Automorphism group of the derangement graph*

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Abstract

In this paper, we prove that the full automorphism group of the derangement graph Γ_n $(n \geq 3)$ is equal to $(R(S_n) \rtimes \mathrm{Inn}(S_n)) \rtimes Z_2$, where $R(S_n)$ and $\mathrm{Inn}(S_n)$ are the right regular representation and the inner automorphism group of S_n respectively, and $Z_2 = \langle\varphi\rangle$ with the mapping $\varphi : \sigma^\varphi = \sigma^{-1}$, $\forall\, \sigma \in S_n$. Moreover, all orbits on the edge set of Γ_n $(n \geq 3)$ are determined.

Keywords: derangement graph, automorphism group, Cayley graph, symmetric group

1 Introduction

For a finite, simple and undirected graph Γ, we use $V(\Gamma), E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set and full automorphism group, respectively. Let G be a finite group and S a subset of G not containing the identity element 1. The *Cayley graph* $\Gamma := \mathrm{Cay}(G, S)$ on G with respect to S is defined by

$$V(\Gamma)=G,\ \ E(\Gamma)=\{(g, sg) \mid g\in G,\ \ s\in S\}.$$

If $S = S^{-1}$, then $\mathrm{Cay}(G, S)$ can be viewed as an undirected graph by identifying an undirected edge $\{g, h\}$ with two directed edges (g, h) and (h, g). It is easy to see from the definition that there are two obvious facts: (1) Γ is regular of vertex degree $|S|$; (2) Γ is connected if and only if $G = \langle\, S\rangle$.

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A bijection of a finite set Ω to itself is a *permutation* of Ω. Let S_n be the *symmetric group* of permutations of $X = \{1, 2, \cdots, n\}$, and let $\mathcal{D}_n := \{\sigma \in S_n \mid x^\sigma \neq x, \forall x \in X\}$ denote the derangements on X , namely the set of fixed point free permutations of S_n. The graph $\Gamma_n := \mathrm{Cay}(S_n, \mathcal{D}_n)$ is called the *derangement graph* on X . Moreover, denote $|\mathcal{D}_n|$ by D_n for the convenience of writing.

There are many nice structures and properties on the derangement graph Γ_n which are discovered by researchers. For example, Renteln [17] proved that it is connected for $n \geq 4$, the clique number $\omega(\Gamma_n) = n$, and the chromatic number $\chi(\Gamma_n) = n$. Imrich [13] and Hamidoune [11] independently proved that the vertex connectivity $\kappa(\Gamma_n) = D_n$. Eggleton and Wallis [4], and Rasmussen and Savage [16] observed that Γ_n is Hamiltonian. Deza and Frankl [3] proved that the maximum independent number $\alpha(\Gamma_n) = (n-1)!$. Moreover, the structure of maximum independent set of Γ_n, namely a coset of the stabilizer of a point, has been determined by several authors ([1, 10, 15, 19]). Ku and Wong [14] conjectured that $\frac{-D_n}{n-1}$ is the smallest eigenvalue of Γ_n, which has been confirmed by Renteln [17]. Deng and Zhang [2] proved that $\frac{n-3}{n-1}D_{n-2}$ is the second largest eigenvalue of Γ_n.

On the other hand, it is interesting and difficult to determine the full automorphism group of a graph. However, there are some known results on the automorphism groups of Cayley graphs with small degree. For example, Godsil [9] gave the automorphism groups of some cubic Cayley graphs. Feng and Xu [7] determined the automorphism groups of tetravalent Cayley graphs on regular p-groups. Recently, Zhang et al. [22] determined the automorphism groups of cubic Cayley graphs of order $2pq$. For other results on the automorphism groups of Cayley graphs, we refer the readers to [5, 6, 11, 12, 18, 20, 21]. Motivated by the known results and nice structures of the derangement graph, in this paper, we characterize the full automorphism group of the derangement graph. The main result can be stated as follows:

Theorem 1.1. *For $n \geq 3$,*

$$\mathrm{Aut}(\Gamma_n) = (R(S_n) \rtimes \mathrm{Inn}(S_n)) \rtimes Z_2,$$

where $R(S_n)$ and $\mathrm{Inn}(S_n)$ are the right regular representation and the inner automorphism group of S_n respectively, and $Z_2 = \langle \varphi \rangle$ with the mapping $\varphi : \sigma^\varphi = \sigma^{-1}$, $\forall\, \sigma \in S_n$.

The rest of this paper is organized as follows. In Section 2, we gather some definitions and known results needed later. In Section 3, we present the proof of Theorem 1.1, i.e., characterize the full automorphism group of the derangement graph Γ_n ($n \geq 3$). In Section 4, we determine all the edge-orbits of Γ_n, which implies that Γ_n is not edge-transitive.

2 Preliminaries

Let G be a finite group and Ω a finite set. Suppose that, for each $\alpha \in \Omega$ and $g \in G$, there corresponds a member of Ω, denoted by α^g. We say that this correspondence defines an *action* of G on Ω, or G *acts* on Ω, if the following conditions hold: (i) $\forall\ \alpha \in \Omega$, $\alpha^1 = \alpha$, where 1 is the identity element of G; (ii) $\forall\ \alpha \in \Omega$, $\forall\ g, h \in$G , $(\alpha^g)^h = \alpha^{gh}$. Furthermore, if $\{g \in G : \alpha^g = \alpha, \forall \alpha \in \Omega\} = 1$, we say the action of G on Ω is *faithful*, or G acts *faithfully* on Ω.

The action of G on Ω induces naturally an equivalence relation $\sim_G$ which is defined as follows: $\alpha \sim_G \beta$ if and only if $\alpha^g = \beta$ for some $g \in G$. The equivalence classes of $\sim_G$ are said to be *G-orbits* on Ω. If there is only one G-orbit on Ω, then G is said to be *transitive* on Ω, or G acts *transitively* on Ω. In particular, a graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ or $E(\Gamma)$ respectively.

For a group G, let $\mathrm{Aut}(G), \mathrm{Inn}(G)$ and $R(G)$ be the automorphism group, the inner automorphism group and the right regular representation of G, respectively. We need the following known results.

Proposition 2.1. [18, III, Theorem 2.18–2.20] *If $n \geq 2$ and $n \neq 6$, then $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$. If $n = 6$, then $|\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)| = 2$, and for each $\alpha \in \mathrm{Aut}(S_6)\backslash\mathrm{Inn}(S_6)$, α maps a transposition to a product of three disjoint transpositions.*

Proposition 2.2. [8] *Let $N_{Aut(Cay(G,S))}(R(G))$ be the normalizer of $R(G)$ in $\mathrm{Aut}(Cay(G,S))$. Then*

$$N_{Aut(Cay(G,S))}(R(G)) = R(G) \rtimes \mathrm{Aut}(G,S) \leq \mathrm{Aut}(Cay(G,S)),$$

where $\mathrm{Aut}(G,S) = \{\phi \in \mathrm{Aut}(G) \mid S^\phi = S\}$.

Proposition 2.3. [1] *All the maximum-size independent sets of the derangement graph Γ_n $(n \geq 2)$ are $B_{i,j} = \{\sigma \in S_n \mid i^\sigma = j\}$, $i, j = 1, 2, \cdots, n$.*

3 Proof of Theorem 1.1

In this section, we completely determine the full automorphism group of the derangement graph.

Lemma 3.1. *Let $B = \{B_{i,j} \mid i, j = 1, 2, \cdots, n\}$ with $B_{i,j} = \{\sigma \in S_n \mid i^\sigma = j\}$. Then $\mathrm{Aut}(\Gamma_n)$ induces an action on B and this action is faithful. In particular, any $\phi \in \mathrm{Aut}(\Gamma_n)$ is a permutation on B.*

Proof. Obviously, any $\phi \in \mathrm{Aut}(\Gamma_n)$ maps a maximum-size independent set of Γ_n to a maximum-size independent set of Γ_n. So by Proposition 2.3, for any $B_{i,j} \in B$ and $\phi \in \mathrm{Aut}(\Gamma_n)$, we have $B_{i,j}^\phi \in B$.

Next if $\phi \in \mathrm{Aut}(\Gamma_n)$ satisfies $B_{i,j}^\phi = B_{i,j}$ for each $B_{i,j} \in B$, then ϕ is the identity map. In fact,

$$\forall\, \sigma \in S_n,\ \{\sigma\} = B_{1,1^\sigma} \cap B_{2,2^\sigma} \cap \cdots \cap B_{n,n^\sigma}.$$

So

$$\begin{aligned}\{\sigma^\phi\} &= (B_{1,1^\sigma} \cap B_{2,2^\sigma} \cap \cdots \cap B_{n,n^\sigma})^\phi \\ &\subseteq B_{1,1^\sigma}^\phi \cap B_{2,2^\sigma}^\phi \cap \cdots \cap B_{n,n^\sigma}^\phi \\ &= B_{1,1^\sigma} \cap B_{2,2^\sigma} \cap \cdots \cap B_{n,n^\sigma} \\ &= \{\sigma\}.\end{aligned}$$

Thus ϕ is the identity map, that is, $\mathrm{Aut}(\Gamma_n)$ acts faithfully on B. This implies that each $\phi \in \mathrm{Aut}(\Gamma_n)$ is a permutation on B. $\square$

Lemma 3.2. *Let $R_k = \{B_{k,1}, B_{k,2}, \cdots, B_{k,n}\}$ and $C_l = \{B_{1,l}, B_{2,l}, \cdots, B_{n,l}\}$. For any $x_1, x_2, \cdots, x_n \in B$, if $x_1 \cup x_2 \cup \cdots \cup x_n = S_n$, then there exists some k or $l \in \{1, 2, \cdots, n\}$ such that $\{x_1, x_2, \cdots, x_n\} = R_k$ or C_l.*

Proof. First we claim that $B_{i,j} \cap B_{i',j'} = \emptyset$ if and only if exactly one of $i = i'$ and $j = j'$ holds. In fact, If exactly one of $i = i'$ and $j = j'$ holds, then $B_{i,j} \cap B_{i',j'} = \emptyset$. If $i \neq i'$ and $j \neq j'$, then $B_{i,j} \cap B_{i',j'} = \{\sigma \in S_n \mid i^\sigma = j \text{ and } i'^\sigma = j'\} \neq \emptyset$. If $i = i'$ and $j = j'$, then $B_{i,j} = B_{i',j'}$, so $B_{i,j} \cap B_{i',j'} \neq \emptyset$.

Note that $\forall\, i$, $|x_i| = (n-1)!$ and $|S_n| = n!$. Hence

$$x_1 \cup x_2 \cup \cdots \cup x_n = S_n \quad \Rightarrow \quad x_i \cap x_j = \emptyset,\ \forall i, j,\ i \neq j.$$

Applying the above claim, we obtain $\{x_1, x_2, \cdots, x_n\} = R_k$ or C_l. $\square$

Lemma 3.3. *Let $\Omega = \{R_1, R_2, \cdots, R_n, C_1, C_2, \cdots, C_n\}$. Then $\mathrm{Aut}(\Gamma_n)$ induces an action on Ω and this action is faithful. In particular, any $\phi \in \mathrm{Aut}(\Gamma_n)$ is a permutation on Ω.*

Proof. First for any $R_k \in \Omega$ and $\phi \in \mathrm{Aut}(\Gamma_n)$,

$$R_k = \{B_{k,1}, B_{k,2}, \cdots, B_{k,n}\} \Rightarrow R_k^\phi = \{B_{k,1}^\phi, B_{k,2}^\phi, \cdots, B_{k,n}^\phi\},$$

$$B_{k,1}^\phi \cup B_{k,2}^\phi \cup \cdots \cup B_{k,n}^\phi = (B_{k,1} \cup B_{k,2} \cup \cdots \cup B_{k,n})^\phi = S_n^\phi = S_n.$$

By Lemma 3.2, we have $R_k^\phi \in \Omega$.

Similarly, for any $C_l \in \Omega$ and $\phi \in \mathrm{Aut}(\Gamma_n)$, $C_l^\phi \in \Omega$.

Next suppose that $\phi \in \mathrm{Aut}(\Gamma_n)$ satisfies $R_k^\phi = R_k$ and $C_l^\phi = C_l$ for any $k, l \in \{1, 2, \cdots, n\}$. To prove the Lemma, it suffices to show that ϕ is the identity map.

Note that

$$\forall\, B_{i,j} \in B,\ \{B_{i,j}\} = R_i \cap C_j.$$

So

$$\{B_{i,j}^\phi\} = (R_i \cap C_j)^\phi \subseteq R_i^\phi \cap C_j^\phi = R_i \cap C_j = \{B_{i,j}\}.$$

By Lemma 3.1, ϕ is the identity map, that is, $\mathrm{Aut}(\Gamma_n)$ acts faithfully on Ω. This implies that each $\phi \in \mathrm{Aut}(\Gamma_n)$ is a permutation on Ω. $\square$

Lemma 3.4. *Let $R = \{R_1, R_2, \cdots, R_n\}$ and $C = \{C_1, C_2, \cdots, C_n\}$. For any $\phi \in \mathrm{Aut}(\Gamma_n)$, the following (i)-(ii) hold:*

(i) There exists some i such that $R_i^\phi \in R$ if and only if $R_i^\phi \in R$ for any i;

(ii) There exists some j such that $C_j^\phi \in C$ if and only if $C_j^\phi \in C$ for any j.

Proof. (i) Suppose that there exist $i, j\ (\neq i) \in \{1, 2, \cdots, n\}$ such that $R_i^\phi \in R$ and $R_j^\phi \in C$. Note that

$$R_i \cap R_j = \emptyset \ if\ i \neq j \text{ and } |R_k \cap C_l| = 1 \text{ for any } k, l.$$

So

$$|R_i \cup R_j| = 2n \Rightarrow |R_i^\phi \cup R_j^\phi| = |(R_i \cup R_j)^\phi| = 2n.$$

On the other hand,

$$R_i^\phi \in R,\ R_j^\phi \in C \Rightarrow |R_i^\phi \cap R_j^\phi| = 1 \Rightarrow |R_i^\phi \cup R_j^\phi| = 2n - 1,$$

which is a contradiction. Thus the assertion holds.

(ii) is similar to the proof of (i). □

Lemma 3.5.

$$|\mathrm{Aut}(\Gamma_n)| \leq 2(n!)^2.$$

Proof. By Lemma 3.3, any $\phi \in \mathrm{Aut}(\Gamma_n)$ is a permutation of Ω. Using Lemma 3.4, we obtain the following disjoint alternatives:

(i) $R^\phi = R$ and $C^\phi = C$;

(ii) $R^\phi = C$ and $C^\phi = R$.

So we have

$$\mathrm{Aut}(\Gamma_n) = \{\phi \in \mathrm{Aut}(\Gamma_n) \mid R^\phi = R,\ C^\phi = C\} \cup \{\phi \in \mathrm{Aut}(\Gamma_n) \mid R^\phi = C,\ C^\phi = R\}.$$

Hence

$$\begin{aligned} |\mathrm{Aut}(\Gamma_n)| &\leq |\{\phi \in \mathrm{Aut}(\Gamma_n) \mid R^\phi = R,\ C^\phi = C\}| + |\{\phi \in \mathrm{Aut}(\Gamma_n) \mid R^\phi = C,\ C^\phi = R\}| \\ &\leq (n!)^2 + (n!)^2 \\ &= 2(n!)^2. \end{aligned}$$

Thus the assertion holds. □

Now we are ready to prove the main result.

Proof of Theorem 1.1. First we show that the mapping $\varphi:\ S_n \to S_n$ defined as $\sigma^\varphi = \sigma^{-1}$ is an automorphism of Γ_n. In fact, obviously, φ is a bijection between S_n and S_n. Moreover,

$$\begin{aligned} (\sigma, \tau) \in E(\Gamma_n) &\Leftrightarrow \forall\ i \in \{1, 2, \cdots, n\},\ i^\sigma \neq i^\tau \\ &\Leftrightarrow \forall\ i \in \{1, 2, \cdots, n\},\ (i^{\sigma^{-1}})^\sigma \neq (i^{\sigma^{-1}})^\tau \\ &\Leftrightarrow \forall\ i \in \{1, 2, \cdots, n\},\ i \neq i^{\sigma^{-1}\tau} \\ &\Leftrightarrow \forall\ i \in \{1, 2, \cdots, n\},\ i^{\tau^{-1}} \neq (i^{\sigma^{-1}\tau})^{\tau^{-1}} \\ &\Leftrightarrow \forall\ i \in \{1, 2, \cdots, n\},\ i^{\tau^{-1}} \neq i^{\sigma^{-1}} \\ &\Leftrightarrow (\sigma^\varphi, \tau^\varphi) = (\sigma^{-1}, \tau^{-1}) \in E(\Gamma_n). \end{aligned}$$

This implies that φ is an automorphism of Γ_n.

Next we claim that $R(S_n) \rtimes \mathrm{Inn}(S_n) \leq \mathrm{Aut}(\Gamma_n)$. In fact, by Proposition 2.1, we have

$$\mathrm{Aut}(S_n, \mathcal{D}_n) = \{\phi \in \mathrm{Aut}(S_n) \mid \mathcal{D}_n^\phi = \mathcal{D}_n\} = \mathrm{Inn}(S_n),$$

where $\mathcal{D}_n = \{\sigma \in S_n \mid x^\sigma \neq x, \forall x \in \{1, 2, \cdots, n\}\}$. Then by Proposition 2.2, $R(S_n) \rtimes \mathrm{Inn}(S_n) \leq \mathrm{Aut}(\Gamma_n)$.

Note that S_n ($n \geq 3$) is a centerless group. So $\mathrm{Inn}(S_n) \cong S_n/Z(S_n) = S_n$, where $Z(S_n)$ is the center of S_n. Thus $(n!)^2 = |R(S_n) \rtimes \mathrm{Inn}(S_n)| \leq |\mathrm{Aut}(\Gamma_n)| \leq 2(n!)^2$ (by Lemma 3.5), that is, the index of $R(S_n) \rtimes \mathrm{Inn}(S_n)$ in $\mathrm{Aut}(\Gamma_n)$ is at most 2, which implies that $R(S_n) \rtimes \mathrm{Inn}(S_n)$ is a normal subgroup of $\mathrm{Aut}(\Gamma_n)$. In addition, it is easy to see that $\varphi \notin R(S_n)$ and $\varphi \notin \mathrm{Inn}(S_n)$ for $n \geq 3$. Hence $(R(S_n) \rtimes \mathrm{Inn}(S_n)) \rtimes Z_2 \leq \mathrm{Aut}(\Gamma_n)$ (where $Z_2 = \langle \varphi \rangle$ is a cyclic group of order 2) and $2(n!)^2 = |(R(S_n) \rtimes \mathrm{Inn}(S_n)) \rtimes Z_2| \leq |\mathrm{Aut}(\Gamma_n)| \leq 2(n!)^2$, which shows that $\mathrm{Aut}(\Gamma_n) = (R(S_n) \rtimes \mathrm{Inn}(S_n)) \rtimes Z_2$. The assertion holds. □

4 Edge-orbits of the derangement graph

It is well known that any permutation can be decomposed as a product of disjoint permutation cycles. For any $\sigma \in S_n$, write

$$\sigma = (a_{1\,1}\, a_{1\,2} \, \cdots \, a_{1\,n_1})(a_{2\,1} \, \cdots \, a_{2\,n_2}) \cdots (a_{s\,1} \, \cdots \, a_{s\,n_s}),$$

a product of disjoint cycles (including 1-cycles), with $n_1 \geq n_2 \geq \cdots \geq n_s$ and $n_1 + n_2 + \cdots + n_s = n$, and we call $(n_1, n_2, \cdots, n_s)$ the *cycle−shape* of σ.

For any $\sigma \in S_n$ and $\phi \in \mathrm{Inn}(S_n) \times Z_2$ (where $Z_2 = \langle \varphi \rangle$ is same as Theorem 1.1), obviously σ and σ^ϕ have the same cycle-shape. Note that φ commutes with each element in $\mathrm{Inn}(S_n)$. Thus, $\mathrm{Inn}(S_n) \rtimes Z_2 = \mathrm{Inn}(S_n) \times Z_2$.

Lemma 4.1. *Let* $\mathrm{Aut}(\Gamma_n)$ $(n \geq 3)$ *act naturally on* $E(\Gamma_n)$*. For any* $(1, \tau)$, $(1, \sigma) \in E(\Gamma_n)$, $(1, \tau)$ *and* $(1, \sigma)$ *belong to the same* $\mathrm{Aut}(\Gamma_n)$*-orbit if and only if* τ *and* σ *have the same cycle-shape.*

Proof. ($\Rightarrow$) If $(1, \tau)$ and $(1, \sigma)$ belong to the same $\mathrm{Aut}(\Gamma_n)$-orbit, then we have the following disjoint alternatives:

(i) There exists $\phi \in \mathrm{Aut}(\Gamma_n)$ such that $1^\phi = 1$ and $\tau^\phi = \sigma$;

(ii) There exists $\phi \in \mathrm{Aut}(\Gamma_n)$ such that $\tau^\phi = 1$ and $1^\phi = \sigma$.

By Theorem 1.1, we can always assume that $\phi = R(g) \cdot \phi'$, where $\phi' \in \mathrm{Inn}(S_n) \times Z_2$.

If the case (i) happens, then

$$1^\phi = 1 \Rightarrow 1^{R(g)\cdot\phi'} = 1 \Rightarrow g^{\phi'} = 1 \Rightarrow g = 1.$$

$$\tau^\phi = \sigma \Rightarrow \tau^{R(g)\cdot\phi'} = \sigma \Rightarrow (\tau g)^{\phi'} = \sigma \Rightarrow \tau^{\phi'} = \sigma.$$

So τ and σ have the same cycle-shape.

If the case (ii) happens, then

$$\tau^\phi = 1 \Rightarrow \tau^{R(g)\cdot\phi'} = 1 \Rightarrow (\tau g)^{\phi'} = 1 \Rightarrow \tau g = 1 \Rightarrow g = \tau^{-1}.$$

$$1^\phi = \sigma \Rightarrow 1^{R(g)\cdot\phi'} = \sigma \Rightarrow g^{\phi'} = \sigma \Rightarrow (\tau^{-1})^{\phi'} = \sigma.$$

So τ^{-1} and σ have the same cycle-shape, that is, τ and σ have the same cycle-shape.

($\Leftarrow$) If τ and σ have the same cycle-shape, then there exists some $\phi \in \mathrm{Inn}(S_n) \leq \mathrm{Aut}(\Gamma_n)$ such that $\tau^\phi = \sigma$. Hence $(1, \tau)^\phi = (1^\phi, \tau^\phi) = (1, \sigma)$, that is, $(1, \tau)$ and $(1, \sigma)$ belong to the same $\mathrm{Aut}(\Gamma_n)$-orbit. □

Using Lemma 4.1, we have the following result:

Corollary 4.2. *Let* $\mathrm{Aut}(\Gamma_n)$ $(n \geq 3)$ *act naturally on* $E(\Gamma_n)$*. For any* (σ_1, τ_1), $(\sigma_2, \tau_2) \in E(\Gamma_n)$, (σ_1, τ_1) *and* (σ_2, τ_2) *belong to the same* $\mathrm{Aut}(\Gamma_n)$*-orbit if and only if* $\tau_1\sigma_1^{-1}$ *and* $\tau_2\sigma_2^{-1}$ *have the same cycle-shape.*

Proof. (σ_1, τ_1) and (σ_2, τ_2) belong to the same $\mathrm{Aut}(\Gamma_n)$-orbit $\Leftrightarrow$ there exists some $\phi \in \mathrm{Aut}(\Gamma_n)$ such that $(\sigma_1, \tau_1)^\phi = (\sigma_2, \tau_2) \Leftrightarrow (1, \tau_1\sigma_1^{-1})^{R(\sigma_1)\,\phi\,R(\sigma_2^{-1})} = (1, \tau_2\sigma_2^{-1}) \Leftrightarrow (1, \tau_1\sigma_1^{-1})$ and $(1, \tau_2\sigma_2^{-1})$ belong to the same $\mathrm{Aut}(\Gamma_n)$-orbit. By Lemma 4.1, the assertion holds. □

By the definition of the derangement graph Γ_n, we have $(\sigma, \tau) \in E(\Gamma_n)$ if and only if $\tau\sigma^{-1} \in \mathcal{D}_n$. Therefore, applying Corollary 4.2, the $\mathrm{Aut}(\Gamma_n)$-orbits on $E(\Gamma_n)$ are in bijective correspondence with the set of all possible cycle-shapes of permutations in $\mathcal{D}_n$. So we obtain the main result in this section as follows:

Theorem 4.3. *Let* $\mathrm{Aut}(\Gamma_n)$ $(n \geq 3)$ *act naturally on* $E(\Gamma_n)$*. All* $\mathrm{Aut}(\Gamma_n)$*-orbits are* $O_{(n_1,n_2,\cdots,n_s)} = \{(\sigma, \tau) \in E(\Gamma_n) \mid \tau\sigma^{-1}$ *has cycle-shape* $(n_1, n_2, \cdots, n_s), n_s \geq 2\}$*. In particular, the number of edge-orbits of* Γ_n $(n \geq 3)$ *equals to the cardinality of the set* $\{\{n_1, n_2, \cdots, n_s\} \mid n = n_1 + n_2 + \cdots + n_s,\ n_i \geq 2,\ 1 \leq i \leq s\}$.

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