The Signless Laplacian Spectral Radius of Unicyclic and Bicyclic Graphs with a Given Girth *

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Abstract

Let $\mathcal{U}(n,g)$ and $\mathcal{B}(n,g)$ be the set of unicyclic graphs and bicyclic graphs on n vertices with girth g, respectively. Let $\mathcal{B}_1(n,g)$ be the subclass of $\mathcal{B}(n,g)$ consisting of all bicyclic graphs with two edge-disjoint cycles and $\mathcal{B}_2(n,g) = \mathcal{B}(n,g)\backslash\mathcal{B}_1(n,g)$. This paper determines the unique graph with the maximal signless Laplacian spectral radius among all graphs in $\mathcal{U}(n,g)$ and $\mathcal{B}(n,g)$, respectively. Furthermore, an upper bound of the signless Laplacian spectral radius and the extremal graph for $\mathcal{B}(n,g)$ are also given.

Keywords: Unicyclic graphs; Bicyclic graphs; Signless Laplacian spectral radius; Girth

1 Introduction

Throughout the paper, let $G = (V, E)$ be a connected undirected simple graph with $V = V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E = E(G) = \{e_1, e_2, \cdots, e_m\}$. The order of a graph is the cardinality of its vertex set. Especially, if $m = n$ or $m = n + 1$, then G is called a unicyclic or bicyclic graph, respectively. The girth $g = g(G)$ of G is the length of the shortest cycle in G. Denote by $d(u, v)$ the distance between the vertices u and v of G, which is the length of the shortest path joining the vertex u with v. Suppose that

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$U \subseteq V(G)$, $u \in V(G)$, and $u \notin U$, the distance between u and U, denoted by $d(u, U)$, is the minimum distance between u and a vertex in U. Let $\Delta = \Delta(G)$ be the maximum degree of G. Let $A(G)$ and $D(G)$ be the adjacency matrix and diagonal matrix of vertex degree of G, respectively. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. The Laplacian spectral radius $\lambda_1(G)$ is the largest eigenvalue of $L(G)$ and the signless Laplacian spectral radius or Q-spectral radius $q_1(G)$ is the largest eigenvalue of $Q(G)$. Moreover, if G is connected, by the Perron-Frobenius Theorem, we have that Q-spectral radius is simple and has a unique unit positive eigenvector. We refer to such an eigenvector as Perron vector of G.

The adjacency matrix $A(G)$ and Laplacian matrix $L(G)$ are studied extensively and the main results are referred to [1] and [14], respectively. Recently, the problem about determining the extremal graphs with the maximal signless Laplacian spectral radius for a class of graphs attracts people's attention. Some properties of signless Laplacian spectra of graphs and some possibilities for developing the spectral theory of graphs based on $Q(G)$ are discussed in [3, 4, 5]. Fan and Yang studied the signless Laplacian spectral radius of graphs with a given number of pendent vertices in [9]. Feng and Yu studied the signless Laplacian spectral radius of unicyclic graphs with a given number of pendent vertices or independence number in [7]. Liu, Tan and Liu studied the (signless) Laplacian spectral radius of unicyclic and bicyclic graphs with n vertices and k pendent vertices in [12]. Zhai, Yu and Shu determined the extremal graph with the maximal Laplacian spectral radius among all bicyclic graphs with a given girth in [15]. In this paper, we determine the unique graph with maximal signless Laplacian spectral radius among all unicyclic and bicyclic graphs with a given girth g, respectively. Furthermore, the upper bound of the signless Laplacian spectral radius and the the extremal graph for all bicyclic graphs with a given girth g are also obtained.

Let $\mathcal{U}(n, g)$ and $\mathcal{B}(n, g)$ be the set of unicyclic and bicyclic graphs on n vertices with a fixed girth g, respectively. Denote by $\mathcal{B}_1(n, g)$ the subclass of $\mathcal{B}(n, g)$ consisting of all bicyclic graphs with two edge-disjoint cycles and by $\mathcal{B}_2(n, g) = \mathcal{B}(n, g) \backslash \mathcal{B}(n, g)$ consisting of bicyclic graphs with three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints. Let P_n (resp.C_n) be the path (cycle) on n vertices. Denote by $B_{p,q}^k$ the graph obtained from two disjoint cycles C_p and C_q by identifying a vertex u of C_p with a vertex v of C_q and attaching k pendent edges to $u(v)$. Denote $P_{p,q,r}^k$ the graphs consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints and k pendent edges at one of the common endpoints, which is shown in Figure 1.

2 Preliminaries

Let G be a bicyclic graph. The the base of G, denoted by $B(G)$, is the minimal bicyclic subgraph of G. Clearly, $B(G)$ is the unique bicyclic subgraph of G containing no pendent vertices, and G can be obtained from $B(G)$ by planting trees to some vertices of $B(G)$. A hanging tree of v in G, denoted by $T(v)$, is a rooted tree with v as its root vertex. For a better classification, in the following discussion, a vertex set of a hanging tree does not include its root vertex.

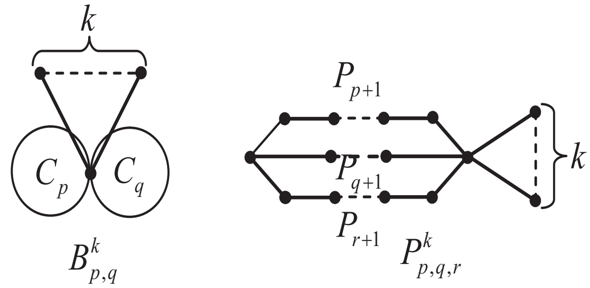

Figure 1: The bicyclic graphs $B^k_{p,q}$ and $P^k_{p,q,r}$.

Bicyclic graphs have two types of bases which are shown in Figure 2. Denote by $B(p,l,q)$ the graph obtained by joining a new path $u_1u_2\cdots u_l$ between two vertex-disjoint cycles C_p and C_q, where $u_1 \in V(C_p)$ and $u_l \in (C_q)$. In particular, $B(p,1,q) \cong C_puvC_q$ for $u \in V(C_p)$ and $u \in V(C_q)$, where C_puvC_q denotes the graph obtained from C_p and C_q by identifying a vertex u of C_p with a vertex v of C_q. Denote by $P(p,q,r)$ the graph consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints, that is, $P(p,q,r) \cong P^0_{p,q,r}$.

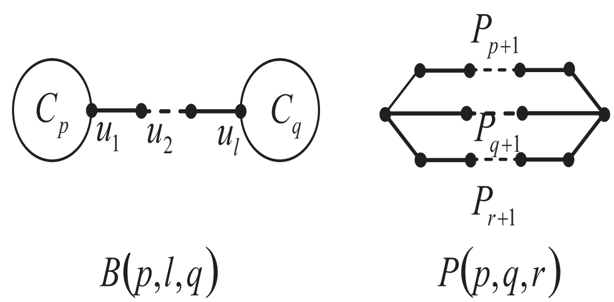

Figure 2: The bases of bicyclic graphs.

Clearly, $\mathcal{B}_1(n,g)$ and $\mathcal{B}_2(n,g)$ can also be defined as follows:
$\mathcal{B}_1(n,g) = \{G \in \mathcal{B} | B(G) = B(p,l,q)$ for some $l \geq 1$ and $p,q \geq 3\}$.
$\mathcal{B}_2(n,g) = \{G \in \mathcal{B} | B(G) = P(p,q,r)$ for some $p,q,r \geq 1\}$.

Lemma 2.1. [13] Let G be a graph on n vertices. Then $q_1(G) \leq \max\{d_u + m_u : u \in V(G)\}$, where $m_u = (\sum_{uv\in E(G)} d_v)/d_u$ is the average degree of neighbors of u, the equality holds if and only if G is regular or semi-regular bipartite.

Lemma 2.2. [11] Let G be a connected graph and u, v be the two vertices of G. Suppose $v_1, v_2, \cdots, v_s \in N(v)\backslash\{N(u) \cup u\}(1 \leq s \leq d_v)$ and G^* is the graph obtained from G by deleting the edges vv_i and adding $uv_i(1 \leq i \leq s)$. Let $X = (x_1, x_2, \cdots, x_n)^t$ be the principal eigenvector of $Q(G)$, where x_i corresponds to $v_i (1 \leq i \leq n)$. If $x_u \geq x_v$, then $q_1(G) < q_1(G^*)$.

Now we consider the graph G_{uv} obtained from a connected graph G by subdividing the edge uv, that is, by replacing edge uv with edges uw, wv, where w is an additional vertex. We call the following two types of paths internal paths: (a) A sequence of vertices $v_0, v_1, \cdots, v_k (k \geq 2)$, where $v_0, v_1, \cdots, v_k$ are distinct and $v_0 = v_{k+1}$ of degree at least 3, $d_{v_i} = 2$, for $i = 1, 2, \cdots, k$ and $v_{i-1}, v_i (i = 1, 2, \cdots, k+1)$ are adjacent; (b) A sequence of vertices $v_{0,1}, \cdots, v_k (k \geq 0)$ such that v_{i-1}, v_i are adjacent, $d_{v_0} \geq 3, d_{v_{k+1}} \geq 3$ and $d_{v_i} = 2$ where $1 \leq i \leq k$.

Lemma 2.3. [3, 8] Let G be a connected graph and uv be some edge on the internal path of G as we defined above. If we subdivide uv, that is, substitute it by uw, wv with a new vertex w, and denote the new graph by G_{uv}, then $q_1(G_{uv}) < q_1(G)$.

Lemma 2.4. [3] Suppose G is a nontrivial simple and connected graph. Let v be some vertex of G. For nonnegative integers k, l, let $G(k, l)$ denote the graph obtained from G by adding pendant paths of length k, l at v. If $k \geq l \geq 1$, then $q_1(G(k, l)) > q_1(G(k+1, l-1))$.

Lemma 2.5. [6] Let G be a connected graph. Suppose v_1, v_2 are vertices each of degree at least 3 and v_1v_2 is an edge of G. Let G^* be the connected graph obtained from G by contracting v_1v_2, that is, deleting the edge and identifying vertices v_1, v_2. Then $q_1(G) < q_1(G^*)$.

Lemma 2.6. [2] Let G be a graph on n vertices with at least an edge and the maximum degree of G be Δ. Then we have $q_1(G) \geq \Delta + 1$. The equality holds if only if G is a star.

Lemma 2.7. [13] Let G be a simple and connected graph on n vertices, its degree sequence is $d_{v_1}, d_{v_2}, \cdots, d_{v_n}$. Then we have

(1) $q_1(G) \leq \max\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v} : uv \in E\}$.

(2) $q_1(G) \leq \max\{d_u + d_v : uv \in E\}$.

Lemma 2.8. [10] Let G be a connected graph. Suppose v_1, v_k are vertices each of degree at least 3 and $N(v_1) \cap N(v_k) = \emptyset$. Suppose further that the unique path $P = v_1v_2 \cdots v_k$ from v_1 to v_k is an internal path. Let G' be a connected graph obtained from G by collapsing the entire internal path (*i.e.*, delete all edges $v_1v_2, v_2v_3, \cdots, v_{k-1}v_k$ and identify the vertices $v_1, v_2, \cdots, v_k$). Then $q_1(G) < q_1(G')$.

Lemma 2.9. [6] Let G be a connected graph and P be a pendant path in G. Suppose e is an edge in P and G^* is the graph obtained from G by subdividing e, then we have $q_1(G) < q_1(G^*)$.

3 Main Results

In this section, the extremal graphs with the maximal signless Laplacian spectral radius among all graphs in $\mathcal{U}(n, g)$ and $\mathcal{B}(n, g)$ and the upper bound of the signless Laplacian spectral radius for $\mathcal{B}(n, g)$ will be presented.

Theorem 3.1. For every pair of positive integers n, g with $3 \leq g \leq n-1$, the graph $U_{g,n-g}$ is the extremal graph with the maximal signless Laplacian spectral radius among all unicyclic graphs with a given girth g, where the graph $U_{g,n-g}$ is obtained from the cycle C_g by attaching $n-g$ pendent edges to some vertex of the cycle.

Proof. For any unicyclic graph $G \in \mathcal{U}(n, g)$, we only need to consider the trees attached to some vertices on the cycle C_g of G. Firstly, if there exists at least one vertex on the hanging trees having degree at least 3, we choose the vertex w which has the minimal value $d(w, C_g)$ among all such vertices described above, then by Lemma 2.8 collapse the entire internal path joining the vertex w with the the vertex on the cycle C_g, then by Lemma 2.9 we get a new graph G' by subdividing some pendent edges several times to keep the order of G' the same as the order of G. Through these edges operations above, G' has a larger signless Laplacian spectral radius and smaller number of the vertices with degree at least 3 on the trees. Continuing the operations as above, we can get all trees attached consist of pendent paths. Secondly, by Lemma 2.2 comparing the eigencomponents corresponding to the root vertices attached by pendent paths and reattaching all the pendent paths to the root vertex corresponding to the largest eigencomponent. Finally, by Lemma 2.4, all pendent paths with the length equal to or greater than 2 are transformed into pendent edges with the same number of vertices as the order of all the original pendent paths, which results in a graph with a larger signless Laplacian spectral radius. Then the proof of the Theorem 3.1 is completed. $\square$

Theorem 3.2. For every pair of positive integers n, g with $3 \leq g \leq \frac{n+1}{2}$, $B_{g,g}^{n-2g+1}$ is the extremal graph with the maximal signless Lapalcian spectral radius among all graphs in $\mathcal{B}_1(n, g)$.

Proof. As the discussion of the proof of Theorem 3.1, we know that the graph G^* with the maximal signless Laplacian spectral radius among all graphs in $\mathcal{B}_1(n, g)$ is obtained from $B(G^*)$ by attaching some pendent edges to some vertex v of $B(G^*)$. $B(G^*)$, as we define above, has the same form as $B(p, l, q)$, where a path $P = u_1 u_2 \cdots u_l$ joining the vertex u_1 on the cycle C_g with the vertex u_l on the cycle C_q for $l \geq 1$. Firstly, it suffices to show $u_1 = u_l = u$, namely $l = 1$. Assume to the contrary that $l \geq 2$. By Lemmas 2.4 and 2.6, when we delete the whole path P and identify all the vertices into a vertex u, then by Lemmas 2.9 and 2.4, we will get a graph G' with $q_1(G') \geq q_1(G^*)$, where G' is obtained by attaching $n-(g+q)+1$ pendent edges to some vertex v of $B(g, 1, q)$. This is a contradiction. Secondly, we can versify that $q = g$ in the graph G^*. If $q \geq g+1$, then by Lemma 2.3 we can transform the cycle C_q into a cycle having the same length g, then by Lemmas 2.9 and 2.4 we can get a graph G' with $q_1(G') \geq q_1(G^*)$. This is a contradiction. Finally, by Lemma 2.2, we show $u = v$. Assume that $u \neq v$. Let $N(u) = \{w_1, w_2, \cdots, w_t, \cdots, w_{d_u}\}$, let t be the number of pendant vertices of the neighbors of u, $N(v) = \{s_1, s_2, \cdots, s_{d_v}\}$, $s_1, s_2 \in V(C_p)$ and $d_v = 4$. If $x_u \geq x_v$, then we can get a new graph $G' = G^* - \{vs_1, vs_2\} + \{us_1, us_2\}$ with $q_1(G^*) < q_1(G')$. This is a contradiction. If $x_v > x_u$, then we also can get a new graph $G'' = G^* - \{uw_1, uw_2, \cdots, uw_t\} + \{vw_1, vw_2, \cdots, vw_t\}$ with $q_1(G^*) < q_1(G'')$. This is also a contradiction. Hence the proof of Theorem 3.2 is completed. $\square$

Theorem 3.3. For every pair of positive integers n, g with $3 \leq g \leq \frac{2(n+1)}{3}$, $P^{n-\lceil 3g/2\rceil}_{\lfloor g/2\rfloor,\lceil g/2\rceil,\lceil g/2\rceil}$ is the extremal graph with the maximal signless Laplacian spectral radius among all graphs in $\mathcal{B}_2(n,g)$.

Proof. We discuss the two facts as follows:

Fact 1. Let G^* have the maximal signless Laplacian spectral radius among all graphs in $\mathcal{B}_2(n,g)$. When $n \geq \lceil 3g/2\rceil - 1$, then G^* is the graph obtained from $P(\lfloor g/2\rfloor, \lceil g/2\rceil, \lceil g/2\rceil)$ by attaching $n - \lceil 3g/2\rceil + 1$ pendent edges to a unique vertex.

Proof of Fact 1. The proof need distinguish two case that g is even or odd.

Case 1. We firstly consider the case that g is even. Set $a = g/2$ and suppose that $B(G^*) = P(p,q,r)$, where $p \leq q \leq r$ and $p+q = 2a$. It suffices to show that $p = q = r = a$. Note $p+q+r \geq 3a$ and $n \geq (p+q+r) - 1$.

If $n = 3a-1$, then G^* can not contain any pendent vertices and $p+q+r = 3a$. Since $p+q = 2a$, $r = a$ and hence $p \leq q \leq a$. If $p \leq a-1$, then $q \geq a+1$, which contradicts to $q \leq a$. Thus $p = q = r = a$.

If $n = 3a$, then $p+q+r \leq 3a+1$. Since $p+q = 2a$, $r \leq a+1$. This implies that $q \leq a+1$. So $(p,q,r) \in \{(a,a,a),(a,a,a+1),(a-1,a+1,a+1)\}$. Assume, for a contradiction, that $(p,q,r) \in \{(a,a,a+1),(a-1,a+1,a+1)\}$, then G^* can not contain any pendent edges, in other words, G^* is isomorphic to one of $P(a,a,a+1)$ and $P(a-1,a+1,a+1)$. When $a = 2$, straightforward calculations show that $q_1(P(2,2,3)) = 4.932$, $q_1(P(1,3,3)) = 5$ and $q_1(P^1_{2,2,2}) = 5.5141$. So we have $\max\{q_1(P(2,2,3)), q_1(P(1,3,3))\} < q_1(P^1_{2,2,2})$, a contradiction. When $a \geq 3$, G^* can not contain a pair of adjacent 3-vertices. So if by Lemma 2.7,

$$q_1 \leq \max\{d_u + d_v : uv \in E(G)\} = 5.$$

However, $q_1(P^1_{a,a,a}) > \Delta+1 = 5$ since in this case G^* is not a star. Thus $q_1(G^*) < q_1(P^1_{a,a,a})$, a contradiction. Therefore, $p = q = r = a$.

Now it remains the case $n \geq 3a+1$. If $r = a$, clearly $p = q = a$ since $p+q = 2a$ and $p \leq q \leq a$. Next we suppose that $r \geq a+1$,and set $k = n-(p+q+r)+1$, namely the number of pendent vertices in G^*. Then $k \leq n-3a$. If k is fixed, then we can find that $\max\{d_u + m_u | u \in V(G)\}$ attains the maximal value just when $p = 1$ and k pendent edges are incident to a 3-vertex of $P(p,q,r)$. In this case,

$$\max\{d_u + m_u | u \in V(G)\} = k+3+\frac{k+7}{k+3} = k+4+\frac{4}{k+3}.$$

By Lemma 2.7, we have

$$q_1(G^*) < \max\{d_u + m_u | u \in V(G)\} = k+4+\frac{4}{k+3}$$

since in this case G^* can not neither be regular nor semi-regular. Note that $k+4+\frac{4}{k+3}$ is increasing with nonnegative integer k. Thus

$$q_1(G^*) < n-3a+4+\frac{4}{n-3a+3} \leq n-3a+5$$

since $n \geq 3a+1$. However, by Lemma 2.1, $q_1(P_{a,a,a}^{n-3a+1}) > \Delta + 1 = n - 3a + 5$, a contradiction.

Case 2. Now we consider the case that g is odd. The proof is similar to Case 1. Suppose that $B(G^*) = P(p,q,r)$, where $p \leq q \leq r$ and $p + q = g$. It suffices to show that $p = \lfloor g/2 \rfloor = \frac{g-1}{2}$, $q = r = \lceil g/2 \rceil = \frac{g+1}{2}$. Note that $p+q+r \geq g + \frac{g+1}{2} = \frac{3g+1}{2}$ and $n \geq \frac{3g+1}{2} - 1$.

If $n = \frac{3g+1}{2} - 1$, then G can not contain any pendent edges and $p+q+r = \frac{3g+1}{2}$. Since $p+q = g$, $r = \frac{g+1}{2}$ and $p \leq q \leq \frac{g+1}{2}$. If $p \leq \frac{g-1}{2}$, then $q \geq \frac{g+1}{2} + 1$, which contradicts to $q \leq r$. So the only case is that $p = \lfloor g/2 \rfloor = \frac{g-1}{2}$, $q = \lceil g/2 \rceil = \frac{g+1}{2}$ and $r = \lceil g/2 \rceil = \frac{g+1}{2}$.

If $n = \frac{3g+1}{2}$, then $p+q+r \leq \frac{3g+1}{2}+1$. Since $p+q = g$, $r \leq \frac{g+1}{2}+1$. This implies that $q \leq \frac{g+3}{2}$ and $p \geq \frac{g-3}{2}$ for $g \geq 5$. So $(p,q,r) \in \{(\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+1}{2}), (\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+3}{2}), (\frac{g-3}{2}, \frac{g+3}{2}, \frac{g+3}{2})\}$. Moreover, if $(p,q,r) \in \{\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+3}{2}), (\frac{g-3}{2}, \frac{g+3}{2}, \frac{g+3}{2}\}$, then G^* can not contain any pendent edges, in other words, G^* is isomorphic to one of $P(\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+3}{2})$ and $P(\frac{g-3}{2}, \frac{g+3}{2}, \frac{g+3}{2})$. When $g = 3$, the conclusion holds clearly. When $g = 5$, straightforward calculations show that $q_1(P(\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+3}{2})) = q_1(2,3,4) = 4.7728$, $q_1(P(\frac{g-3}{2}, \frac{g+3}{2}, \frac{g+3}{2})) = q_1(1,4,4) = 4.9032$, $q_1(P^1_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}) = q_1(P^1_{2,2,2}) = 5.3552$. So we have that $\max\{q_1(2,3,4), q_1(1,4,4)\} < q_1(P^1_{2,2,2})$, a contradiction. When $g \geq 7$, G^* can not contain a pair of adjacent 3-vertices. So by Lemma 2.7,

$$q_1(G^*) \leq \max\{d_u + d_v | uv \in E(G)\} = 5.$$

However, $q_1(P^1_{\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+1}{2}}) > \Delta + 1 = 5$ since $P^1_{\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+1}{2}}$ is not a star, a contradiction. Therefore $p = \lfloor g/2 \rfloor = \frac{g-1}{2}$, $q = r = \lceil g/2 \rceil = \frac{g+1}{2}$.

Now it remains the case that $n \geq \frac{3g+1}{2} + 1$. If $r = \frac{g+1}{2}$, clearly, $p = \frac{g-1}{2}$, $q = \frac{g+1}{2}$ since $p+q = g$ and $p \leq q \leq r$. Next we suppose $r \geq \frac{g+1}{2}+1$ and set $k = n-(p+q+r)+1$, namely the number of pendent vertices in G^*. Then $k \leq n - \frac{3g+1}{2}$. We can find $\max\{d_u + m_u | u \in V(G)\}$ attains the maximal value just when $g = 3$, namely $p = 1$, and k pendent edges are incident to a 3-vertex of $P(p,q,r)$. In this case, if k is fixed, then we can find that

$$\max\{d_u + m_u | u \in V(G)\} = k + 3 + \frac{k+7}{k+3} = k + 4 + \frac{4}{k+3}.$$

By Lemma 2.1, we have

$$q_1(G^*) < k + 4 + \frac{4}{k+3} \leq n - \frac{3g+1}{2} + 4 + \frac{4}{n - \frac{3g+1}{2} + 3} \leq n - \frac{3g+1}{2} + 5$$

since the function $k + 4 + \frac{4}{k+3}$ is increasing with the nonnegative number k and $k \leq n - \frac{3g+1}{2}$. However, $q_1(P^{n-\frac{3g+1}{2}+1}_{\frac{g-1}{2}, \frac{g+1}{2}, \frac{g+1}{2}}) > \Delta + 1 = n - \frac{3g+1}{2} + 5 > q_1(G^*)$, a contradiction.

Fact 2. Let G^* have the maximal signless Laplacian spectral radius among all graphs in $\mathcal{B}_2(n,g)$, when $n \geq \lceil 3g/2 \rceil - 1$. Then $G^* \cong P^{n-\lceil 3g/2 \rceil+1}_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}$ and $q_1(G^*) < n - \lceil 3g/2 \rceil + 5 + \frac{4}{n - \lceil 3g/2 \rceil + 4}$.

Proof of Fact 2. By Fact 1, we have G^* is obtained from by attaching $n - \lceil 3g/2 \rceil + 1$ pendent edges to a unique vertex u of $P(\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil)$. Next we only need to prove that u is a 3-vertex of $P(\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil)$. Assume to contrary that u is a 2-vertex of $P(\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil)$. For convenience, set $k = n - \lceil 3g/2 \rceil + 1$. If $k = 0$, clearly $G^* \cong P^0_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}$.

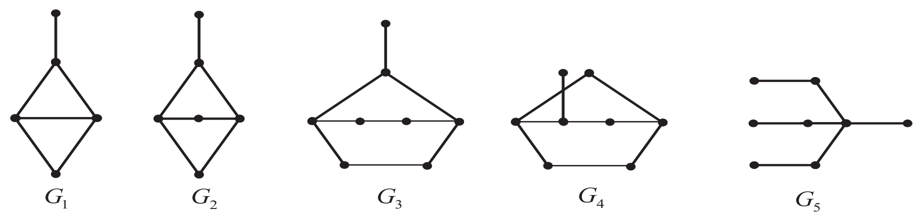

Figure 3: Several special graphs.

Now we consider the case $k = 1$. If $g \in \{3, 4, 5\}$, then G^* is isomorphic to one of G_i ($i \in \{1, 2, 3, 4\}$, see Figure 3). Straightforward calculations show that $q_1(G_1) = 5.4679 < q_1(P^1_{1,2,2}) = 5.7785$, $q_1(G_2) = 5.2361 < q_1(P^1_{2,2,2,}) = 5.5141$, $\max\{q_1(G_3), q_1(G_4)\} = \max\{5.0664, 4.9891\} < q_1(P^1_{2,3,3}) = 5.3552$, a contradiction. If $g \geq 6$, then u can not be simultaneously adjacent to the two 3-vertices of $P(\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil)$ and the two 3-vertices are not adjacent. So $d_u m_u \leq 1 + 2 + 3 = 6$. Besides, $d_{v_i} m_{v_i} \leq 2 + 2 + 3 = 7$ if v_i is a 3-vertex different from u, $d_{v_i} m_{v_i} \leq 3 + 3 = 6$ if v_i is a 2-vertex and $d_{v_i} m_{v_i} = 3$ if v_i is a pendent vertex. This implies that

$$\max\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} | uv \in E(G^*)\} \leq \max\{\frac{31}{6}, \frac{26}{5}, \frac{20}{4}\} = \frac{26}{5},$$

since $d_u + d_v \in \{4, 5, 6\}$ for each edge $uv \in E(G)$. By Lemma 2.7, we have $q_1(G^*) \leq \frac{26}{5}$. However, when $g \geq 6$, G_5 is a subgraph of $P^1_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}$. Thus $q_1(P^1_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}) > q_1(G_5) = 3 + \sqrt{5} > \frac{26}{5}$. This is, $q_1(G^*) < q_1(P^1_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil})$, a contradiction. Also we can consider another more simpler method. When $k = 1$, we observe that any maximal graph G on girth $g \geq 6$ by Lemma 2.2 has a smaller signless Laplacian spectral radius than that of the maximal graph of girth $g = 5$, namely G_3, that is $q_1(G) < q_1(G_3) = 5.0664 < \frac{26}{5}$ by Lemma 2.3. So the result is also proved.

Next we consider the case $k \geq 2$. We have that $\max\{d_u + m_u | u \in V(G^*)\}$ attains the maximal value just when u is simultaneously adjacent to the two 3-vertices of $P(\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil)$. In this case,

$$\max\{d_u + m_u | u \in V(G^*)\} = k + 2 + \frac{k + 6}{k + 2} \leq k + 4$$

since $k \geq 2$. By Lemma 2.7, we have $q_1(G^*) < k + 4$ since G^* is neither regular nor semi-regular. However, by Lemma 2.6, $q_1(P^{n - \lceil 3g/2 \rceil + 1}_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}) > \Delta + 1 = k + 4$ since $P^{n - \lceil 3g/2 \rceil + 1}_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}$ is not a star. Note that $P^{n - \lceil 3g/2 \rceil + 1}_{\lfloor g/2 \rfloor, \lceil g/2 \rceil, \lceil g/2 \rceil}$ is neither regular nor semi-regular bipartite except that $P^0_{2,2,2} \cong K_{2.3}$ is a semi-regular bipartite graph. Thus if $(n,g) \neq (5,4)$,

$$q_1(P^{n-\lceil 3g/2\rceil+1}_{\lfloor g/2\rfloor,\lceil g/2\rceil,\lceil g/2\rceil}) < \max\{d_u+m_u | u \in V(P^{n-\lceil 3g/2\rceil+1}_{\lfloor g/2\rfloor,\lceil g/2\rceil,\lceil g/2\rceil})\} \leq n-\lceil \frac{3g}{2}\rceil+5+\frac{4}{n-\lceil\frac{3g}{2}\rceil+4}$$

since $k+3+\frac{4}{k+2}$ is increasing with nonnegative integer k.

As for $(n,g)=(5,4)$, we know that

$$q_1(P^0_{2,2,2}) = 5 < \frac{16}{3} = n-\lceil\frac{3g}{2}\rceil+5+\frac{4}{n-\lceil\frac{3g}{2}\rceil+4}. \qquad \square$$

Theorem 3.4 Let G^* have the maximal signless Laplacian spectral radius among all graphs in $\mathcal{B}(n,g)$, where $n \geq \lceil 3g/2\rceil - 1$. Then $G^* \cong P^{n-\lceil 3g/2\rceil+1}_{\lfloor g/2\rfloor,\lceil g/2\rceil,\lceil g/2\rceil}$ and $q_1(P^{n-\lceil 3g/2\rceil+1}_{\lfloor g/2\rfloor,\lceil g/2\rceil,\lceil g/2\rceil}) < n-\lceil 3g/2\rceil+5+\frac{4}{n-\lceil 3g/2\rceil+4}$.

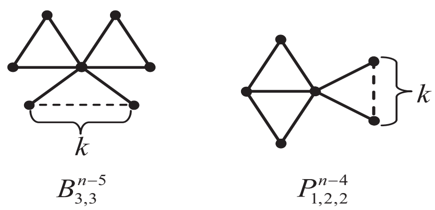

Figure 4: Graphs $B^{n-5}_{3,3}$ and $P^{n-4}_{1,2,2}$.

Proof. We distinguish the two cases as follows:

Case 1. If $g=3$, the corresponding graphs $B^{n-5}_{3,3}$ and $P^{n-4}_{1,2,2}$ are shown in Figure 4, then by Lemma 2.2, we have $q_1(B^{n-5}_{3,3}) < q_1(P^{n-4}_{1,2,2})$.

Case 2. If $g \geq 4$, we have to show $q_1(B^{n-2g+1}_{g,g}) < q_1(P^{n-\lceil 3g/2\rceil+1}_{\lfloor g/2\rfloor,\lfloor g/2\rfloor,\lfloor g/2\rfloor})$. According to the results above, $q_1(B^{n-2g+1}_{g,g}) < n-2g+6+\frac{4}{n-2g+5} \leq n-2g+7$ since in this case $n \geq 2g-1$. However $q_1(P^{n-\lceil 3g/2\rceil+1}_{\lfloor g/2\rfloor,\lfloor g/2\rfloor,\lfloor g/2\rfloor}) > \Delta+1 = n-\lceil 3g/2\rceil+5$. Since $g \geq 4$, $n-\lceil 3g/2\rceil+5-(n-2g+7) = \lfloor g/2\rfloor - 2 \geq 0$.

This completes the whole proof of Theorem 3.4. $\square$

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