

An inductive construction for Hamilton cycles in Kneser graphs

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Abstract

The Kneser graph $K(n, r)$ has as vertices all r -subsets of an n -set with two vertices adjacent if the corresponding subsets are disjoint. It is conjectured that, except for $K(5, 2)$, these graphs are Hamiltonian for all $n \geq 2r + 1$. In this note we describe an inductive construction which relates Hamiltonicity of $K(2r + 2s, r)$ to Hamiltonicity of $K(2r' + s, r')$. This shows (among other things) that Hamiltonicity of $K(2r + 1, r)$ for all $3 \leq r \leq k$ implies Hamiltonicity of $K(2r + 2, r)$ for all $r \leq 2k + 1$. Applying this result extends the range of values for which Hamiltonicity of $K(n, r)$ is known. Another consequence is that certain families of Kneser graphs ($K(\frac{27}{13}r, r)$ for instance) contain infinitely many Hamiltonian graphs.

1 Introduction

Let S be a finite set. We write $K(S, r)$ for the graph with vertex set $S^{(r)} = \{X \subseteq S : |X| = r\}$, and two vertices adjacent if the corresponding sets are disjoint. Up to isomorphism $K(S, r)$ depends only on the cardinality of S and so for $n \in \mathbb{N}$ we write $K(n, r)$ for $K(\{1, 2, \dots, n\}, r)$.

The graphs $K(n, r)$ are known as *Kneser graphs*. It is conjectured that $K(n, r)$ is Hamiltonian for all pairs (n, r) with $n \geq 2r + 1$ except for $(5, 2)$ (this is Petersen's graph). However, this is only known in general when n is much larger than $2r + 1$. Specifically, Chen [2] showed that $K(n, r)$ is Hamiltonian whenever

$$n \geq \frac{3r + 1 + \sqrt{5r^2 - 2r + 1}}{2} = (2.62 \dots + o(1))r.$$

Hamiltonicity is also known for several small values (see [3] and references therein). The strongest results in this direction being that $K(n, r)$ is Hamiltonian for all $n \leq 27$, proved by Savage and Shields [3] using a computer search.

Intuitively the graph $K(2r + s, r)$ should be more likely to be Hamiltonian when s is large since it has higher degree and higher edge density then. However, it is not obvious that this is true. For instance, does the Hamiltonicity of the family of graphs $K(2r + 2, r)$ follow from the Hamiltonicity of the family of graphs $K(2r + 1, r)$ (this intriguing family of Kneser graphs is known as the *odd graphs* [1])?

We describe a construction which proves this and more:

Theorem 1. *Let $r \geq 3$ and $n = 2r + s$ be even. If $K(2a + \frac{s}{2}, a)$ is either Hamiltonian or $K(5, 2)$ for all $1 \leq a \leq \lfloor \frac{r}{2} \rfloor$ then $K(n, r)$ is Hamiltonian.*

The exception made for $K(5, 2)$ is clearly needed since Petersen's graph is notoriously non-Hamiltonian. In our constructions we get round this by using a Hamilton path in $K(5, 2)$ and modifying our arguments slightly.

We give two appealing corollaries of our main theorem.

Corollary 2. *If $K(2r + 1, r)$ is Hamiltonian for all $3 \leq r \leq k$ then $K(2r + 2, r)$ is Hamiltonian for all $3 \leq r \leq 2k + 1$. In particular $K(2r + 2, r)$ is Hamiltonian for all $r \leq 27$.*

It appears that the best previously known result is that $K(2r + 2, r)$ is Hamiltonian when $r \leq 13$ [3].

Proof. This follows immediately from Theorem 1 with $s = 2$ and the fact that $K(2r + 1, r)$ is Hamiltonian for all $3 \leq r \leq 13$ [3]. $\square$

Corollary 3. *If $K(2r + 1, r)$ is Hamiltonian for all $3 \leq r \leq k$ then $K(\frac{2k+1}{k}r, r)$ is Hamiltonian for infinitely many values of r .*

A particular consequence of this and the fact that $K(2r + 1, r)$ is Hamiltonian for all $3 \leq r \leq 13$ [3] is that $K(\frac{27}{13}r, r)$ is Hamiltonian for infinitely many r .

Proof. We claim that under the assumptions given the graph $K(2r + 2^l, r)$ is Hamiltonian for all $l \geq 0$, $3 \leq r \leq k2^l$. Given this claim, setting $r = k2^l$ provides infinitely many Hamiltonian graphs in $K(\frac{2k+1}{k}r, r)$.

We will prove the claim by induction on l . The base case $l = 0$ is the statement that $K(2r + 1, r)$ for all $3 \leq r \leq k$. For the induction step we may assume that $K(2r + 2^{l-1}, r)$ is Hamiltonian for all $3 \leq r \leq k2^{l-1}$ which, by Theorem 1, implies that $K(2r + 2^l, r)$ is Hamiltonian for all $3 \leq r \leq k2^l$. $\square$

2 Notation

For $a, b \in \mathbb{N}$ we write $[a, b]$ for $\{a, a + 1, \dots, b\}$ and $[a]$ for $[1, a]$.

The key idea is to relate the graphs $K(n, r)$ and $K(\frac{n}{2}, a)$ by splitting the ground set $[n]$ into $\frac{n}{2}$ pairs, and considering how an r -subset of $[n]$ intersects these. To this end we need

some notation. Let $n = 2m$ be even. Let $p_i = \{2i - 1, 2i\}$ so that the sets $p_1, p_2, \dots, p_m$ partition $[n]$. If $X \subseteq [n]$ we define:

$$X(i) = \begin{cases} 0 & \text{if } X \cap p_i = \emptyset \\ -1 & \text{if } X \cap p_i = \{2i - 1\} \\ 1 & \text{if } X \cap p_i = \{2i\} \\ 2 & \text{if } X \cap p_i = p_i \end{cases}$$

and identify X with the m -tuple $(X(1), X(2), \dots, X(m)) \in \{-1, 0, 1, 2\}^m$.

We also define

$$\begin{aligned} A(X) &= \{i \in [m] : X(i) = 2\}, \\ B(X) &= \{i \in [m] : X(i) = \pm 1\}, \end{aligned}$$

and

$$f_X : B(X) \rightarrow \{\pm 1\}$$

to be the function with $f_X(i) = X(i)$.

It is clear that $A(X) \cap B(X) = \emptyset$ and $|X| = 2|A(X)| + |B(X)|$. Moreover, the triple $(A(X), B(X), f_X)$ determines the set X . In what follows it will be convenient to swap between the representation of a set X by the $X(i)$ and by $(A(X), B(X), f_X)$.

Let $S_{a,b} = \{X \subseteq [n] : |A(X)| = a, |B(X)| = b\}$. It is clear that

$$[n]^{(r)} = \bigcup_{a=0}^{\lfloor \frac{r}{2} \rfloor} S_{a, r-2a},$$

and that this is a disjoint union. When a is an integer we denote by $K_a(n, r)$ the subgraph of $K(n, r)$ induced by $S_{a, r-2a}$. When A is a set of integers we denote by $K_A(n, r)$ the subgraph of $K(n, r)$ induced by $\bigcup_{a \in A} S_{a, r-2a}$.

A permutation of $[n]$ induces an automorphism of the graph $K(n, r)$. If the permutation preserves the partition of $[n]$ into the pairs $p_1, \dots, p_m$ then the automorphism induced preserves the sets $S_{a, r-2a}$. In what follows when we refer to permuting or reordering the pairs $p_1, \dots, p_m$ we mean applying such an automorphism.

3 Construction

Throughout $n = 2m$ will be even. Our strategy is to find cycles in each of $K_a(n, r)$ and then to join them up. Specifically, we first construct a Hamilton cycle $\mathcal{H}_0$ in $K_0(n, r)$. Next we partition each of $K_a(n, r)$ $a \geq 1$ into a small number of cycles. Finally, we join the cycles which partition $K_1(n, r)$ to $\mathcal{H}_0$, join the cycles which partition $K_2(n, r)$ to this new cycle, and so on until we have a Hamilton cycle in $K_{\{0, 1, \dots, \lfloor \frac{r}{2} \rfloor\}}(n, r) = K(n, r)$.

The existence of $\mathcal{H}_0$ is the content of Lemma 1. The partition of $K_a(n, r)$ into cycles is the content of Lemma 2. In each case we need some extra properties to carry out the joining process.

If $c \in \{\pm 1\}^k$ we denote the k -tuple $-c$ by $\bar{c}$.

Lemma 1. For all $r \geq 3$ and even $n \geq 2r + 2$ there is a Hamilton cycle $\mathcal{H}$ in $K_0(n, r)$ with the property that for every $c \in \{\pm 1\}^{r-2}$ there is an edge XY of $\mathcal{H}$ with

$$X = (c, 0, \pm 1, \pm 1, \underbrace{0, \dots, 0}_{m-r-1})$$

$$Y = (\bar{c}, \pm 1, 0, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}).$$

Lemma 2. Let $n = 2m$ and $1 \leq a \leq \lfloor \frac{r}{2} \rfloor$ with $(n, r, a) \neq (10, 4, 2)$. If the graph $K(2a + m - r, a)$ is Hamiltonian or is $K(5, 2)$ then the graph $K_a(n, r)$ can be partitioned into cycles $C_1, C_2, \dots, C_k$ where $k \leq 2^{r-2a}$. Moreover, this can be done in such a way that:

1. for $1 \leq i \leq k$, each of the cycles C_i contains an edge XY with

$$X = (c(i), \underbrace{2, \dots, 2}_a, \underbrace{0, \dots, 0}_a, \underbrace{0, \dots, 0}_{m-r})$$

$$Y = (\bar{c(i)}, \underbrace{0, \dots, 0}_a, \underbrace{2, \dots, 2}_a, \underbrace{0, \dots, 0}_{m-r})$$

where $c(1), c(2), \dots, c(k)$ are distinct elements of $\{\pm 1\}^{r-2a}$,

and

2. there is some reordering of the pairs $p_1, \dots, p_m$ with respect to which for every $d \in \{\pm 1\}^{r-2a-2}$ there is an edge XY in one of the C_i with

$$X = (d, 0, \underbrace{0, \dots, 0}_a, \pm 1, \underbrace{2, \dots, 2}_a, \pm 1, \underbrace{0, \dots, 0}_{m-r-1})$$

$$Y = (\bar{d}, \pm 1, \underbrace{2, \dots, 2}_a, 0, \underbrace{0, \dots, 0}_a, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}).$$

We postpone the proofs of these until the next section and move on to using them to prove Theorem 1.

We repeatedly use the concept of inserting one cycle in another. Suppose that we have two vertex disjoint cycles

$$X = x_1, x_2, \dots, x_k$$

$$Y = y_1, y_2, \dots, y_l$$

in a graph G . If, for some i, j , we have $x_i y_j, x_{i+1} y_{j+1} \in E(G)$ then we can form a new cycle by adding these edges and removing the edges $x_i x_{i+1}$ and $y_j y_{j+1}$. Specifically, this new cycle is

$$x_1, x_2, \dots, x_{i-1}, x_i, y_j, y_{j-1}, \dots, y_1, y_l, \dots, y_{j+1}, x_{i+1}, x_{i+2}, \dots, x_k.$$

We refer to this construction as inserting cycle Y into cycle X . Note that we can insert several cycles into X provided that the edges of X that we delete to do the insertion of each are distinct.

Proof of Theorem 1. We assume that $(n, r) \neq (10, 4)$ since our construction does not work in this case. However, it is known that $K(10, 4)$ is Hamiltonian so this is not a problem. We will show, by induction on i , that a cycle containing all vertices of $K_{\{0,1,\dots,i\}}(n, r)$ can be constructed, containing for each $d \in \{\pm 1\}^{r-2i-2}$ an edge XY with

$$\begin{aligned} X &= (d, 0, \underbrace{0, \dots, 0}_i, \pm 1, \underbrace{2, \dots, 2}_i, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}) \\ Y &= (\bar{d}, \pm 1, \underbrace{2, \dots, 2}_i, 0, \underbrace{0, \dots, 0}_i, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}). \end{aligned}$$

When $i = \lfloor \frac{r}{2} \rfloor$ this gives a Hamilton cycle in $K(n, r)$ and so this claim implies Theorem 1.

Lemma 1 shows that this is true for $i = 0$.

Now take $0 \leq i \leq \lfloor \frac{r}{2} \rfloor$ and suppose that the claim is true for i . Let $\mathcal{H}_i$ be a cycle in $K_{\{0,1,\dots,i\}}(n, r)$ as guaranteed by the induction hypothesis. Take a partition of $K_{i+1}(n, r)$ into cycles $C_1, \dots, C_t$ as described by Lemma 2 (with $a = i + 1$). Note that the conditions of this Lemma are met since $\frac{s}{2} = m - r$ and we know that $K(2a + \frac{s}{2}, a)$ is either Hamiltonian or $K(5, 2)$. By the first part of Lemma 2 each cycle in this partition contains an edge XY with:

$$\begin{aligned} X &= (c, \underbrace{2, \dots, 2}_{i+1}, \underbrace{0, \dots, 0}_{i+1}, \underbrace{0, \dots, 0}_{m-r}) \\ Y &= (\bar{c}, \underbrace{0, \dots, 0}_{i+1}, \underbrace{2, \dots, 2}_{i+1}, \underbrace{0, \dots, 0}_{m-r}) \end{aligned}$$

for some $c \in \{\pm 1\}^{r-2i-2}$, with distinct vectors $c(1), \dots, c(t)$ playing the role of c in cycles $C_1, \dots, C_t$ respectively.

Further, by the induction hypothesis, for each $d \in \{\pm 1\}^{r-2i-2}$ there is an edge XY in $\mathcal{H}_i$ with

$$\begin{aligned} X &= (d, 0, \underbrace{0, \dots, 0}_i, \pm 1, \underbrace{2, \dots, 2}_i, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}) \\ Y &= (\bar{d}, \pm 1, \underbrace{2, \dots, 2}_i, 0, \underbrace{0, \dots, 0}_i, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}). \end{aligned}$$

In particular for each $c(j)$ we can find such an edge with $d = \overline{c(j)}$. These edges can be used to insert each of the cycles $C_1, \dots, C_t$ into $\mathcal{H}_i$. The fact that the $c(j)$ are distinct

means that we do not ever have to use the same edge to insert more than one cycle. The result is a cycle containing all vertices of $K_{\{0,1,\dots,i+1\}}(n,r)$ which we will denote by $\mathcal{H}'_{i+1}$.

Now we take a permutation of the pairs $p_1, \dots, p_m$ with the properties of Lemma 2 part 2 and apply it to each vertex in $\mathcal{H}'_{i+1}$. Note that under this permutation $\mathcal{H}'_{i+1}$ is mapped to another cycle containing all vertices of $K_{\{0,1,\dots,i+1\}}(n,r)$. We denote this new cycle by $\mathcal{H}_{i+1}$. By Lemma 2, for each $d \in \{\pm 1\}^{r-2i-2}$ there is an edge XY in one of the cycles $C_1, \dots, C_t$ of the form:

$$\begin{aligned} X &= (d, 0, \underbrace{0, \dots, 0}_{i+1}, \pm 1, \underbrace{2, \dots, 2}_{i+1}, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}) \\ Y &= (\bar{d}, \pm 1, \underbrace{2, \dots, 2}_{i+1}, 0, \underbrace{0, \dots, 0}_{i+1}, \pm 1, \underbrace{0, \dots, 0}_{m-r-1}). \end{aligned}$$

These edges are not destroyed by the insertion of $C_1, \dots, C_t$ into $\mathcal{H}_i$ because the only edges destroyed in this process are of the form XY with $B(X) = B(Y)$. Hence $\mathcal{H}_{i+1}$ satisfies the properties required and the proof is complete. $\square$

4 Proofs

Let $\mathcal{B} = B_1, \dots, B_s$ be a sequence of b -subsets of $[m]$. For each $1 \leq i \leq s$ let $\mathcal{A}^{(i)} = A_1^{(i)}, \dots, A_t^{(i)}$ be a sequence of a -subsets of $[m] \setminus B_i$. We will use these sequences together with a single function $f : B_1 \rightarrow \{\pm 1\}$ to construct a sequence $\mathcal{P} = P_1, \dots, P_{st}$ of r -subsets of $[n]$ where $r = 2a + b$ (in fact in all our applications it will be a path in $K(n, r)$). We will define the P_k by specifying $B(P_k)$, $A(P_k)$ and f_{P_k} . Firstly, for $1 \leq k \leq st$ let $k = (x-1)t + y$ with $1 \leq x \leq s$ and $1 \leq y \leq t$. We set

$$B(P_k) = B_x; \quad A(P_k) = A_y^{(x)}.$$

We set $f_{P_1} = f$ and given $f_{P_{k-1}}$ define f_{P_k} inductively as follows.

If $B_{P_k} \setminus B_{P_{k-1}} = \{r_1, \dots, r_l\}$, $B_{P_{k-1}} \setminus B_{P_k} = \{s_1, \dots, s_l\}$ with $r_1 < r_2 < \dots < r_l$ and $s_1 < s_2 < \dots < s_l$ then

$$f_{P_k}(x) = \begin{cases} -f_{P_{k-1}}(x) & \text{if } k \in B_{P_k} \cap B_{P_{k-1}} \\ f_{P_{k-1}}(s_i) & \text{if } x = r_i. \end{cases}$$

Note that if $B_{P_k} = B_{P_{k-1}}$ then $f_{P_k}(x) = -f_{P_{k-1}}(x)$.

We denote the sequence $\mathcal{P}$ by $\mathcal{P}(\mathcal{B}; f; \mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)})$ and refer to this construction as the $\mathcal{P}$ construction.

An important point of the definition of the f is that it is reversible; if $B(P_{k-1}), B(P_k)$ and f_{P_k} are given then $f_{P_{k-1}}$ is uniquely determined by the same rule as above.

We will use this construction to build paths in $K(n, r)$ and so we will need to know when consecutive sets in the sequence are disjoint. The following simple lemma will be used.

Lemma 3. *If $\mathcal{B}, f$ and $\mathcal{A}^{(i)}$ are as above and in addition we have:*

1. $A_1^{(i)} \cap B_{i-1} = \emptyset$ for all $2 \leq i \leq s$,
2. $A_t^{(i)} \cap B_{i+1} = \emptyset$ for all $1 \leq i \leq s-1$,
3. *each sequence $\mathcal{A}^{(i)}$ is a path in $K([m] \setminus B_i, a)$ (that is $A_k^{(i)} \cap A_{k+1}^{(i)} = \emptyset$ for all $1 \leq i \leq s$, $1 \leq k \leq t-1$),*

then $\mathcal{P}(\mathcal{B}; f; \mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)})$ is a path in $K(n, r)$.

Proof. Let X, Y be sets. We have that $X \cap Y = \emptyset$ unless one of the following holds for some i :

1. $(X(i), Y(i)) = (\pm 1, 2)$ or $(2, \pm 1)$,
2. $(X(i), Y(i)) = (2, 2)$,
3. $(X(i), Y(i)) = (1, 1)$ or $(-1, -1)$.

For consecutive elements of $\mathcal{P}(\mathcal{B}; f; \mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)})$ the first of these possibilities is ruled out by conditions 1 and 2 of the Lemma. The second possibility is ruled out by condition 3 of the Lemma. The third possibility is ruled out by the definition of f . $\square$

For S a finite set we write $G(S, r)$ for the graph with vertex set $S^{(r)}$ and two vertices A and B adjacent if $|A \cap B| = r - 1$. Up to isomorphism $G(S, r)$ depends only on the cardinality of S and so for $n \in \mathbb{N}$ we shall write $G(n, r)$ for $G([n], r)$.

Lemma 4. *For all $1 \leq r \leq n$ the graph $G(n, r)$ contains:*

1. *a Hamilton path which starts at the vertex $[r]$, ends at the vertex $[n - r + 1, n]$, and contains*

$$[r - 1] \cup \{r\}, [r - 1] \cup \{r + 1\}, [r - 1] \cup \{r + 2\}, \dots, [r - 1] \cup \{n\}$$

as consecutive vertices,

2. *a Hamilton cycle which contains*

$$[r - 1] \cup \{r\}, [r - 1] \cup \{r + 1\}, [r - 1] \cup \{r + 2\}, \dots, [r - 1] \cup \{n\}$$

as consecutive vertices.

Where if $n = r$ we interpret the single vertex $[n]$ as being both a Hamilton path and a Hamilton cycle with the required properties.

Proof. We use induction on n . The result clearly holds for $n = 2$ (interpreting the single edge $G(2, 1)$ as a cycle). We may assume, by part 1 of the inductive hypothesis, that for all $1 \leq k \leq n - r + 1$ there is a Hamilton path in $G([k + 1, n], r - 1)$ from $[k + 1, k + r - 1]$ to $[n - r + 2, n]$. Adding k to each vertex gives a path P_k in $G(n, r)$ from $[k, k + r - 1]$ to $\{k\} \cup [n - r + 2, n]$ containing all the vertices $v \in G(n, r)$ with $\min(v) = k$. Denote the reverse of a path P by $r(P)$. Now the path required by part 1 is given by $P_1, r(P_2), P_3, r(P_4), \dots, P_{n-r+1}$, and the cycle required by part 2 is given by $r(P_1), P_2, r(P_3), P_4, \dots, P_{n-r+1}$ (note that P_{n-r+1} is a single vertex). $\square$

Proof of Lemma 2. Let $b = r - 2a$, $s = \binom{m}{b}$ and $t = \binom{m-b}{a}$.

We use the $\mathcal{P}$ construction to define 2^b paths in $K(n, r)$ each containing st vertices. These will form the building blocks for our cycles.

We start by taking $\mathcal{B} = B_1, \dots, B_s$ to be a Hamilton cycle in $G(m, b)$. By Lemma 4 (and relabelling elements of the ground set) we may take this cycle to have $B_1 = [1, b - 2] \cup \{m - 1, m\}$ and $B_{s-k} = [1, b - 1] \cup \{m - k\}$ for $m - b \geq k \geq 0$.

We now define sets F_i, L_i for $1 \leq i \leq s$ inductively as follows:

- $F_1 = [m - a - 1, m - 2]$.
- Given F_i we take L_i to be the lexicographically least element of $[m]^{(a)}$ with

$$L_i \cap (F_i \cup B_i \cup B_{i+1}) = \emptyset$$

(such a set exists since $m - |F_i \cup B_i \cup B_{i+1}| \geq m - (a + b + 1) \geq r + 1 - (r - a + 1) = a$).

- Given L_i we take F_{i+1} to be the lexicographically least element of $[m]^{(a)}$ with

$$F_{i+1} \cap (L_i \cup B_i \cup B_{i+1}) = \emptyset$$

(again such a set exists since $m - |L_i \cup B_i \cup B_{i+1}| \geq a$).

We claim that $L_s = [b, b + a - 1]$.

To see this note that for all $t \in [b, b + a - 1]$ we have that $t \in B_{s-m+t}$ but $t \notin B_x$ for $x > s - m + t$. Hence by construction $t \notin F_{s-m+t+1}$ and, since L_x is chosen to be the lexicographically smallest suitable set, $t \in L_{s-m+t+1}$. It follows similarly that for all $s - m + t \leq x \leq s$ we have $t \notin F_x$ and $t \in L_x$. In particular $L_s = [b, b + a - 1]$.

The important consequence of this claim is that $F_1 \cap (L_s \cup B_s \cup B_1) = \emptyset$ (since $b + a - 1 < m - a - 1$) and so the property we demand in the construction of the F_i, L_i holds going “around the corner” from B_s to B_1 .

Assuming now that $(m - b, a) \neq (5, 2)$ we let $\mathcal{A}^{(i)} = A_1^{(i)}, \dots, A_t^{(i)}$ be a Hamilton cycle in $K([m] \setminus B_i, a)$ with $A_1^{(i)} = F_i$ and $A_t^{(i)} = L_i$. Such a cycle exists by the hypotheses of the Lemma. Indeed, $K([m] \setminus B_i, a)$ is $K(\frac{n}{2} - b, \frac{r-b}{2})$ for some $0 \leq b \leq r - 2$ or equivalently is $K(r - b + \frac{n-2r}{2}, \frac{r-b}{2})$ for some $\lfloor \frac{r}{2} \rfloor \geq \frac{r-b}{2} \geq 1$.

By Lemma 3, we have that $\mathcal{P}(\mathcal{B}; f; \mathcal{A}^{(1)}, \dots, \mathcal{A}^{(s)}) = P_1, P_2, \dots, P_{st}$ is a path in $K_a(n, r)$ for each of the 2^b choices for $f : B_1 \rightarrow \{\pm 1\}$. Moreover, each choice of f results in a distinct path and (by the reversibility of the definition of f in the $\mathcal{P}$ construction) these 2^b

paths partition $K_a(n, r)$. Finally, we can construct a bijection from the set of last vertices of these paths to the set of first vertices of these paths by mapping $(A(P_{st}), B(P_{st}), f)$ to $(A(P_1), B(P_1), g)$ where

$$g(x) = \begin{cases} -f(x) & \text{if } x \in X_{st} \cap X_1 \\ f(b-1) & \text{if } x = m-1. \end{cases}$$

(Note that $P_1 = P_{st} \setminus \{b-1\} \cup \{m-1\}$ so this is the same process as used to construct the f_X in the $\mathcal{P}$ construction.)

This results in the paths being joined to form at most 2^{r-2a} cycles $C_1, \dots, C_k$ which cover $K(n, r)$.

Finally, we observe that there is an ordering of the pairs $p_1, \dots, p_m$ for which we have edges of the form required by part 1 of the Lemma. This follows from the fact that if we permute the pairs $p_1, \dots, p_m$ so that there is an edge XY in C_1 with $A(X) = [r-2a+1, r-a]$, $A(Y) = [r-a+1, r]$, $B(X) = B(Y) = [r-2a]$ then there will be an edge of this form in each of the C_i . Similarly there is a permutation of $[m]$ resulting in edges of the form required by part 2 of the Lemma. Every possible d occurs because the pattern of $A(X), B(X)$ going round the cycle does not depend on f_X . That is, if X is followed by Y in some C_i then $A(Y), B(Y)$ depend only on $A(X), B(X)$ and not on f_X .

If $(m-b, a) = (5, 2)$ then we use the same method but instead of a Hamilton cycle in $K(5, 2)$ use a Hamilton path. This means the choice of F_i and L_i must be changed. Specifically, given F_i we choose L_i to be an arbitrary element of $[m]^{(2)}$ with $|F_i \cap L_i| = 1$ and $L_i \cap (B_i \cup B_{i+1}) = \emptyset$. The method for choosing F_i is as before. Now there is a Hamilton path in $K([m] \setminus B_i, 2)$ from F_i to L_i and we use this as our $\mathcal{A}^{(i)}$. We can still choose L_s so that $F_1 \cap (L_s \cup B_s \cup B_1)$ unless $F_s = \{m-5, m-4\}$ or $\{m-3, m-2\}$. We can avoid these possibilities for F_s unless $L_{s-1} = \{m-5, m-4\}$ or $\{m-3, m-2\}$. However, there is no possible F_{s-1} for which these are the only choices for L_{s-1} and so we always have a suitable L_{s-1}, F_s, L_s .

□

Let $L_1, \dots, L_h, R_1, \dots, R_h$ be pairwise disjoint k -sets with $w_i, x_i \in L_i$ and $y_i, z_i \in R_i$. Let $H(k, h) = (V, E)$ be the graph with

$$V = \bigcup_{i=1}^h L_i \cup \bigcup_{j=1}^h R_j$$

and

$$E = \bigcup_{i=1}^h \{lr : l \in L_i, r \in R_i\} \cup \bigcup_{i=1}^h \{y_i w_{i+1}, z_i x_{i+1}\}$$

where we identify w_{h+1}, x_{h+1} with w_1, x_1 respectively. We refer to the vertices w_i, x_i, y_i, z_i as *linking vertices*.

So, $H(k, h)$ comprises h pairwise disjoint copies of the complete bipartite graph $K_{k,k}$ arranged in a cycle with two independent edges going from the right-hand part of each $K_{k,k}$ to the left-hand part of its successor around the cycle.

We also (with a slight abuse of notation) denote by $H'(k, h)$ any graph constructed in the same way as $H(k, h)$ but allowing the L_i and R_i to be k -sets or $(k - 1)$ -sets (but requiring that $|L_i| = |R_i|$ for all i).

Lemma 5. *For all $h \geq 2$ and $k \geq 3$ each of the graphs $H(k, h)$ and $H'(k, h)$ contains a Hamilton cycle. Moreover, if M is a set of independent edges, each lying in one of the complete bipartite graphs induced by $L_i \cup R_i$ for $1 \leq i \leq h$ and with $w_1 y_1, x_1 y_1 \notin M$, then the Hamilton cycles can be chosen so as to include all edges in M .*

Proof. We start from the linking vertex $y_1 \in R_1$. This is joined to a vertex $w \in L_2$. Since M is an independent set of edges and there are 2 linking vertices in R_2 , there is some linking vertex $y \in R_2$ with $wy \notin M$. We take a path through all vertices of $L_2 \cup R_2$ which starts at w , contains all edges in M which lie in the bipartite graph induced by $L_2 \cup R_2$, and ends at y . Continuing inductively we produce a path ending at a linking vertex $z \in R_h$. This can be joined to a vertex $x \in L_1$ and the cycle can be completed here (using the fact that the edge from x to y_1 is not in M).

An identical argument applies to $H'(k, h)$. □

We will prove Lemma 1 for odd r by finding a copy of $H(r + 1, 2^{r-1})$ in $K_0(n, r)$ and specifying a set M of independent edges in it satisfying the conditions of Lemma 5. By this Lemma we can find a Hamilton cycle $\mathcal{C}$ in this graph containing all the edges in M . We then partition the remaining vertices of $K_0(n, r)$ into cycles which can be inserted into $\mathcal{C}$ using edges in M .

If r is even the strategy is identical except that a copy of $H'(r + 1, 2^{r-1})$ must be used.

Proof of Lemma 1. Suppose first that r is odd. Let V be the set of vectors in $\{-1, 0, 1\}^m$ which satisfy:

1. $v_1, \dots, v_r \in \{\pm 1\}$,
2. $v_{r+1} = 1$,
3. $v_{r+2} = \dots = v_m = 0$,
4. $|\{i : v_i = 1\}| \equiv 0 \pmod{2}$.

Note that $|V| = 2^{r-1}$.

For $v \in V$ define

$$L_v = \{X \in K_0(n, r) : |\{i : 1 \leq i \leq r + 1, X(i) = v_i\}| = r\},$$

$$R_v = \{X \in K_0(n, r) : |\{i : 1 \leq i \leq r + 1, X(i) = -v_i\}| = r\}.$$

So L_v comprises sets X with a single 0 among the first $r + 1$ coordinates and with $X(i)$ agreeing with v_i on the remaining r , while R_v comprises sets X with a single 0 among the first $r + 1$ coordinates and with $X(i)$ agreeing with $-v_i$ on the remaining r .

The subgraph of $K_0(n, r)$ induced by $L_v \cup R_v$ is isomorphic to $K_{r+1, r+1}$. For example, if $m = 4, r = 3, v = (+1, -1, -1, +1)$ we have:

$$\begin{aligned} L_v &= \{(+1, -1, -1, 0), (+1, -1, 0, +1), (+1, 0, -1, +1), (0, -1, -1, +1)\} \\ R_v &= \{(-1, +1, +1, 0), (-1, +1, 0, -1), (-1, 0, +1, -1), (0, +1, +1, -1)\}. \end{aligned}$$

If $m = 7, r = 3, v = (+1, -1, -1, +1, 0, 0, 0)$ then $(0, 0, 0)$ is appended to each of the elements of L_v, R_v above (and the graph induced is still $K_{4,4}$).

The fact that r is odd means that the L_v and R_v are pairwise disjoint. We will refer to the $K_{r+1, r+1}$ subgraphs described here as (L, R) -blocks.

We can order the elements of V cyclically as $v_1, v_2, \dots, v_{2^{r-1}}$ so that consecutive elements differ in exactly two coordinates. To achieve this take a cyclic ordering of $\{\pm 1\}^r$ with consecutive elements differing in exactly one coordinate (that is a Hamilton cycle in the hypercube or Gray code). Taking alternate elements of this cycle and appending $+1$ followed by $m - r - 1$ 0s to each gives the required cyclic ordering. We note now (because we will need it later) that we may choose this ordering so that the coordinate 1 does not change value between v_1 and v_2 .

If u is the predecessor of v in this ordering then, because u and v differ in only two coordinates, we can find $w, x \in R_u$ and $y, z \in L_v$ with $wy, xz \in E(K_0(n, r))$. It follows that we have a copy of $H(r + 1, 2^{r-1})$ in $K_0(n, r)$.

For each $c \in \{\pm 1\}^{r-2}$ choose an edge XY within some (L, R) -block of the form

$$\begin{aligned} X &= (c, 0, \pm 1, +1, \underbrace{0, \dots, 0}_{m-r-1}) \\ Y &= (\bar{c}, \pm 1, 0, -1, \underbrace{0, \dots, 0}_{m-r-1}). \end{aligned}$$

For each such c there are two choices for such an edge, and so we can make our choice avoiding edges in the block induced by $L_{v_1} \cup R_{v_1}$. Call this set of (necessarily independent) edges M_1 .

If $m = r + 1$ then our copy of $H(r + 1, 2^{r-1})$ is a spanning subgraph of $K_0(n, r)$. By Lemma 5 there is a Hamilton cycle in this graph containing all edges in M_1 . This satisfies the conditions required by Lemma 1.

If $m > r + 1$ then we need to deal with the vertices in $T = [m]^{(r)} \setminus [r + 1]^{(r)}$.

Let $\mathcal{B} = B_1, \dots, B_l$ be any ordering of the sets in T and let $\mathcal{A}^{(i)} = \emptyset$ for $1 \leq i \leq l$. Now for each $f : B_1 \rightarrow \{\pm 1\}$ we have that $\mathcal{P}(\mathcal{B}; f; \mathcal{A}^{(1)}, \dots, \mathcal{A}^{(l)})$ is a path in $K_0(n, r)$ containing l vertices. Between them these 2^b paths contain every vertex in T exactly once. We can join these paths into cycles as in the proof of Lemma 2. That is, we join the last vertex $(B_l, \emptyset, f)$ to the first vertex $(B_1, \emptyset, g)$ where g is constructed from f, B_l, B_1 exactly as f_{P_k} is constructed from $f_{P_{k-1}}, B_{k-1}, B_k$ in the definition of the $\mathcal{P}$ construction.

We may assume (by choosing $\mathcal{B}$ suitably) that $B_1 = [3, r + 2], B_2 = \{2\} \cup [4, r + 2]$. This means that each cycle contains an edge $(B_1, \emptyset, f)(B_2, \emptyset, g)$. If $f(r + 1) = -1$ then pick any edge in the block $L_v \cup R_v$ with

$$v = (\pm 1, g(2), -f(3), \dots, -f(r + 1), 0, \dots, 0).$$

If $f(r+1) = 1$ then pick any edge in the block $L_v \cup R_v$ with

$$v = (\pm 1, -g(2), f(3), \dots, f(r+1), 0, \dots, 0).$$

Let this set of edges be M_2 . We can choose these edges so that $M_2 \cap M_1 = \emptyset$, $M_1 \cup M_2$ is independent, and the edges in M_2 are not incident with any linking vertices in the block induced by $L_{v_1} \cup R_{v_1}$. By Lemma 5 there is a Hamilton cycle in our copy of $H(r+1, 2^{r-1})$ containing all edges in $M_1 \cup M_2$. Inserting the cycles partitioning T into this using the edges in M_2 gives a Hamilton cycle in $K_0(n, r)$ which satisfies the conditions required by Lemma 1.

If r is even then the same argument holds except for the fact that the L_v and R_v are no longer pairwise disjoint; for example $R_{(+1, -1, -1, -1, +1)}$ and $L_{(-1, +1, +1, +1, +1)}$ both contain $(+1, -1, -1, -1, 0)$. We can get round this by deleting the set X with $X(r+1) = 0$ from L_v and R_v for all v ending with $+1, +1$. This gives $H'(r+1, 2^{r-1})$ as a subgraph of $K_0(n, r)$. Notice that we never delete a set which will play the part of w, x, y, z in linking the bipartite blocks since the two coordinates which vary between the u and v associated with consecutive bipartite blocks never include $r+1$. We claim that those vertices in the set

$$U = \{X : B(X) = [r], |\{i : X(i) = 1\}| \equiv 0 \pmod{2}\}$$

have not been included. Indeed, if $X \in U$ then modifying X by setting $X(r+1)$ to be 1 does not produce a vector in V . It follows that X is not in any L_v . Similarly, modifying X by setting $X(r+1)$ to be -1 and negating all coordinates does not produce a vector in V and so X is not in any R_v .

Notice that U consists of 2^{r-1} vertices comprising 2^{r-2} pairs (X, Y) with $X(i) = -Y(i)$.

We let M_3 be the set of all edges XY within (L, R) -blocks with $B(X) = B(Y) = [2, r+2]$. By our choice of ordering of V the edges in M_3 are not incident with any linking vertices in the block induced by $L_{v_1} \cup R_{v_1}$. We now choose M_1 and M_2 as before, ensuring also that M_1, M_2, M_3 are pairwise disjoint and that $M_1 \cup M_2 \cup M_3$ is an independent set. By Lemma 5 there is a Hamilton cycle in our copy of $H'(r+1, 2^{r-1})$ containing all edges in $M_1 \cup M_2 \cup M_3$. We insert the pairs of vertices in U into this using the edges in M_3 . Finally, as before, we insert the cycles partitioning T into this using the edges in M_2 . This gives a Hamilton cycle in $K_0(n, r)$ satisfying the conditions required by Lemma 1. $\square$

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