Subplanes of order 3 in Hughes Planes

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To Professor Spyros Magliveras on his 70th birthday

Abstract

In this study we show the existence of subplanes of order 3 in Hughes planes of order q^2, where q is a prime power and $q \equiv 5$ (*mod* 6). We further show that there exist finite partial linear spaces which cannot embed in any Hughes plane.

1 Introduction

L. Puccio and M. J. de Resmini [5] showed that subplanes of order 3 exist in the Hughes plane of order 25. (We refer always to the *ordinary* Hughes planes; equivalently, all our nearfields are regular.) Computations of the second author [2] show that among the known projective planes of order 25 (including 99 planes up to isomorphism/duality), exactly four have subplanes of order 3. These four planes are the ordinary Hughes plane and three closely related planes. Recently, Caliskan and Magliveras [1] showed that there are exactly 2 orbits on subplanes of order 3 in the Hughes plane of order 121. In this study we show that every Hughes plane of order q^2, where q is a prime power and $q \equiv 5$ (*mod* 6), has subplanes of order 3.

We begin with the construction of the Hughes plane $H(q^2)$ of order q^2, q an odd prime power, as given by Rosati [6] and Zappa [9]. Throughout this paper, $\mathbb{K}$ denotes a finite field of order q^2, and $\mathbb{F}$ its subfield of order q, where q is an odd prime power. For any $\theta \in \mathbb{K}$ with $\theta \notin \mathbb{F}$, we have $\mathbb{K} = \mathbb{F}[\theta]$ and $\{1, \theta\}$ is a basis for $\mathbb{K}$ over $\mathbb{F}$. We will always choose θ such that $\theta^2 = d \in \mathbb{F}$, where d is a nonsquare in $\mathbb{F}$. We now define the *regular nearfield* N of order q^2, where N has the same elements as $\mathbb{K}$ and the same *addition*. However, *multiplication* in N is defined as follows: $a \circ b = ab$ if a is a square in $\mathbb{K}$, and $a \circ b = ab^q$ otherwise. Let $V = \{(x, y, z) \mid x, y, z \in N\}$ be the 3-dimensional left vector space over N. Define the *set of points* (*set of lines*) of $H(q^2)$ to be the set of all equivalence classes of elements of $V \setminus \{(0,0,0)\}$, under the equivalence $(x,y,z) \sim (k \circ x, k \circ y, k \circ z)$ ($[a,b,c] \sim [k \circ a, k \circ b, k \circ c]$) for $k \in N^*$. It is occasionally convenient to 'normalize' the vector representatives (x,y,z) for points (using left-multiplication by elements of N^*) so that their first nonzero coordinate is 1; coordinates for lines may be similarly normalized.

We may take $\{1,\theta\}$ as a basis for N as a vector space over $\mathbb{F}$. The *incidence relation* for $H(q^2)$ is defined as follows : Point (x,y,z) is incident with line $[a,b,c]$, where $a = a_1 + a_2\theta$, $b = b_1 + b_2\theta$, and $c = c_1 + c_2\theta$, if and only if $xa_1 + yb_1 + zc_1 + (xa_2 + yb_2 + zc_2) \circ \theta = 0$. It is well known that different choices of θ give isomorphic planes of order q^2.

In order to implement nearfield multiplication in N, the following is useful for readily identifying squares in $\mathbb{K}$.

Lemma 1.1 *Consider a quadratic extension $\mathbb{K} = \mathbb{F}[\theta] \supset \mathbb{F}$ where $\mathbb{F}$ is a field of odd order q, and $\theta^2 = d \in \mathbb{F}$. A typical element $x = a + b\theta$ (where $a, b \in \mathbb{F}$) is a square in $\mathbb{K}$, iff its norm $x^{q+1} = a^2 - db^2$ is a square in $\mathbb{F}$.*

Proof: We may assume $x \neq 0$. The element $x \in \mathbb{K}$ is a square in $\mathbb{K}$ iff $x^{(q^2-1)/2} = 1$ iff $(x^{q+1})^{(q-1)/2} = 1$, iff the element $x^{q+1} \in \mathbb{F}$ is a square in $\mathbb{F}$. Note that $x^{q+1} = x^q x = (a - b\theta)(a + b\theta) = a^2 - db^2$. $\square$

It has long been recognized by M. J. de Resmini and others that Hughes planes have subplanes of order 2; for completeness we include a proof of this fact in Section 2. On the other hand, this is not totally surprising since for a quadrilateral to generate a subplane of order 2 only requires a single algebraic condition to hold. In order for a quadrilateral to generate a subplane of order 3, several inequivalent conditions must hold. We show the existence of subplanes of order 3 in the Hughes plane $H(q^2)$ in Section 3 in case $q \equiv 5 \pmod{12}$, and in Section 4 in case $q \equiv 11 \pmod{12}$.

2 Subplanes of order 2

We require the following technical lemma.

Lemma 2.1 *Let $\mathbb{F}$ be a finite field of odd order q, and let $d \in \mathbb{F}$ be a nonsquare.*

(a) If $q \equiv 1 \pmod 4$ then there exists a nonzero element $b \in \mathbb{F}$ such that $b^4 + db^2 + d^2$ is a nonsquare in $\mathbb{F}$.

(b) If $q \equiv 3 \pmod 4$ then there exist $(q+1)/2$ nonzero values of $b \in \mathbb{F}$ such that $b^2 + 1$ is a nonsquare in $\mathbb{F}$.

Proof: (a) The equation $x^2 + dxz + d^2z^2 = dy^2$ defines a nondegenerate conic in the classical projective plane coordinatized by $\mathbb{F}$, with homogeneous coordinates (x,y,z). Since d is a nonsquare in $\mathbb{F}$, all $q+1$ points of this conic must have $xz \neq 0$ and so all points of the conic have the form $(x,y,1)$ with $x \neq 0$. No more than two such points share the same x-coordinate, so the points $(x,y,1)$ of the conic have at least $(q+1)/2$ distinct nonzero

x-coordinates. Since $\mathbb{F}$ contains only $(q-1)/2$ nonsquares, the conic must contain a point of the form $(b^2, y, 1)$ with $b \neq 0$.

(b) The equation $x^2+y^2+z^2=0$ defines a nondegenerate conic in the classical projective plane coordinatized by $\mathbb{F}$. Since -1 is a nonsquare in $\mathbb{F}$, all $q+1$ points of the conic have the form $(x, 1, z)$ in homogeneous coordinates with $xz \neq 0$. No more than two such points $(x, 1, \pm z)$ share the same x-coordinate, yielding $(q+1)/2$ values of x for which x^2+1 equals a nonsquare $-z^2$. □

Theorem 2.2 *Every Hughes plane has a subplane of order 2.*

Proof: Let d be a nonsquare in $\mathbb{F}$, so that $\mathbb{K} = \mathbb{F}[\theta]$ where $\theta \in \mathbb{K}$ satisfies $\theta^2 = d$. We consider two cases.

Suppose first that $q \equiv 1 \bmod 4$. In this case -1 is a square in $\mathbb{F}$, and θ is a nonsquare in $\mathbb{K}$ since its norm $\theta^q\theta = (-\theta)\theta = -d$ is a nonsquare in $\mathbb{F}$. Choose $b \in \mathbb{F}$ such that $b^4 + db^2 + d^2$ is a nonsquare in $\mathbb{F}$ as in Lemma 2.1(a). Write $c = (b/d) + (1/b) \in \mathbb{F}$, so that $1 \pm c\theta$ is a nonsquare in $\mathbb{K}$ by Lemma 1.1. The seven points $p_0, p_1, \ldots, p_6$ of the Hughes plane with coordinates

$$(1,0,0),\ (0,1,0),\ (1,-d/b,\theta),\ (1,\theta,b),\ (1/b,-(b/d)\theta,1),\ (1,b+\theta,0),\ (1,b,\theta)$$

and the seven lines $\ell_0, \ell_1, \ldots, \ell_6$ with coordinates

$$[0,\theta,-b],\ [0,0,1],\ [\theta,0,-1],\ [0,-b,\theta],\ [-b,0,1],\ [-b-\theta,1,1+c\theta],\ [-b-\theta,1,1]$$

satisfy $p_i \in \ell_j$ iff $j - i \in \{0,1,3\}$ mod 7. This gives a subplane of order 2 in the Hughes plane of order q^2.

Now suppose that $q \equiv 3 \bmod 4$. In this case we may take $d = -1$, a nonsquare in $\mathbb{F}$, and θ is a square in $\mathbb{K}$ since its norm $\theta^{q+1} = -d = 1$ is a square in $\mathbb{F}$. By Lemma 2.1(b), there exists $b \in \mathbb{F}$ such that b^2+1 is a nonsquare in $\mathbb{F}$. By Lemma 1.1, the elements $1 \pm b\theta$ and $b \pm \theta$ are nonsquares in $\mathbb{K}$. The seven points of the Hughes plane

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,\theta,0),\ (0,1,1-b\theta),\ (1,\theta,b+\theta),\ (1,0,b+\theta)$$

and the seven lines

$$[0,0,1],\ [1,0,0],\ [1,\theta,0],\ [-b-\theta,-1+b\theta,1],\ [0,-1+b\theta,1],\ [-b-\theta,0,1],\ [0,1,0]$$

give a subplane of order 2, where as before we have $p_i \in \ell_j$ iff $j - i \in \{0,1,3\}$ mod 7. □

3 Case: $q \equiv 5$ $(mod\ 12)$

Let $q \equiv 5$ $(mod\ 12)$. We may take $d = -3$, a nonsquare in $\mathbb{F}$, and $\mathbb{K} = \mathbb{F}[\theta]$ where $\theta^2 = -3$. There is an element $i \in \mathbb{F}$ satisfying $i^2 = -1$, since $q \equiv 1$ $(mod\ 4)$. Also $\omega = (-1+i\theta)/2 \in \mathbb{K}$ is a primitive cube root of unity, and the other is $\omega^2 = (-1-i\theta)/2$. Furthermore, $\zeta = i\omega = (-i+i\theta)/2 \in \mathbb{K}$ is a primitive 12-th root of unity. We compute that $\zeta^2 = (1+\theta)/2$, $\zeta^4 = \omega = (-1+\theta)/2$, and $\zeta^5 = i\omega^2 = (-i - i\theta)/2$. Moreover, $\zeta + \zeta^7 = \zeta^2 + \zeta^8 = \zeta^4 + \zeta^{10} = \zeta^5 + \zeta^{11} = 0$, since $\zeta^6 = -1$. Hence, $\zeta^7 = (i - i\theta)/2$, $\zeta^8 = (-1-\theta)/2$, $\zeta^{10} = (1-\theta)/2$, and $\zeta^{11} = (i + i\theta)/2$. The following Lemma follows easily from Lemma 1.1.

Lemma 3.1 *$1 \pm \theta$ are squares and θ, $3 \pm \theta$ not squares in $\mathbb{K}$.*

We now define α, a set of 13 points, and β, a set of 13 lines, as follows :

$$
\alpha : \quad
\begin{array}{rl}
p_1 & (0,0,1) \\
p_2 & (0,1,0) \\
p_3 & (0,1,\zeta) \\
p_4 & (0,1,\zeta^7) \\
p_5 & (1,0,0) \\
p_6 & (1,0,\zeta^2) \\
p_7 & (1,0,\zeta^8) \\
p_8 & (1,\zeta,0) \\
p_9 & (1,\zeta,\zeta^2) \\
p_{10} & (1,\zeta,\zeta^8) \\
p_{11} & (1,\zeta^7,0) \\
p_{12} & (1,\zeta^7,\zeta^2) \\
p_{13} & (1,\zeta^7,\zeta^8)
\end{array}
\qquad\qquad
\beta : \quad
\begin{array}{rl}
\ell_1 & [0,0,1] \\
\ell_2 & [0,1,0] \\
\ell_3 & [0,1,\zeta^5] \\
\ell_4 & [0,1,\zeta^{11}] \\
\ell_5 & [1,0,0] \\
\ell_6 & [1,0,\zeta^4] \\
\ell_7 & [1,0,\zeta^{10}] \\
\ell_8 & [1,\zeta^5,0] \\
\ell_9 & [1,\zeta^5,\zeta^4] \\
\ell_{10} & [1,\zeta^5,\zeta^{10}] \\
\ell_{11} & [1,\zeta^{11},0] \\
\ell_{12} & [1,\zeta^{11},\zeta^4] \\
\ell_{13} & [1,\zeta^{11},\zeta^{10}]
\end{array}
$$

Theorem 3.2 *Let q be a prime power, $q \equiv 5 \pmod{12}$. Then α is the set of points, and β the set of lines, of a subplane of order 3 in the Hughes plane $H(q^2)$. This subplane is invariant under the polarity $(x,y,z) \leftrightarrow [x^q, y^q, z^q]$ of $H(q^2)$.*

Proof: It is known that all elements of $\mathbb{F}$ are squares in $\mathbb{K}$. We use the Lemma 3.1 and the incidence relation described by Rosati [6] to determine whether p_i and ℓ_j are incident for each pair of a point p_i, $1 \le i \le 13$, in α and a line ℓ_j, $1 \le j \le 13$, in β. This gives rise to the following incidence matrix M:

$$
M = \begin{pmatrix}
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0
\end{pmatrix}
$$

An easy computation shows that $MM^T = J_{13} + 3I_{13}$, where J_{13} denotes the 13×13 matrix in which every entry is a "1" and I_{13} the 13×13 identity matrix.

By Rosati [7], the map $(x, y, z) \leftrightarrow [x^q, y^q, z^q]$ is a polarity of $H(q^2)$. One easily checks that this map interchanges α and β. This completes the proof of Theorem 3.2. $\square$

4 Case: $q \equiv 11$ $(mod\ 12)$

Let us now assume that $q \equiv 11$ $(mod\ 12)$, so that both -1 and -3 are nonsquares in $\mathbb{F}$, and in particular 3 is a square in $\mathbb{F}$.

Lemma 4.1 *There exists $c \in \mathbb{F}$ such that $c^2 - c + 1$ is a nonsquare in $\mathbb{F}$.*

Proof: By the Chevalley-Warning Theorem [8, p.5], there exist $a, b, c \in \mathbb{F}$, not all zero, such that $c^2 - bc + b^2 + a^2 = 0$. Clearly $b \neq 0$, so $(c/b)^2 - (c/b) + 1 = -(a/b)^2$, a nonsquare in $\mathbb{F}$. $\square$

Fixing $c \in \mathbb{F}$ as in Lemma 4.1, we readily obtain the following from the Lemma 1.1.

Lemma 4.2 *The elements $\theta, 1 \pm \theta$ and $3 \pm \theta$ are squares in $\mathbb{K}$. The elements $c - 2 \pm c\theta$, $c + 1 \pm (c-1)\theta$ and $2c - 1 \pm \theta$ are nonsquares in $\mathbb{K}$.*

We shall use Lemma 4.2 along with the fact that $c \notin \{0, 1\}$. Now we define α', a set of 13 points, and β', a set of 13 lines, as follows :

	p_1	$(1, \omega, \omega^2)$		ℓ_1	$[1, \omega, \omega^2]$
	p_2	$(1, 0, -\omega)$		ℓ_2	$[0, -\omega, 1]$
	p_3	$(-\omega, 1, 0)$		ℓ_3	$[1, 0, -\omega]$
	p_4	$(0, -\omega, 1)$		ℓ_4	$[-\omega, 1, 0]$
	p_5	$(1/(c-1), \omega, \omega^2)$		ℓ_5	$[\omega^2, c/(1-c), \omega]$
α' :	p_6	$(-c, \omega, \omega^2)$	β' :	ℓ_6	$[\omega^2, c-1, \omega]$
	p_7	$((1-c)/c, \omega, \omega^2)$		ℓ_7	$[\omega^2, -1/c, \omega]$
	p_8	$(\omega^2, (1-c)/c, \omega)$		ℓ_8	$[\omega, \omega^2, c/(1-c)]$
	p_9	$(\omega^2, 1/(c-1), \omega)$		ℓ_9	$[\omega, \omega^2, c-1]$
	p_{10}	$(\omega^2, -c, \omega)$		ℓ_{10}	$[\omega, \omega^2, -1/c]$
	p_{11}	$(\omega, \omega^2, 1/(c-1))$		ℓ_{11}	$[c-1, \omega, \omega^2]$
	p_{12}	$(\omega, \omega^2, -c)$		ℓ_{12}	$[-1/c, \omega, \omega^2]$
	p_{13}	$(\omega, \omega^2, (1-c)/c)$		ℓ_{13}	$[c/(1-c), \omega, \omega^2]$

Theorem 4.3 *Let q be a prime power, $q \equiv 11$ $(mod\ 12)$. Then α' is the set of points, and β' the set of lines, of a subplane of order 3 in the Hughes plane $H(q^2)$.*

Proof: By Lemma 4.1 and 4.2, our computation gives rise to the following incidence matrix M', where $M'M'^T = J_{13} + 3I_{13}$. This proves Theorem 4.3. □

$$M' = \begin{pmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0
\end{pmatrix}$$

5 Further Substructures of Hughes Planes

No subplanes of order 3 have ever been found in Hughes planes of order q^2 for $q \equiv 1$ (mod 6); and computational evidence for small values of q suggests that subplanes of order 3 do not occur in this case. It is also an open problem whether there exists a Hughes plane with a subplane of order 4. However, the following argument, first used in [3], shows that there exist finite partial linear spaces which cannot embed in any Hughes plane.

First, some terminology: Let L be a finite partial linear space (a point-line incidence structure, in which every line has at least two points, and any two distinct points lie on at most one line of L). As before, we denote by $H(q^2)$ a Hughes plane of order q^2. We say that $f : L \to H(q^2)$ is an *embedding* if f injectively maps points of L to points of $H(q^2)$, and f injectively maps lines of L to lines of $H(q^2)$, such that $f(P)$ lies on $f(\ell)$ (in $H(q^2)$) if and only if the point P lies on the line ℓ (in L). (Replacing "if and only if" by "if" in the latter definition, does not change the essential difficulty of the embedding problem, or the validity of Theorem 5.1 below; see [3, Lemma 1].) In this language, our main result (above) is that the projective plane of order 3 embeds in $H(q^2)$ whenever $q \equiv 5$ (mod 6).

Theorem 5.1 *There exists a finite partial linear space which does not embed in any Hughes plane.*

Proof: Let L_0 be a finite partial linear space which does not embed in any Desarguesian plane of odd order. (We may take L_0 to be a projective plane of order 2, or a configuration violating Desargues' Theorem.) Let Γ_0 be the incidence graph of L_0, i.e. the graph whose vertices correspond to points and lines of L_0; and whose edges correspond to incident point-line pairs of L_0. Thus Γ_0 is a bipartite graph with no 4-cycle. By [4, Theorem 6.3] (see also [3, Lemma 2]), there exists a bipartite graph Γ having no 4-cycle, such that for every 2-coloring of the edges of Γ, there exists a subgraph isomorphic to Γ_0, all of whose edges have the same color. We may regard Γ as the point-line incidence graph of a partial linear space L.

Suppose that q is an odd prime power and that $f : L \to H(q^2)$ is an embedding. For each point P_i and line ℓ_j of L, denote $f(P_i) = (x_i, y_i, z_i)$ and $f(\ell_j) = [a_j, b_j, c_j]$. Here we require the nonzero vectors (x_i, y_i, z_i) and $[a_j, b_j, c_j]$ in N^3 to be 'normalized' to have first nonzero coordinate equal to 1, as described in the Introduction. Now write

$$(a_j, b_j, c_j) = (a_{j1} + a_{j2}\theta, b_{j1} + b_{j2}\theta, c_{j1} + c_{j2}\theta),\ (a_{jk}, b_{jk}, c_{jk}) \in \mathbb{F}^3$$

for all j, k, where $\{1, \theta\}$ is a fixed basis for $\mathbb{K}$ over $\mathbb{F}$.

Assuming $P_i \in \ell_j$, we color the incident point-line pair (P_i, ℓ_j) red or blue according as $a_{j2}x_i + b_{j2}y_i + c_{j2}z_i \in \mathbb{K}$ is a square or a nonsquare.

Case 1: Γ has a subgraph isomorphic to Γ_0, all of whose edges are red. In this case the map

$$P_i \mapsto (x_i, y_i, z_i),\ \ \ell_j \mapsto (a_j, b_j, c_j)$$

restricts to an embedding of Γ_0 in a Desarguesian plane of order q^2, since

$$a_jx_i + b_jy_i + c_jz_i = (a_{j1}x_i + b_{j1}y_i + c_{j1}z_i) + (a_{j2}x_i + b_{j2}y_i + c_{j2}z_i) \circ \theta = 0$$

for every red incident point-line pair $P_i \in \ell_j$. (The fact that vertices of Γ_0 are mapped injectively to points and lines of the Desarguesian plane, follows from the fact that the vectors (x_i, y_i, z_i) have first nonzero coordinate 1 and so represent distinct 1-dimensional subspaces of $\mathbb{K}^3$; similarly for the vectors $[a_j, b_j, c_j]$.) This contradicts the choice of Γ_0.

Case 2: Γ has a subgraph isomorphic to Γ_0, all of whose edges are blue. In this case the map

$$P_i \mapsto (x_i, y_i, z_i),\ \ \ell_j \mapsto (a_j^q, b_j^q, c_j^q)$$

restricts to an embedding of Γ_0 in a Desarguesian plane of order q^2, since

$$a_j^qx_i + b_j^qy_i + c_j^qz_i = (a_{j1}x_i + b_{j1}y_i + c_{j1}z_i) + (a_{j2}x_i + b_{j2}y_i + c_{j2}z_i) \circ \theta = 0$$

for every blue incident point-line pair $P_i \in \ell_j$. (As in Case 1, injectivity of the embedding of Γ_0 in the Desarguesian plane of order q^2, follows from the fact that the vectors (x_i, y_i, z_i) and $[a_j, b_j, c_j]$ in N^3 are normalized.) Again, this contradicts the choice of Γ_0. $\square$

The proof of Theorem 5.1 reveals a straightforward strategy for trying to embed a given finite partial linear space L (such as a finite projective plane) in a Hughes plane $H(q^2)$: Choose an appropriate 2-coloring of the incident point-line pairs of L (i.e. the edges of its incidence graph Γ), such that both of the resulting monochromatic subgraphs of Γ correspond to partial linear spaces embeddable in a Desarguesian plane of order q^2. Unfortunately there are exponentially many 2-colorings of the edges of Γ to consider; and even for a projective plane of order 4, with 105 incident point-line pairs, this seems a daunting task. On the other hand, it is easy to 2-color these 105 incident point-line pairs without rendering any monochromatic subplane of order 2; so the argument of Theorem 5.1 seems ineffective in ruling out subplanes of order 4 in Hughes planes.

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