How many faces can polycubes of lattice tilings by translation of $\mathbb{R}^3$ have?

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Abstract

We construct a class of polycubes that tile the space by translation in a lattice-periodic way and show that for this class the number of surrounding tiles cannot be bounded. The first construction is based on polycubes with an L-shape but with many distinct tilings of the space. Nevertheless, we are able to construct a class of more complicated polycubes such that each polycube tiles the space in a unique way and such that the number of faces is $4k + 8$ where $2k + 1$ is the volume of the polycube. This shows that the number of tiles that surround the surface of a space-filler cannot be bounded.

Keywords: tilings of $\mathbb{R}^3$, tilings by translation, lattice periodic tilings, space-fillers

1 Introduction

Searching tilings in dimension 3 in order to construct material with various properties of rigidities, thinness and weakness is a challenge for metallurgists, crystallographers, mathematicians and artists [13, 16]. We focus on space-fillers by translation that is a single tile that tiles the space by translation. In the literature, one can find the five solids of Fedorov that are convex space-fillers and that tile the space by translation [8]. If we relax the condition of convexity and allow also other isometries than translation, many definitions of space-fillers appear with nice properties [13, 4].

We consider in this article lattice tiling by translation that is tiling with a single tile T and such that the positions of copies of T in the tiling are given by integral combinations of non-collinear vectors forming a base of $\mathbb{R}^2$ for plane tilings (or of $\mathbb{R}^3$ for space tilings). Of course, there are infinitely many metrically different shapes in lattice tilings of the plane but if one consider them up to affine transformations then there are only two basic tiles namely the square and the hexagon [8, 1]. Thus for tilings of the plane the number of faces is either 4 for the square-like tiles or 6 for the hexagon-like tiles.

We are also able to design fast algorithms to decide whether or not a polyomino tiles the plane [12, 2]. In fact, for rectifiable shapes of tiles with or without a condition of convexity the tile tiles either like a square or like an hexagon. This result is based on investigation of the contour word of polyominoes and factorizations of this word with two forms one for the square shape and one for the hexagon shape [1]. Our goal was to generalize this nice characterization to dimension 3 and we design an algorithm to find all lattice tilings by polycubes [10]. But the case of dimension 3 is much more difficult than for dimension 2. For example, we found in a previous article a family of tiles such that each tile of this family tiles the space by translation but not like a lattice [11]. In dimension 3, we consider basic tiles constructed from the set of tiles that tiles the space up to affine transformations. For each basic tile, the number of faces is given by counting the number of copies that surround the origin tile of the lattice tiling. In this article, we present a construction of a family of non convex polycubes such that each tile of the family tiles the space by translation with a great number of contacts with other tiles. In fact, we show that the tiling covers the surface of the tile located at the origin by $4k + 8$ translated copies of the original tile of volume $2k + 1$. That is, in our construction, the associated polycube has $4k + 8$ faces.

This shows that the number of basic tiles for tiling is not finite and there will be no way for setting a Beauquier-Nivat Theorem in dimension 3. Thus the number of tiles that surround the surface of a space-filler can not be bounded.

2   Fedorov solids

In 1891, Fedorov found 5 convex space-fillers [8, 24] associated with regular lattices (see Fig. 1). Notice that the most complicated of these objects appears in various contexts: it is the solid of Kelvin [18] proposed in the context of minimizing the volume of a space-filler when its surface is normalized to 1 and it is also known in algebraic combinatorics by the name of permutahedron [3, 25].

Nevertheless, we are able to find polyhedra that tile the space with more faces than the Fedorov solids [19]. The following non-convex polyhedron (see Fig. 2) constructed by Cyril Stanley Smith in 1953 [21] tiles the space by translation and in order to cover its surface we must surround the original polyhedra by 16 distinct copies. This means that the basic tile contains 16 faces. In fact, the construction is a modification of a permutahedron which is the Fedorov solid with the maximum number of faces, namely 14. Of course, in the construction the author shows that it is really a space-filler (see Fig. 3).
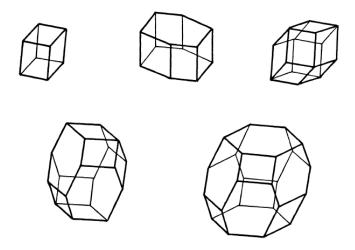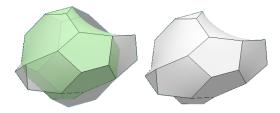
Figure 1: The 5 solids of Fedorov

Figure 2: Smith's space-filler

In the case of convex space-fillers congruent to a single tile up to isometry, there is an upper bound for the number of faces of basic tiles which is due to Delone and Sandakova [5]. In the article [20] the best upper bound for this case is 92.

Nevertheless, if we remove the convexity constraint and focus only on lattice tilings by translation, some natural questions come up: Is the number of surrounding faces of a space-filler bounded [6]? Can we find a finite number of basic tiles as in dimension 2 where by the theorem of Beauquier and Nivat the basic tiles are either a square or an hexagon?

In the literature, we find a huge number of constructions of space-fillers [14, 15] with various constructions based on parallelohedra, zonohedra, monohedra, isozonohedra and otherhedra [16, 7, 25]. We find also a more algebraic approach for tilings of the plane and the space including a whole study of cross-like tiles in all dimensions [22, 21].

In order to avoid specific constructions of polyhedra such as permutohedra, associahedra, stereohedra and so on, we focus on a more discrete model by constructing polycubes [9] (simply connected finite unions of unit cubes) that are space-fillers by translation.
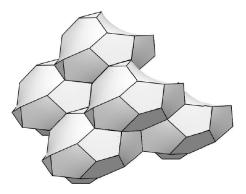
Figure 3: Tiling of the space by Smith's space-filler

3 $L-$shape with a maximum of contacts

In this section, we construct a polycube surrounded by a great number of translated copies of itself.

Theorem 3.1. *For all k, there exists a polycube $P_k = L_{2k+1}$ with volume $2k + 1, k \geq 1$ that tiles the space by translation and for which the number of faces is $4k + 8$.*

Proof. Let L_{2k+1} be a polycube with L-shape form constructed from an origin cube at position $(0, 0, 0)$ and 2 branches of length k one called X_k with cubes in position $(i, 0, 0)$ for $i = 1, \cdots, k$ and the second one called Y_k with cubes in position $(0, j, 0)$ for $j = 1, \cdots, k$. This tile L_{2k+1} has volume $2k + 1$ and tiles the space layer by layer.

  The first layer with height equal to zero is tiled as a two dimensional tiling with L_{2k+1}. First, we put a tile L_{2k+1} in position $(0, 0, 0)$. Thus the next translated position is given by $T(L_{2k+1}) = L_{2k+1} + v_1$ where $v_1 = (1, 1, 0)$. And we choose a second vector of translation in the first layer namely $v_2 = (2k + 1, 0, 0)$. All the layer with height equal to zero is tiled by integral combinations of v_1 and v_2.

  As the tiling of each layer is unique up to translation, we find that the number of faces in each layer is 6. Now we would like to maximize the number of faces between two consecutive layers. To do that we translate the first layer by $(0, k, 1)$. If we compute the number of faces we find $2k + 1$. Indeed, the upper face in position $(0, 0, 0)$ is in contact with the cube $(0, k, 0)$ of the tile $L_{2k+1} + (0, k, 1)$. The upper face of $(-1, -1, 0)$, that is the cube $(1, 0, 0)$ of L_{2k+1}, is in contact with the cube $(0, k - 1, 0)$ of the tile $L_{2k+1} + (0, k, 1)$. By induction, the upper face in position $(-i, -i, 0)$, that is the cube $(i, 0, 0)$ of L_{2k+1}, is in contact with the cube $(0, k - i, 0)$ of the tile $L_{2k+1} + (0, k, 1)$ for $i = 0, \cdots, k$. Thus we find $k + k$ contacts that give $2k$ faces for each branch, 2 contacts for the origin cube and of course the 6 contacts in the layer. In total by addition of the faces in the layer, we find $2k + 2k + 2 + 6 = 4k + 8$ faces for the $L-$shape with $2k + 1$ cubes (see Fig. 4 for an example with $k = 2$). $\qquad\square$

  Remark that this tiling is not unique because we have many possibilities for the positions of the consecutive layers.
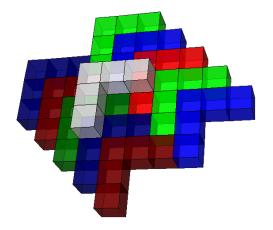
Figure 4: Tiling of the 0 layer by an $L-$shape and contacts with a tile in the 1 layer

4 Space-fillers that tile in a unique way

Next, we show how to construct a family of polycubes that give for each polycube a unique tiling with a maximum of contacts.

Theorem 4.1. *For all k, there exists a polycube $P'_k = G_{2k+1}$ with volume $2k+1, k \geq 3$ that tiles the space by translation in a unique way and for which the number of faces is $4k+8$.*
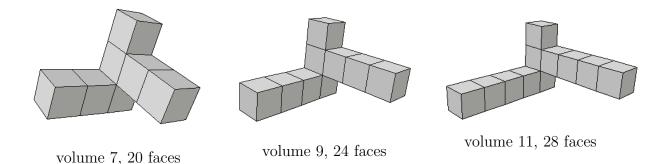

volume 7, 20 faces   volume 9, 24 faces   volume 11, 28 faces

Figure 5: Polycubes that tile the space in a unique way for $k = 3, 4$ and 5

Proof. Let G_{2k+1} be a polycube with 2 branches of length $k \geq 3$ one called X_k with cubes in position $(i, 0, 0)$ for $i = 0, \cdots, k-1$ and the second one called Y_k with cubes in position $(0, j, 1)$ for $j = 0, \cdots, k-1$ and a single cube in position $(0, 0, 2)$. This tile G_{2k+1} has volume $2k+1$ (see Fig. 5 for examples with $k = 3, 4, 5$) and tiles the space layer by layer.

The first layer with height equal to zero is tiled in the following way. First, we put the branch $(i, 0, 0)$ for $i = 0, \cdots, k-1$ of the polycube. We then translate the polycube

at the origin by the vector $(-1, 0, -1)$ and place the second branch $Y_k + (-1, 0, -1)$ that is the cubes in position $(-1, j, 0)$ for $j = 0, \cdots, k - 1$. The only remaining position for the translation of the single cube (in position $(0, 0, 2)$) in order to have an L-shape with equal branches will be $(0, 0, 2) + (-1, k, -2) = (-1, k, 0)$. Thus, we use the translation $(-1, k, -2)$ as the second vector of translation (see Fig. 6 where the red polycube is the polycube at the origin, the green one is the translated copy by (-1,0,-1) and the blue one is the translated copy by (-1,3,-2)). In fact, we just construct an L-shape with $2k + 1$ cubes with two branches of length k. This construction is unique because we find a shape in two dimensions with two branches of length k and a single cube, and the tiling in two dimension with an L-shape is unique if $k \geq 1$. We just have to translate, in the 0 layer, this L-shape by the usual translation by $(1, 1, 0)$. Thus the basis for our tiling of the space is $v_1 = (-1, 0, -1)$, $v_2 = (-1, k, -2)$ and $v_3 = (1, 1, 0)$.
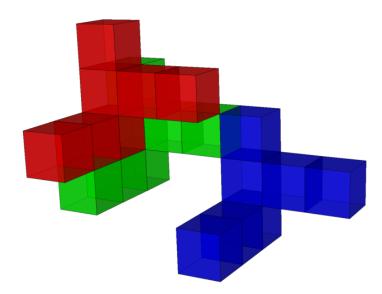

Figure 6: The polycube at the origin (in red) and two translated copies

The first layer is tiled with an L-shape with combinations of tiles having the single cube, X_k or Y_k in this layer (see Fig. 7). All the layers has the same tiling up to a translation and we find the position of tiles in the layers by translating the layer at height equal to zero by translations $(-i, 0, -i)$ with $i \in \mathbb{Z}$.

The number of contacts is given by a direct examination of a tiling of an L-shape with maximization of the contacts. We find (by the same reasoning as in the proof of the previous Theorem) $2k + 2k + 2 + 6 = 4k + 8$ faces for the G-shape with $2k + 1$ cubes. $\quad\square$
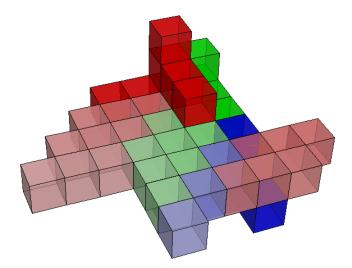
Figure 7: The layer 0 and 3 polycubes

5 Conclusion

In summary, we define a family of space fillers with volume $2k + 1$ such that the surface of the origin tile is covered by $4k + 8$ translated copies of the original tile. This shows that if we relax the convexity constraint we find a non finite number of basic tiles and thus there is no way of constructing a Beauquier-Nivat Theorem in dimension 3.

Many interesting questions remain open. In particular, we think that the polycubes in Theorem 4.1 are the minimal shapes in volume that tile the space by translation with maximal numbers of contacts. That is there is no other polycube with the same volume $2k + 1$ that tiles the space by translation and with a greater number (greater than $4k + 8$) of translated polycubes that surrounded the origin polycube. It will be also interesting, following the work of Vallentin [24], to find the analogues of such polycubes in dimension 4 or more.

Acknowledgements

References

[1] D. Beauquier and M. Nivat, On translating one polyomino to tile the plane, *Disc. Comput. Geom.* 6 (1991) 575–592.

[2] S. Brlek, X. Provençal, On the tiling by translation problem, *Discrete Appl. Math.*, 157, n. 3 (2009) 464–475.

[3] J. H. Conway and N. J. A. Sloane, Low-Dimensional Lattices. VI. Voronoi Reduction of Three-Dimensional Lattices. Proceedings: Mathematical and Physical Sciences, 436, No. 1896 (Jan. 8, 1992) 55–68.

[4] L. Danzer, A family of 3D-spacellers not permitting any periodic or quasiperiodic tiling, *In Aperiodic '94 (G. Chapuis, Ed., World Scientific, Singapore)*, (1995) 11–17.

[5] B. N. Delone and N. N. Sandakova, Theory of stereohedra. (Russian) *Trudy Mat. Inst. Steklov*, 64 (1961) 28–51.

[6] N. Dolbillin and M. Tanemura, How Many Facets on Average can a Tile Have in a Tiling?, *Forma*, 21 (2006) 177–196.

[7] P. Engel, Über Wirkungsbereichsteilungen von kubischer Symmetrie, *Zeitschrift fr Kristallographie*, 154, n. 3-4 (1981) 199–215.

[8] Fedorov, Regular partition of plane and space, (Russian; translation of Fedorov [1899]) Nauka, Leningrad, 1979.

[9] S. W. Golomb, Checker boards and polyominoes, *Amer. Math. Monthly*, 61, n. 10 (1954) 675–682.

[10] I. Gambini, An algorithm for the enumeration of lattice tilings of the space by polycubes, in preparation.

[11] I. Gambini and L. Vuillon, Non lattice periodic tilings of $\mathbb{R}^3$ by single polycubes, LAMA research report.

[12] I. Gambini and L. Vuillon, An algorithm for deciding if a polyomino tiles the plane by translations, *Theoretical Informatics and Applications*, 41 (2007), n 2, 147–155.

[13] B. Grünbaum, G.C. Shephard, Tilings with congruent tiles, *Bull. Amer. Maths. Soc.*, 3, n. 3 (1980) 951–974.

[14] B. Grünbaum, Tilings by some nonconvex parallelohedra, *Geombinatorics*, 19 (2010), 100–107.

[15] B. Grünbaum, The Bilinski dodecahedron, and assorted parallelohedra, zonohedra, monohedra, isozonohedra and otherhedra, *Math. Intelligencer*, 32, n. 4 (2010) 5–15.

[16] A. J. Guttmann (ed.), Polygons, Polyominoes and Polycubes, *Lecture Notes in Physics*, 775, Springer, Berlin, 2009.

[17] K. Keating and A. Vince, Isohedral polyomino tilings of the plane, *Disc. Comput. Geom.* 21(1999) 615–630.

[18] W. T. Kelvin, On the division of space with minimum partitional area. *Philos. Mag. Lett.*, 24(151):503, 1887.
http://zapatopi.net/kelvin/papers/on_the_division_of_space.html.

[19] E. A. Lord, A. L. Mackay, S. Ranganathan, New Geometries for New Materials. (2006). Cambridge University Press.

[20] P. Sabariego and F. Santos, On the Number of Facets of Three-Dimensional Dirichlet Stereohedra IV: Quarter Cubic Groups (2010), Arxiv 0708.2114

[21] C. S. Smith, *Acta Metallurgica*, 1, May 1953, p295.

[22] S. K. Stein and S. Szabo, Algebra and Tiling, *Math. Assoc. of America*, Washington, DC 1994.

[23] S. K. Stein, Factoring by subsets, *Pacific J. Math.* 22 (1967), 523–541.

[24] F. Vallentin, Sphere coverings, lattices, and tilings (in low dimensions), Ph.D. thesis, Technical University Munich, Germany, 2003, 128 pages.

[25] G. M. Ziegler, Lectures on Polytopes, New York: Springer-Verlag, 1995.