path of $\epsilon^*(T)$ is the reverse of the promotion path of $\epsilon^* \circ \partial(T)$ (Remark 2.6), the statement follows. $\square$

The above lemma, when specialized to staircase-shaped tableaux, implies the following:

Proposition 3.5. Let $S \in SYT(sc_k)$. The promotion path of $\iota(S)$ always passes through the box with entry $n = k(k+1)/2$.

Proof. Suppose n is in box (i, j) of $S \in SYT(sc_k)$. Since S is of staircase shape, we have $\epsilon^*(S) = \epsilon(S)^t$ (Fact 2.22). The above lemma then says the dual-promotion path of $\epsilon(S)$ ends on box (j, i) of $\epsilon(S)$, which is "glued" exactly below box (i, j) of S by the construction of ι. Now we use the observation that the promotion path of $\iota(S)$, when restricted to the lower staircase portion, corresponds to the dual-promotion path of $\epsilon(S)$. The result follows.

$\square$

This proves our first main result on the embedding ι.

Theorem 3.6. For $S \in SYT(sc_k)$, $\iota \circ \partial(S) = \partial \circ \iota(S)$.

By the above theorem and the definition of evacuation, we have that

Theorem 3.7. For $S \in SYT(sc_k)$, $\iota \circ \epsilon(S) = \epsilon \circ \iota(S)$.

Remark 3.8. It can be show either independently or as a corollary of Theorem 3.6 that

$$\iota \circ \partial^*(S) = \partial^* \circ \iota(S).$$

On the other hand, it is *not* true that $\iota \circ \epsilon^*(S) = \epsilon^* \circ \iota(S)$. On the contrary by Fact 2.20 we know that

$$\iota \circ \epsilon(S) = \epsilon^* \circ \iota(S).$$

It is not hard to see that

$$\iota \circ \epsilon^*(S) = \epsilon \circ \iota(S^t).$$

4 Descent vectors

4.1 Descent vectors of rectangular tableaux

Rhoades [11] invented the notion of "extended descent" in order to describe the promotion action on rectangular tableaux:

Definition 4.1. Let $R \in SYT(r^c)$, and $n = c \cdot r$. We say i is an **extended descent** of R if either i is a descent of R, or $i = n$ and 1 is a descent of $\partial(R)$. The **extended descent set** of R, denoted by $\mathrm{Des_e}(R)$, is the set of all extended descents of R.

Example 4.2. In the case that $R_1 = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & 7 \\ \hline 4 & 9 & 11 \\ \hline 8 & 10 & 12 \\ \hline \end{array}$, $\mathrm{Des_e}(R_1) = \{1, 3, 6, 7, 9, 11\}$. Here $12 \notin \mathrm{Des_e}(R_1)$ because 1 is not a descent of $\partial(R_1) = \begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 3 & 4 & 8 \\ \hline 5 & 6 & 10 \\ \hline 9 & 11 & 12 \\ \hline \end{array}$.

In the case that $R_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 9 \\ \hline 6 & 8 & 11 \\ \hline 7 & 10 & 12 \\ \hline \end{array}$, $\mathrm{Des_e}(R_2) = \{2, 4, 5, 6, 9, 11, 12\}$. Here $12 \in \mathrm{Des_e}(R_2)$ because 1 is a descent of $\partial(R_2) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline 7 & 9 & 10 \\ \hline 8 & 11 & 12 \\ \hline \end{array}$.

It is often convenient to think of $\mathrm{Des_e}(R)$ as an array of n boxes, where a dot is put at the i-th box of this array if and only if i is an extended descent of R. In this form, we will call $\mathrm{Des_e}(R)$ the **descent vector** of R. Furthermore, we identify ("glue together") the left edge of the left-most box and the right edge of the right-most box so that the array $\mathrm{Des_e}(R)$ forms a circle. It therefore makes sense to talk about rotating $\mathrm{Des_e}(R)$ to the right, where the content of the i-th box goes to the $(i+1)$-st box (mod n), or similarly, rotating to the left.

Example 4.3. Continuing the above example,

$$\mathrm{Des_e}(R_1) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet & & \bullet & & & \bullet & \bullet & & \bullet & & \bullet & \\ \hline \end{array}$$

and

$$\mathrm{Des_e}(R_2) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & \bullet & & \bullet & \bullet & \bullet & & & \bullet & & \bullet & \bullet \\ \hline \end{array}.$$

We would like to point out that the map $\mathrm{Des_e} : SYT(r^c) \to (0,1)^n$ is not injective and that the pre-images of $\mathrm{Des_e}$ are not equinumerous in general.

Rhoades [11] showed a nice property of the promotion action on the extended descent set. In the language of descent vectors, it has the following visualization:

Theorem 4.4 (Rhoades, [11])**.** If R is a standard tableau of rectangular shape, then the promotion ∂ rotates $\mathrm{Des_e}(R)$ to the right by one position.

Example 4.5. Continuing the above example, if $R_3 = \partial(R_2) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline 7 & 9 & 10 \\ \hline 8 & 11 & 12 \\ \hline \end{array}$ then

$$\mathrm{Des_e}(R_3) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet & & \bullet & & \bullet & \bullet & \bullet & & & \bullet & & \bullet \\ \hline \end{array}.$$

The action of evacuation ϵ on descent vectors is also very nice: (Note that dual-evacuation ϵ^* is the same as evacuation ϵ on rectangular tableaux.)

Theorem 4.6. Let $R \in SYT(r^c)$ and $n = c \cdot r$. Then evacuation ϵ rotates $\mathrm{Des_e}(R)$ to the right by one position and then flips the result of the rotation. More precisely, the i-th box of $\mathrm{Des_e}(\epsilon(R))$ is dotted if and only if the $(n-i)$-th (mod n) box of $\mathrm{Des_e}(R)$ is dotted.

Proof. We first note that $\epsilon(R) = \epsilon^*(R)$ (Fact 2.20). Then we note that $\mathrm{Des}(R)$ is the set of left descents of the column reading word w_R of R (Remark 2.15). Now, i is a left descent of w_R if and only if $n-i$ is a left descent of $w_R^\sharp$ (Remark 2.12). Therefore $i \in \mathrm{Des}(R)$ if and only if $n - i \in \mathrm{Des}(\epsilon(R))$.

If $n \in \mathrm{Des_e}(R)$, then $1 \in \mathrm{Des}(\partial(R))$ (Definition 4.1), thus $n-1 \in \mathrm{Des}(\epsilon \circ \partial(R))$ by the previous paragraph, thus $n-1 \in \mathrm{Des}(\partial^{-1} \circ \epsilon(R))$ (Fact 2.17), thus $n \in \mathrm{Des_e}(\epsilon(R))$ (Theorem 4.4). Since ϵ is an involution, the converse is also true. $\square$

Example 4.7. Continuing the above example, $\epsilon(R_3) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline 7 & 9 & 11 \\ \hline 8 & 10 & 12 \\ \hline \end{array}$ and

$$\mathrm{Des_e}(\epsilon(R_3)) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & \bullet & & & \bullet & \bullet & \bullet & & \bullet & & \bullet & \bullet \\ \hline \end{array}.$$

It is clear that this action is an involution.

4.2 Descent vectors of staircase tableaux

For staircase tableaux, we give the following construction of descent vectors.

Definition 4.8. Let $S \in SYT(sc_k)$ and $n = |sc_k| = k(k+1)/2$. Then $\mathrm{Des_e}(S)$ is an array of $2n$ boxes. The rules of placing dots into these boxes are the following.

- If $i \in Des(S)$, then put a dot in the i-th box and leave the $(n+i)$-th box empty.
- If $i \notin Des(S)$ and $i < n$, then put a dot in the $(n+i)$-th box and leave the i-th box empty.
- If $1 \in Des(\partial(S))$, then leave the n-th box empty and put a dot in the $(2n)$-th box.
- If $1 \notin Des(\partial(S))$, then leave the $(2n)$-th box empty and put a dot in the n-th box.

We identify the left edge and the right edge of this array.

Example 4.9. In the case that $S_1 = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$,

$$\mathrm{Des_e}(S_1) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet & \bullet & & & \bullet & \bullet & & & \bullet & \bullet & & \\ \hline\end{array}.$$

In the case that $S_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$,

$$\mathrm{Des_e}(S_2) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & \bullet & \bullet & & \bullet & & \bullet & & & \bullet & & \bullet \\ \hline\end{array}.$$

As in the case of rectangular tableaux, the map $\mathrm{Des_e}$ is not injective and the pre-images of $\mathrm{Des_e}$ are not equinumerous in general.

From the definition, we see that the first half and the second half of $\mathrm{Des_e}(S)$ are just complements of each other, that is, for each $i \in [n]$ precisely one of the i-th and $(n+i)$-th boxes is dotted. Thus the second half of $\mathrm{Des_e}(S)$ is redundant. On the other hand, this redundancy demonstrates the link between $\mathrm{Des_e}(S)$ and $\mathrm{Des_e}(\iota(S))$ as stated in Theorem 4.11. First, we need a supporting lemma, whose proof is not hard but rather tedious, so we leave it to the appendix.

Lemma 4.10. Let $S \in SYT(sc_k)$ and $n = |sc_k|$. If the promotion path of S ends with a vertical (up) move, then the corner of n in $\epsilon^*(S)$ is northeast of the corner of n in S. If the promotion path of S ends with a horizontal (left) move, then the corner of n in $\epsilon^*(S)$ is southwest of the corner of n in S.

Theorem 4.11. For $S \in SYT(sc_k)$, $\mathrm{Des_e}(S) = \mathrm{Des_e}(\iota(S))$.

Proof. Parsing through the construction of ι, we see that this claim is the conjunction of the following two statements:

1. for $i \neq n$, $i \in \mathrm{Des}(S)$ if and only if $n - i \notin \mathrm{Des}(\epsilon(S))$; and

2. $1 \in \mathrm{Des}(\partial(S))$ if and only if $n \notin \mathrm{Des}(\iota(S))$.

For the first statement, we note that $i \in \mathrm{Des}(S)$ is equivalent to that i is a left descent of a reading word w_S of S (Remark 2.15), which is equivalent to that $n-i$ is a left descent of the word $w_s^\sharp$ (Remark 2.12), which is equivalent to that $n - i$ is a descent in $\epsilon^*(S)$, which is equivalent to that $n - i$ is not a descent of $\epsilon(S)$ (Fact 2.22).

Now, $1 \in \mathrm{Des}(\partial(S))$ is equivalent to that the promotion path of S ends with a vertical (up) move, which is equivalent to that the corner n in $\epsilon^*(S)$ is northeast of the corner of n in S by Lemma 4.10 , which is equivalent to that $n \notin \mathrm{Des}(\iota(S))$ (Remark 3.3).

□

The above Theorems 4.11, 3.6 and 4.4 imply the following analogy to Theorem 4.4 for staircase tableaux:

Theorem 4.12. If S is a standard tableau of staircase shape, then promotion ∂ rotates $\mathrm{Des}_{\mathrm{e}}(S)$ to the right one position.

Note that if we rotate $\mathrm{Des}_{\mathrm{e}}(S)$ in any direction by n positions we get the complement of $\mathrm{Des}_{\mathrm{e}}(S)$, which is $\mathrm{Des}_{\mathrm{e}}(S^t)$. This agrees with Edelman and Greene's result [3] that $\partial^n(S) = S^t$ and $\partial^n = \partial^{-n}$.

Unlike the case of rectangular tableaux, evacuation ϵ and dual-evacuation ϵ^* act differently on staircase tableaux. Their actions on descent vectors are described below:

Theorem 4.13. Let $S \in SYT(sc_k)$ and $n = k(k+1)/2$. Then evacuation ϵ rotates $\mathrm{Des}_{\mathrm{e}}(S)$ to the right by one position and then flips the result of the rotation. More precisely, the i-th box of $\mathrm{Des}_{\mathrm{e}}(\epsilon(S))$ is dotted if and only if the $(2(n-i))$-th box of $\mathrm{Des}_{\mathrm{e}}(S)$ is dotted.

Theorem 4.14. Let $S \in SYT(sc_k)$ and $n = k(k+1)/2$. Then dual-evacuation ϵ^* rotates $\mathrm{Des}_{\mathrm{e}}(S)$ to the right by $n-1$ positions and then flips the result of the rotation. More precisely, the i-th box of $\mathrm{Des}_{\mathrm{e}}(\epsilon(S))$ is dotted if and only if the $(n-i)$-th box of $\mathrm{Des}_{\mathrm{e}}(S)$ is dotted.

Example 4.15. Let $S_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$, then $\epsilon(S_3) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ and $\epsilon^*(S_3) = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$.

Correspondingly,

$$\mathrm{Des}_{\mathrm{e}}(S_3) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & \bullet & & \bullet & & & \bullet & & \bullet & & \bullet & \bullet \\ \hline\end{array},$$

$$\mathrm{Des}_{\mathrm{e}}(\epsilon(S_3)) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet & & \bullet & & \bullet & & & \bullet & & \bullet & & \bullet \\ \hline\end{array},$$

and

$$\mathrm{Des}_{\mathrm{e}}(\epsilon^*(S_3)) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & \bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & \\ \hline\end{array}.$$

Note that $\mathrm{Des}_{\mathrm{e}}(\epsilon(S_3))$ is the complement of $\mathrm{Des}_{\mathrm{e}}(\epsilon^*(S_3))$. This agrees with the fact that $\epsilon^*(S_3) = \epsilon(S_3)^t$.

Proof of Theorems 4.13 and 4.14. Theorem 4.13 follows directly from Theorem 4.11 and Theorem 3.7.

Theorem 4.14 follows from the fact that $\mathrm{Des}_{\mathrm{e}}(S)$ is the complement of $\mathrm{Des}_{\mathrm{e}}(S^t)$. $\square$

Theorems 4.4 and 4.12 imply that if T is either a rectangular or staircase tableau, $\mathrm{Des}_{\mathrm{e}}(T)$ encodes important information about the promotion cycle that T is in.

Corollary 4.16. If T, either of rectangular or staircase shape, is in a promotion cycle of size C then $\mathrm{Des}_{\mathrm{e}}(T)$ must be periodic with period dividing C. (The period does not have to be exactly C.)

Example 4.17. Let $T = \begin{array}{|c|c|c|}\hline 1&5&9\\ \hline 2&6&10\\ \hline 3&7&11\\ \hline 4&8&12\\ \hline \end{array}$, then

$$\mathrm{Des_e}(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet&\bullet&\bullet& &\bullet&\bullet&\bullet& &\bullet&\bullet&\bullet& \\ \hline \end{array}.$$

We see that $\mathrm{Des_e}(T)$ has a period of 4, thus T must be in a promotion cycle of size either 4 or 12. Indeed, the promotion order of T is 4.

On the other hand, the promotion order of $T = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&7&9\\ \hline 4&8&11\\ \hline 6&10&12\\ \hline \end{array}$ is also 4, while its descent vector

$$\mathrm{Des_e}(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet& &\bullet& &\bullet& &\bullet& &\bullet& &\bullet& \\ \hline \end{array}$$

has period 2.

Equipped with the above knowledge, we can say more about the promotion action on $SYT(sc_k)$. For example:

Corollary 4.18. In the promotion action on $SYT(sc_k)$ there always exists a full cycle, that is, a cycle of the same size as the order of the promotion ∂, in this case $k(k+1)$.

Proof. Consider $T \in SYT(sc_k)$ obtained by filling the numbers 1 to $k(k+1)/2$ down columns, from the leftmost column to the rightmost column. Then $\mathrm{Des_e}(T)$ has period $k(k+1)$, thus T must be in a full cycle. $\square$

For example, for $k = 3$, and $T = \begin{array}{|c|c|c|}\hline 1&4&6\\ \hline 2&5\\ \cline{1-2} 3\\ \cline{1-1} \end{array}$,

$$\mathrm{Des_e}(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \bullet&\bullet& &\bullet& &\bullet& & &\bullet& &\bullet& \\ \hline \end{array}$$

has period 12.

Indeed, computer experiments show that "most" cycles of the promotion action are full cycles. For example, for $k = 3$, there is 1 full-cycle (of size 12) and 1 cycle of size 4. For $k = 4$, there are 38 full-cycles (of size 20) and 2 cycles of size 4. For $k = 5$, there are 9756 full-cycles (of size 30), 16 cycles of size 10, and 4 cycles of size 6. For $k > 5$, the exact promotion cycle structure is not known. We are tempted to guess that as $k \to \infty$, almost all cycles are full-cycles. (Note that this is stronger than stating that almost all staircase tableaux *live in* some full-cycle.)

Corollary 4.19. In the promotion action on $SYT(sc_k)$, let $N = k(k+1)$. If a cycle of length C appears, then C is a divisor of N, but not a divisor of $N/2$.

Proof. The cycle size C is a divisor of N since the order of promotion $|\partial|$ is N.

On the other hand, C cannot be a divisor of $N/2$ since by definition $\mathrm{Des_e}(T)$ can never have period of length that is a divisor of $N/2$. $\square$

5 Some comments and questions

The discovery of ι is a by-product of our attempt to solve an open question posed by Stanley ([20, page 13]) that asks if Rhoades' CSP result on rectangular tableaux can be extended to other shapes, and if there is a more combinatorial proof of this result.

Rhoades' proof uses Kazhdan-Lusztig theory, requiring special properties of rectangular tableaux. It seems that there is not an obvious analogous proof for other shapes.

We decided to try our luck in computer exploration using Sage-Combinat ([22], [13]). The first thing we noticed from the computer data was the nice promotion cycle structure of staircase tableaux, which is not a surprise at all due to Fact 2.21. Thus we decided to focus on staircase tableaux.

It was soon clear to us that brute-force computation of the cycle structure could not proceed very far; we could only handle $SYT(sc_k)$ for $k \leq 5$ on our computer. On the other hand, the promotion cycle structures on rectangular tableaux are extremely easy to compute by Rhoades' result, as the generating function of $\overline{\mathrm{maj}}$ is the $q-$analogue of the hook length formula. So the embedding ι is an effort to study the promotion cycle structure on $SYT(sc_k)$ by borrowing information from the promotion action on $SYT(k^{k+1})$.

Among the cases of promotion action on $SYT(sc_k)$ for which we know the complete cycle structure (that is, $k = 3, 4, 5$), we have found that each has a CSP polynomial that is a product of cyclotomic polynomials of degree $\leq k(k+1)$: For $k = 3, 4, 5$, these polynomials are

$$\begin{gathered}\Phi_2\Phi_4^2\Phi_6\Phi_8\Phi_{12},\\ \Phi_2^3\Phi_3\Phi_4^2\Phi_8\Phi_{10}^2\Phi_{16}\Phi_{20},\ \text{and}\\ \Phi_2^{11}\Phi_6\Phi_{10}^3\Phi_{11}\Phi_{13}\Phi_{22}\Phi_{24}^4\Phi_{30},\end{gathered}$$

respectively.

We note that these polynomials in product form are not unique, for example

$$\Phi_2^2\Phi_4\Phi_6\Phi_{10}\Phi_{12}$$

gives another CSP polynomial for $SYT(sc_3)$.

The study of this product form continues, with the hope of finding a counting formula, the $q-$analogue of which is a CSP polynomial for the promotion action on $SYT(sc_k)$.

For the case $k > 5$, Corollary 4.19 gives a necessary condition for what kind of cycles can appear in the promotion action on $SYT(sc_k)$. We do not know if this condition is sufficient.

We are also eager to know if the embedding ι has any representation-theoretical interpretation.

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A Proof of Lemma 4.10

To prove Lemma 4.10, we first make the observation that the location of the corner that contains n (the n-corner) in S cannot be the same as the n-corner in $\epsilon^*(S)$. This is because, as we had observed in Lemma 3.4, the n-corner in S is the same as the n-corner in $\epsilon^* \circ \partial(S)$; and $\epsilon^*(S) = \partial \circ \epsilon^* \circ \partial(S)$ does not have the same n-corner as that of $\epsilon^* \circ \partial(S)$.

With this observation, Lemma 4.10 is a consequence of the following general fact:

Lemma A.1. Let $T \in SYT(\lambda)$ and $\lambda \vdash n$. If the promotion path of T ends with a vertical (up) move, then the whole dual-promotion path of T must be (weakly) northeast of the promotion path.

If the promotion path of T ends with a horizontal (left) move, then the whole dual-promotion path of T must be (weakly) southwest of the promotion path.

Proof. Without loss of generality, we argue the case where the promotion path of T ends with a vertical move.

Imagine a boy and a girl standing at the most northwest box of T. The boy will walk along the promotion path in reverse towards the southeast, and the girl will walk along the dual-promotion path towards the southeast. They will walk at the same speed.

In the first step, the boy goes south by assumption. The girl may go east or south. If she starts by going east, then she is already strictly northeast of the boy. If she starts by going south with the boy, then she must turn east earlier than the boy turns. (Suppose the boy turns east at box (i, j). By definition of promotion path, this implies that $T[i, j] > T[i-1, j+1]$. If the girl goes south at box $(i-1, j)$ then by definition of dual-promotion path, this implies that $T[i, j] < T[i-1, j+1]$, a contradiction. Therefore, the girl must turn east at box $(i-1, j)$ or earlier.) So either way we see that the girl will be strictly northeast of the boy before the boy makes his first east turn.

If they never meet again then we are done. So we assume that their next meeting position is at the box (s, t), and argue that they will never cross. By induction this will prove the claim.

It is clear that the girl must enter the box (s, t) from north, and the boy must enter the box (s, t) from the west. From box (s, t), the girl can either go south or go east.

Suppose the girl goes south from (s, t). Then the boy must also go south from (s, t). (For if he went east, it would imply that $T[s-1, t+1] < T[s, t]$, which would make the girl go through $(s-1, t+1)$ instead of (s, t).) Then we can use our previous argument to show that the girl must make an east turn before the boy, and stay northeast of the boy.

Suppose the girl goes east from (s, t). Then again the boy must go south from (s, t). (For the girl's behaviour shows that $T[s-1, t+1] > T[s, t]$, but the boy's going east would imply that $T[s-1, t+1] < T[s, t]$.) So the girl stays northeast of the boy. □

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