On a conjecture concerning the Petersen graph

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Submitted: Oct 4, 2010; Accepted: Jan 10, 2011; Published: Jan 19, 2011
Mathematics Subject Classifications: 05C38, 05C40, 05C75

Abstract

Robertson has conjectured that the only 3-connected, internally 4-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord is the Petersen graph. We prove this conjecture in the special case where the graphs involved are also cubic. Moreover, this proof does not require the internal-4-connectivity assumption. An example is then presented to show that the assumption of internal 4-connectivity cannot be dropped as an hypothesis in the original conjecture.

We then summarize our results aimed toward the solution of the conjecture in its original form. In particular, let G be any 3-connected internally-4-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. If C is any girth cycle in G then $N(C)\backslash V(C)$ cannot be edgeless, and if $N(C)\backslash V(C)$ contains a path of length at least 2, then the conjecture is true. Consequently, if the conjecture is false and H is a counterexample, then for any girth cycle C in H, $N(C)\backslash V(C)$ induces a nontrivial matching M together with an independent set of vertices. Moreover, M can be partitioned into (at most) two disjoint non-empty sets where we can precisely describe how these sets are attached to cycle C.

* work supported by NSF Grant DMS-0354554
† work supported by NSA Grant H98230-09-01-0041

1. Introduction and Terminology.

This paper is motivated by the following conjecture due to Robertson:

Conjecture 1.1: The only 3-connected, internally 4-connected, girth 5 graph in which every odd cycle of length greater than 5 has a chord is the Petersen graph.

Since its discovery at the end of the nineteenth century, the Petersen graph has been cited as an example, and even more often as a counterexample, in nearly every branch of graph theory. These occurrences could fill a book and in fact have; see [HoSh]. We will not attempt to give a complete list of the appearances of this remarkable graph in print, but let us mention a few of the more recent applications. Henceforth, we shall denote the Petersen graph by P_{10}.

Let us now adopt the following additional notation. If u and v are distinct vertices in P_{10}, the graph formed by removing vertex v will be denoted $P_{10}\backslash v$ and, if u and v are adjacent, the subgraph obtained by removing edge uv will be denoted by $P_{10}\backslash uv$. Other notation and terminology will be introduced as needed.

It is a well-known fact that every Cayley graph is vertex-transitive, but the converse is false, the smallest counterexample being P_{10}. (Cf. [A].) In their studies of vertex-transitive graphs [LS, Sc], the authors present four interesting classes of non-Cayley graphs and digraphs (generalized Petersen, Kneser, metacirculant and quasi-Cayley) and all four classes contain P_{10}.

The Petersen graph has long played an important role in various graph traversability problems. It is known to be the smallest hypohamiltonian graph [GHR]. It is also one of precisely five known connected vertex-transitive graphs which fail to have a Hamilton cycle. It does, however, possess a Hamilton path. Lovász [L1] asked if *every* connected vertex-transitive graph contains a Hamilton path. This question has attracted considerable attention, but remains unsolved to date. (Cf. [KM1, KM2].)

One of the earliest alternative statements of the 4-color conjecture was due to Tait [Ta]: Every cubic planar graph with no cut-edge is 3-edge-colorable. The Petersen graph P_{10} is the smallest nonplanar cubic graph that is not 3-edge colorable. Some eleven years before the 4-color problem was settled [AH1, AH2], Tutte [Tu1, Tu2] formulated the following stronger conjecture about cubic graphs:

Conjecture 1.2: Every cubic cut-edge free graph containing no P_{10}-minor is 3-edge-colorable.

A cubic graph with no cut-edge which is *not* 3-edge-colorable is called a *snark*. Not surprisingly, in view of the preceding conjecture of Tutte, much effort has been devoted to the study of snarks and many snark families have been discovered. (Cf. [Wa, WW, CMRS].) However, to date, all contain a Petersen minor. A proof has been announced by Robertson, Sanders, Seymour and Thomas [Th, TT], but has not yet appeared.

Note that there is a relationship between the question of Lovász and 3-edge-colorings in that for cubic graphs, the existence of a Hamilton cycle guarantees an edge coloring in three colors. Actually, there are only two known examples of connected cubic vertex-transitive graphs which are not 3-edge-colorable of which P_{10} is one and the other is the cubic graph derived from P_{10} by replacing each vertex by a triangle. (Cf. [Po].) (The latter graph is known as the *inflation* or the *truncation* of P_{10}.)

Note also that a 3-edge-coloring of graph G is equivalent to being able to express the all-1's vector of length $|E(G)|$ as the sum of the incidence vectors of three perfect matchings. Seymour [Se1] was able to prove a relaxation of Tutte's conjecture by showing that every cubic bridgeless graph with no P_{10}-minor has the property that the edge-incidence vector of all-1's can be expressed as an *integral* combination of the perfect matchings of G. Lovász [L2] later derived a complete characterization, in which the Petersen graph plays a crucial role, of the lattice of perfect matchings of *any* graph.

In connection with covering the edges of a graph by perfect matchings, we should also mention the important - and unsolved - conjecture of Berge and Fulkerson [F; see also Se1, Zhan].

Conjecture 1.3: Every cubic cut-edge free graph G contains six perfect matchings such that each edge of G is contained in exactly two of the matchings.

The Petersen graph, in fact, has *exactly* six perfect matchings with this property.

Drawing on his studies of face-colorings, Tutte also formulated a related conjecture for general (i.e., not necessarily cubic) graphs in terms of integer flows.

Conjecture 1.4: Every cut-edge free graph containing no subdivision of P_{10} admits a nowhere-zero 4-flow.

This conjecture too has generated much interest. For cubic graphs, Conjecture 1.2 and Conjecture 1.4 are equivalent since in this case a 3-edge-coloring is equivalent to a 4-flow.

The 5-flow analogue for cubic graphs, however, has been proved by Kochol [Ko].

Theorem 1.5: If G is a cubic cut-edge free graph with no Petersen minor, G has a nowhere-zero 5-flow.

Another partial result toward the original conjecture is due to Thomas and Thomson [TT]:

Theorem 1.6: Every cut-edge free graph without a $P_{10} \backslash e$-minor has a nowhere-zero 4-flow.

This result generalizes a previous result of Kilakos and Shepherd [KS] who had derived the same conclusion with the additional hypothesis that the graphs be cubic.

The original (not necessarily cubic) 4-flow conjecture remains unsolved.

Yet another widely studied problem is the cycle double conjecture. A set of cycles in a graph G is a *cycle double cover* if every edge of G appears in exactly two of the cycles in the set. The following was conjectured by Szekeres [Sz] and, independently, by Seymour [Se2]. It remains open.

Conjecture 1.7: Every connected cut-edge free graph contains a cycle double cover.

The following variation involving P_{10} was proved by Alspach, Goddyn and Zhang [AGZ].

Theorem 1.8: Every connected cut-edge free graph with no P_{10}-minor has a cycle double cover.

For much more on the interrelationships of edge-colorings, flows and cycle covers, the interested reader is referred to [Zhan, Ja].

An embedding of a graph G in 3-space is said to be *flat* if every cycle of the graph bounds a disk disjoint from the rest of the graph. Sachs [Sa] conjectured that a graph G has a flat embedding in 3-space if and only if it contains as a minor none of seven specific graphs related to P_{10}. His conjecture was proved by Robertson, Seymour and Thomas [RST3].

Theorem 1.9: A graph G has a flat embedding if and only if it has no minor isomorphic to one of the seven graphs of the 'Petersen family' obtained from P_{10} by Y-Δ and Δ-Y transformations. (the complete graph K_6 is one of these seven graphs.)

A smallest graph with girth g and regular of degree d is called a (d, g)-*cage*. The unique $(3, 5)$-cage is P_{10}. This observation was proved by Tutte [Tu3] under a more stringent definition of "cage".

Any smallest graph which is regular of degree d and has diameter k (if it exists) is called a *Moore graph* of type (d, k). For $k = 2$, Moore graphs exist only for $d = 2, 3, 7$ and possibly 57. The unique Moore graph of type $(3, 2)$ is P_{10}. (Cf. [HoSi].)

A graph G is said to be *distance-transitive* if for every two pairs of vertices $\{v, w\}$ and $\{x, y\}$ such that $d(u, v) = d(x, y)$ (where d denotes distance), there is an automorphism σ of G such that $\sigma(v) = x$ and $\sigma(w) = y$. There are only twelve finite cubic distance-transitive graphs and P_{10} is the only one with diameter 2 and girth 5. (Cf. [BS].)

Distance-transitive graphs form a proper subclass of another important graph class called *distance-regular* graphs. (Cf. [BCN].) These graphs are closely related to the *association schemes* of algebraic combinatorics.

A closed 2-cell surface embedding of a graph G is called *strong* (or *circular*). The following conjecture is folklore which appeared in literature as early as in 1970s (Cf. [H, LR]).

Conjecture 1.10: Every 2-connected graph has a strong embedding in some surface.

(Note that, for cubic graphs, this conjecture is equivalent to the cycle-double-cover conjecture.) (Cf. [Zhan, Corollary 7.1.2].)

Ivanov and Shpectorov [I, IS] have investigated certain so-called Petersen geometries associated with the sporadic simple groups. The smallest of these geometries is associated with P_{10}.

The following conjecture of Dirac was proved by Mader.

Theorem 1.11 [M1]: Every graph G with at least $3|V(G)| - 5$ edges (and at least 3 vertices) contains a subdivision of K_5.

One of the main tools used in proving this is another of Mader's own theorems.

Theorem 1.12 [M2]: If G has girth at least 5, at least 6 vertices and at least $2|V(G)|-5$ edges, then G either contains a subdivision of $K_5 \backslash e$ or $G \cong P_{10}$.

Our plan of attack is to proceed as follows. In Section 2, we present several lemmas of a technical nature. In Section 3, we prove the conjecture for cubic graphs without using the internal-4-connectivity assumption. We then close the section by presenting infinitely many examples of a graphs which are 3-connected of girth 5 and in which every odd cycle of length greater than 5 has a chord, but which are not the Petersen graph. These examples led us to invoke the additional assumption of internal-4-connectivity.

Let H be a subgraph of a graph G. Denote by $N'(H)$ the set of neighbors of vertices in H which are not themselves in H. We also use $N'(H)$ to denote the subgraph induced by $N'(H)$ (this will not cause any confusion in this paper). Let G be a 3-connected internally-4-connected graph G having girth 5 in which every odd cycle of length greater than 5 has a chord. Let C be a 5-cycle in G. We then proceed to focus our attention on the structure of the subgraph induced by $N'(C)$.

In Section 4, we show that $N'(C)$ cannot be edgeless. In Section 5, we show that if $N'(C)$ contains a path of length at least 3, then $G \cong P_{10}$. In Section 6 we undertake the lengthier task of showing that if $N'(C)$ contains a path of length 2, then $G \cong P_{10}$. In summary then, we will reduce the conjecture to the case when $N'(C)$ is the disjoint union of a nonempty matching and a possibly empty edgeless subgraph. Moreover, the matching must be attached to the 5-cycle C only in certain restricted ways. We will summarize these details in Section 7.

2. Some technical lemmas.

Suppose H is a subgraph of a graph G and $x \in V(G)$. Denote $N(x) = \{v \in V(G) : vx \in E(G)\}$, $N(H) = \{v \in V(G) : uv \in E(G)$ for some $u \in V(H)\}$ and $N'(H) = N(H)\backslash V(H)$. Define $N'(H,x) = N(x)\backslash V(H)$. Note that in general $N(x)$ does not contain x and so $N'(x) = N(x)$ when x is not in $V(H)$. If $V(H) = \{x_1, x_2, ..., x_t\}$, we will write N_i' for $N'(x_i)\backslash V(H) = N(H, x_i)$, where $1 \leq i \leq t$, ignoring the dependency on H. Since all graphs G in this paper are assumed to have girth 5, $N(x)$ is an independent set, for all $x \in V(G)$, and hence any $N'(x)$ in this paper will be independent as well.

Let G be a graph and H a proper subgraph of G. If $e = xy$ is an edge of G not belonging to $H \cup V(N'(H))$, but joining two vertices x and y of $N'(H)$, we call e an *edge-bridge* of $H \cup V(N'(H))$. Let D be a component of $G\backslash(V(H) \cup V(N'(H)))$. If there exists a vertex $w \in N'(H)$ which is adjacent to some vertex of D, we will say that w is a *vertex of attachment* for D in $N'(H)$. If D is a component of $G\backslash(V(H) \cup V(N'(H)))$ and B consists of D, together with all of its vertices of attachment in H, we call B a *non-edge-bridge* of H. Furthermore, any vertex of bridge B which is not a vertex of attachment will be called an *interior vertex* of B. Clearly, any path from an interior vertex of B to a vertex in H passes through a vertex of attachment of B.

We now further classify the non-edge-bridges of $H \cup N'(H)$ as follows. If such a non-edge-bridge has all of its vertices of attachment in the same $N'(x)$, we will call it a *monobridge* and if $x = x_i$ we will often denote it by B_i. Now suppose that $N'(x_i) \cap N'(x_j) = \emptyset$, for all $x_i \neq x_j \in V(H)$. Then if $x_i \neq x_j \in V(H)$, a *bibridge* $B_{i,j}$ of $H \cup N'(H)$ is a bridge which is not a monobridge, but has all of it vertices of attachment in the two sets N_i' and N_j'.

Two distinct vertices x and y in a subgraph H will be called a *co-bridge pair* in H if there exists a non-edge bridge B of $H \cup N'(H)$ such that B has an attachment in $N'(x)$ and an attachment in $N'(y)$. If two vertices of H are not a co-bridge pair, they will be called a *non-co-bridge pair* in H.

Two distinct non-adjacent vertices x and y in a subgraph H will be called *well-connected* in H if x and y are non-adjacent and there exist two induced paths in H joining x and y one of which is of odd length at least 3 and the other of even length at least 2.

Lemma 2.1: Let G be a 3-connected graph of girth five in which every odd cycle of length greater than 5 contains a chord. Let H be a subgraph of G and x, y, two vertices of H such that

(1) x and y are well-connected in H,

(2) $N'(x) \cap N'(y) = \emptyset$ and

(3) there exists no edge-bridge having one endvertex in $N'(x)$ and the other in $N'(y)$.

Then x and y are a non-co-bridge pair in H.

Proof: Suppose, to the contrary, that B is a non-edge bridge of $H \cup N'(H)$ with vertex u a vertex of attachment of B in $N'(x)$ and v a vertex of attachment of B in $N'(y)$. Let

P_{uv} be a shortest path in B joining u and v. Since B is a non-edge bridge, P_{uv} contains at least two edges. Let Q_{xy} and Q'_{xy} be induced paths in H joining x and y and having opposite parity. Then let $P = P_{uv} \cup Q_{xy} \cup \{ux, vy\}$ and $P' = P_{uv} \cup Q'_{xy} \cup \{ux, vy\}$. Then both P and P' are chordless and one of them is an odd cycle of length at least 7, a contradiction. ∎

Lemma 2.2: Suppose G is 3-connected and has girth 5. Let C be any cycle in G of length 5. Then the subgraph induced by $N'(C)$ has maximum degree 2.

Proof: This is an easy consequence of the girth 5 assumption. ∎

Lemma 2.3: Suppose G is 3-connected, has girth 5 and all odd cycles of length greater than 5 have a chord. Then G contains no cycle of length 7.

Proof: Suppose C is a 7-cycle in G. Then C must have a chord which then lies in a cycle of length at most 4, a contradiction. ∎

We will also need the next three results on traversability in $P_{10}, P_{10}\backslash v$ and $P_{10}\backslash uv$. At this point we remind the reader that the Petersen graph is both vertex- and edge-transitive. In the proof of the following two lemmas and henceforth we shall make use of these symmetry properties.

Lemma 2.4: Let P_{10} denote the Petersen graph and let x and y be any two non-adjacent vertices in P_{10}. Then there exist

(i) a unique induced path of length 2 joining x and y;

(ii) exactly two internally disjoint induced paths of length 3 joining x and y; and

(iii) exactly two internally disjoint induced paths of length 4 joining x and y.

(iv) Moreover if z is adjacent to both x and y, then these induced paths of length 3 and 4 do not pass through z.

Proof: This is easily checked. ∎

Lemma 2.5: (i) Let $P_{10}\backslash v$ be the Petersen graph with one vertex v removed. Then for every pair of non-adjacent vertices x and y, there exist induced paths of length 3 and 4 joining them.

(ii) Let $P_{10}\backslash uv$ denote the Petersen graph with a single edge uv removed. Then for every pair of non-adjacent vertices x and y, there exists an induced path of length 4 and either an induced path of length 3 or one of length 5.

(iii) Moreover, in both (i) and (ii) if z is a vertex adjacent to both x and y, these paths do not pass through z.

Proof: The existence of induced paths of length 3 and 4 is a direct consequence of Lemma 2.4 since in P_{10} there are two internally disjoint paths of each type.

If z is incident to both x and y, then any induced path joining x and y and passing through z has length exactly 2. Therefore any induced path joining x and y of length 3 or 4 does not pass through z. ∎

Corollary 2.6: In any of the three graphs $P_{10}, P_{10}\backslash v$ and $P_{10}\backslash uv$, if x_i and x_j are any pair of distinct non-adjacent vertices, then they are well-connected. ∎

3. The cubic case.

In this section we prove the conjecture for graphs which are 3-connected and cubic, have girth 5 and in which every odd cycle of length greater than 5 has a chord. Note that we do not assume internal-4-connectivity in this section.

We begin by treating the case in which for some girth cycle C, $N'(C)$ contains a path of length at least 3. Then by eliminating in sequence five cases corresponding to five possible subgraphs, we arrive at our final result. Although the approach in these five cases is much the same, nevertheless each of the final four makes use of its predecessor in the sequence.

Lemma 3.1: Suppose G is a cubic 3-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. Let C be a 5-cycle in G. Then if $N'(C)$ contains a path of length at least 3, $G \cong P_{10}$.

Proof: Let $C = x_1x_2x_3x_4x_5x_1$ be a 5-cycle in G. Then, since G is cubic and has girth 5, $N'(C)$ must contain exactly five vertices.

Suppose first that $N'(C)$ contains a cycle $y_1y_2y_3y_4y_5y_1$. Then without loss of generality, we may suppose that $y_1 \sim x_1, y_2 \sim x_3, y_3 \sim x_5, y_4 \sim x_2$ and $y_5 \sim x_4$. But then $G \cong P_{10}$.

Suppose next that $N'(C)$ contains a path of length 4 which we denote by $y_1y_2y_3y_4y_5$. Again, without loss of generality, we may suppose that $y_i \sim x_i$, for $i = 1, \ldots, 5$. But now if $y_1 \not\sim y_5$, $\{y_1, y_5\}$ is a 2-cut in G, a contradiction. Hence $y_1 \sim y_5$ and again $G \cong P_{10}$.

Finally, suppose $N'(C)$ contains a 3-path which we will denote by $y_1y_2y_3y_4$. As before, we may suppose that $y_1 \sim x_1, y_2 \sim x_3, y_3 \sim x_5$ and $y_4 \sim x_2$. Since G is cubic, there must then exist a fifth vertex $y_5 \in N'(C)$ such that $y_5 \sim x_4$. Now also since G is cubic, there must exist a vertex $z \in V(G)$, $z \neq x_1, \ldots, x_5, y_1, \ldots, y_5$. By 3-connectivity and Menger's theorem, there must be three paths in G joining z to vertices y_1, y_4 and y_5 respectively. In other words, there must exist a bridge (containing vertex z) with vertices of attachment y_1, y_4 and y_5 in $C \cup N'(C)$. Hence, in particular, vertices x_1 and x_4 are a co-bridge pair. But by Lemma 2.1, these two vertices are a non-co-bridge pair and we have a contradiction. ∎

Next suppose $N'(C)$ contains a path of length 2. Elimination of this case will be the culmination of the next two lemmas.

Lemma 3.2: Let G be a cubic 3-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Then if G contains a subgraph isomorphic to graph J_1 shown in Figure 3.1, $G \cong P_{10}$.

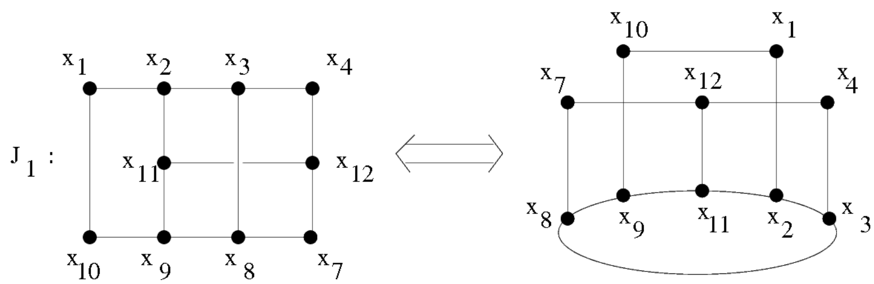

Figure 3.1

Proof: Suppose $G \ncong P_{10}$, but G does contain as a subgraph the graph J_1. We adopt the vertex labeling of Figure 3.1.

Claim 1: The subgraph J_1 must be induced.

It is easy to check that adding any edge different from x_1x_7 and x_4x_{10} results in the formation of a cycle of size less than five, contradicting the girth hypothesis.

So then let us assume x_1 is adjacent to x_7. Then if $C = x_2x_3x_8x_9x_{11}x_2$, $N'(C)$ contains the *induced* path $x_{10}x_1x_7x_{12}x_4$ of length 4, contradicting Lemma 3.1. By symmetry, if we add the edge x_4x_{10}, a similar contradiction is reached. This proves Claim 1.

Claim 2: For $1 \leq i < j \leq 12$, $N'_i \cap N'_j = \emptyset$.

It is routine to check that any possible non-empty intersection of two different N'_is produces either a cycle of length less than 5, thus contradicting the girth hypothesis, or else a 7-cycle, thus contradicting Lemma 2.3. This proves Claim 2.

Claim 3: For $(i, j) \in \{(1, 4), (1, 7), (4, 10), (7, 10)\}$, there is no edge joining N'_i and N'_j.

This is immediate by Lemma 2.3.

For $i = 1, 4, 7, 10$, let y_i denote the (unique) neighbor of x_i which does not lie in J_1.

Then since G is cubic and 3-connected, there must be a bridge B in $G - V(J_1)$ with at least three vertices of attachment from the set $\{y_1, y_4, y_7, y_{10}\}$. It then follows that either $\{x_1, x_7\}$ or $\{x_4, x_{10}\}$ is a co-bridge pair. But these pairs are both well-connected and hence by Lemma 2.1, neither is a co-bridge pair, a contradiction. ∎

Lemma 3.3: Let G be a cubic 3-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Suppose C is a girth cycle in G such that $N'(C)$ contains a path of length 2. (That is, G contains a subgraph isomorphic to graph J_2 shown in Figure 3.2.) Then $G \cong P_{10}$.

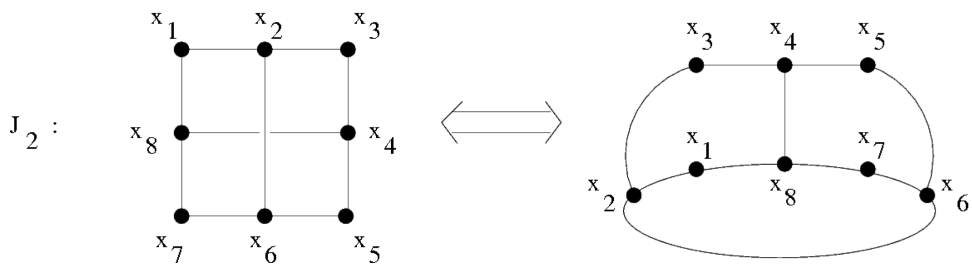

Figure 3.2

Proof: Suppose $G \not\cong P_{10}$, but G does contain a subgraph isomorphic to J_2. We adopt the vertex labeling shown in Figure 3.2.

Claim 1: J_2 is an induced subgraph.

This is immediate via the girth 5 hypothesis.

Claim 2: For $1 \leq i < j \leq 8$, $N'_i \cap N'_j = \emptyset$. For all pairs $\{i,j\} \neq \{1,5\}$ and $\{3,7\}$, this follows from the girth 5 hypothesis and observing that $N'_2 = N'_4 = N'_6 = N'_8 = \emptyset$. Suppose, then, that there exists a vertex $y \in N'_1 \cap N'_5$. Then if we let $C = x_1x_2x_6x_7x_8x_1$ we find that $N'(C)$ contains a path $yx_5x_4x_3$ of length 3, contradicting Lemma 3.1.

So $N'_1 \cap N'_5 = \emptyset$ and by symmetry, $N'_3 \cap N'_7 = \emptyset$ as well. This proves Claim 2.

For $i = 1, 3, 5, 7$, let y_i be the neighbor of x_i not in J_2.

Claim 3: For $\{i,j\} \in \{\{1,3\},\{1,5\},\{1,7\},\{3,5\},\{3,7\},\{5,7\}\}$, there is no edge joining y_i and y_j.

By symmetry, we need only check the pairs $\{1,3\}$ and $\{1,5\}$. If there is an edge joining y_1 and y_3, there is then a subgraph isomorphic to J_1 and we are done by Lemma 3.2. If, on the other hand, $y_1 \sim y_5$, we have a 7-cycle in G, contradicting Lemma 2.3. This proves Claim 3.

It is easily checked that $\{x_1, x_5\}$ and $\{x_3, x_7\}$ are each well-connected and hence by Claim 3 and Lemma 2.1 each is a non-co-bridge pair. On the other hand, since G is cubic and 3-connected, there is a bridge B of the subgraph spanned by $V(J_2) \cup \{y_1, y_3, y_5, y_7\}$ which must have attachments at at least three of the vertices $\{y_1, y_3, y_5, y_7\}$. But it then follows that either $\{x_1, x_5\}$ or $\{x_3, x_7\}$ is a co-bridge pair, a contradiction. ∎

The next two results culminate in the elimination of the case in which there is a matching of size 2 in $N'(C)$.

Lemma 3.4: Suppose G is a cubic 3-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. Suppose G contains the graph L_1 shown in Figure 3.3 as a subgraph. Then $G \cong P_{10}$.

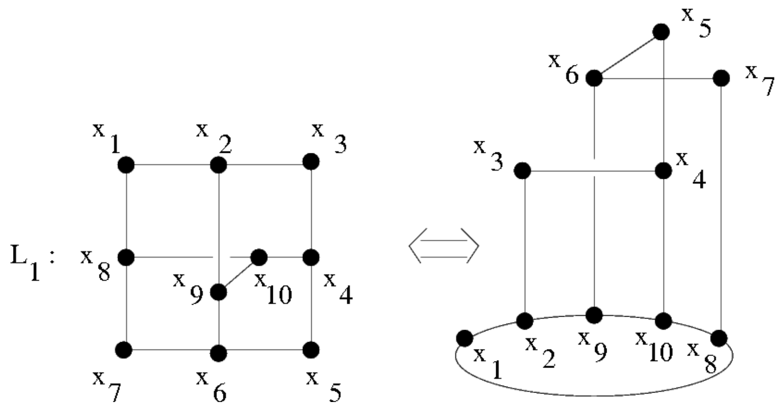

Figure 3.3

Proof: Suppose $G \not\cong P_{10}$, but suppose G does contain the graph L_1 as a subgraph. We assume the vertices of this subgraph L_1 are labeled as in Figure 3.3.

Claim 1: L_1 is an induced subgraph.

By the girth hypothesis, if vertices x_i and x_j are joined by a path of length at most 3, then they are not adjacent. Therefore, by symmetry we need check only the pairs $\{x_1, x_5\}$ and $\{x_3, x_7\}$. However, if $x_1 \sim x_5$, $L_1 \cup x_1x_5 \cong P_{10} \backslash e$. But this graph contains a girth cycle C such that $N'(C)$ contains a path of length 3 and so by Lemma 3.1, $G \cong P_{10}$, a contradiction. So $x_1 \not\sim x_5$. By symmetry, $x_3 \not\sim x_7$ as well and Claim 1 is proved.

Since G is cubic, $N'_2 = N'_4 = N'_6 = N'_8 = N'_9 = N'_{10} = \emptyset$ and each of N'_1, N'_3, N'_5 and N'_7 consists of a single vertex. Let $N'_i = \{y_i\}$ for $i = 1, 3, 5, 7$ and let $L'_1 = \{y_1, y_3, y_5, y_7\}$.

Claim 2: y_1, y_3, y_5 and y_7 are all distinct.

By symmetry, we need only check that $y_1 \neq y_3$ and $y_1 \neq y_5$. The first of these assertions follows immediately via the girth hypothesis. If $y_1 = y_5$, on the other hand, it follows that $y_1x_1x_2x_9x_{10}x_4x_5y_5(= y_1)$ is a chordless 7-cycle, contrary to hypothesis. Thus Claim 2 is true.

Claim 3: L'_1 is independent.

Indeed, if there were an edge joining any two vertices of L'_1, one can find a chordless 7-cycle containing it, which is a contradiction.

Since G is cubic, $G - (L_1 \cup L'_1) \neq \emptyset$. Therefore, there must exist a non-edge bridge B with attachments on at least three of y_1, y_3, y_5 and y_7. But then B must have either both y_1 and y_5 as vertices of attachment or both y_3 and y_7 as vertices of attachment. By symmetry, without loss of generality, let us assume that y_1 and y_5 are vertices of attachment for B. Hence x_1 and x_5 are a co-bridge pair.

On the other hand, the induced paths $x_1x_2x_3x_4x_5$ and $x_1x_2x_9x_{10}x_4x_5$ serve to show that x_1 and x_5 are well-connected, and since there does not exist an edge joining y_1 and y_5, it follows from Lemma 2.1 that $\{x_1, x_5\}$ is a non-co-bridge pair. Hence we have a contradiction and Lemma 3.4 is proved. ∎

Lemma 3.5: Suppose G is a cubic 3-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. Suppose G contains the graph L_2 shown in Figure 3.4 as a subgraph. Then $G \cong P_{10}$.

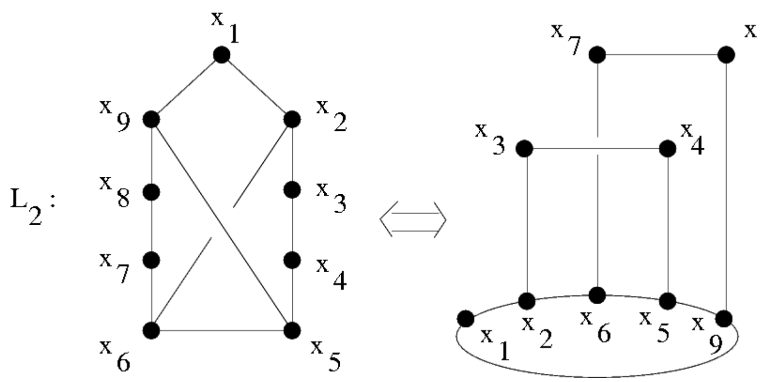

Figure 3.4

Proof: Suppose $G \not\cong P_{10}$, but suppose G does contain the graph L_2 as a subgraph. We assume the vertices of this subgraph L_2 are as labeled in Figure 3.4.

Claim 1: L_2 is an induced subgraph of G.

As before, we need only check pairs of vertices $\{x_i, x_j\}$ which lie at distance at least 4. Hence we need only check the pair $\{x_3, x_8\}$. If x_3 and x_8 are joined by an edge, then the resulting graph is isomorphic to $P_{10}\backslash v$. But this graph contains a girth cycle C such that $N'(C)$ contains a path of length 3 and so by Lemma 3.1, $G \cong P_{10}$, a contradiction. Hence $x_3 \not\sim x_8$ and Claim 1 is true.

Since G is cubic, $N'_2 = N'_5 = N'_6 = N'_9 = \emptyset$ and each of N'_1, N'_3, N'_4, N'_7 and N'_8 consists of a single vertex. Let $N'_i = \{y_i\}$ for $i = 1, 3, 4, 7, 8$ and let $L'_2 = \{y_1, y_3, y_4, y_7, y_8\}$.

Claim 2: y_1, y_3, y_4, y_7 and y_8 are all distinct.

By the girth hypothesis, we need only check that $y_i \neq y_j$ when x_i and x_j are at distance at least 3. By symmetry, then, we need only check the five pairs $\{y_1, y_4\}, \{y_3, y_7\}$, $\{y_3, y_8\}, \{y_4, y_7\}$ and $\{y_4, y_8\}$.

Suppose $y_1 = y_4$. Then consider the 5-cycle $C = x_1x_2x_6x_5x_9x_1$ and note that $N'(C)$ contains the path $y_1(= y_4)x_4x_3$ and by the girth hypothesis, this is an induced path of length 2. But then by Lemma 3.3, $G \cong P_{10}$, a contradiction. Hence $y_1 \neq y_4$.

If $y_3 = y_7$, $y_3x_3x_4x_5x_9x_8x_7y_7(= y_3)$ is a chordless 7-cycle, a contradiction. If $y_3 = y_8$, $y_3x_3x_4x_5x_6x_7x_8y_8(= y_3)$ is a chordless 7-cycle, a contradiction. Suppose next that $y_4 = y_8$. Then the 10-vertex subgraph $L_2 \cup \{x_4y_4, y_4x_7\}$ is isomorphic to L_1 and hence by Lemma 3.4, $G \cong P_{10}$, a contradiction.

Finally, if $y_4 = y_8$, $y_4x_4x_3x_2x_1x_9x_8y_8(= y_4)$ is a chordless 7-cycle, a contradiction. Thus Claim 2 is true.

Claim 3: For $i, j \in \{1, 3, 4, 7, 8\}$, there is no edge joining y_i and y_j, that is, L'_2 is independent.

By the girth 5 hypothesis and symmetry, we need only check the pairs $\{i, j\} = \{1, 3\}, \{1, 4\}, \{3, 7\}, \{3, 8\}, \{4, 7\}$. But if there is an edge joining y_1 and y_3, using induced path $x_1x_9x_5x_4x_3$ we obtain a chordless 7-cycle, a contradiction. Similarly, for the pair $\{1, 4\}$ using induced path $x_1x_2x_6x_5x_4$, for $\{3, 7\}$, using induced path $x_3x_4x_5x_6x_7$, for $\{3, 8\}$, using induced path $x_3x_4x_5x_9x_8$, and for $\{4, 7\}$, using induced path $x_4x_3x_2x_6x_7$, we obtain a chordless 7-cycle, a contradiction in each case. This proves Claim 3.

Since G is cubic, $G - (L_2 \cup L'_2) \neq \emptyset$. Therefore, there must exist a non-edge bridge B with attachments on at least three of y_1, y_3, y_4, y_7 and y_8.

First assume that there is attachment at vertex y_1. Vertices x_1 and x_3 are well-connected using induced paths $x_1x_2x_3$ and $x_1x_9x_5x_6x_2x_3$ and since $x_1 \not\sim x_3$, $\{x_1, x_3\}$ is a non-co-bridge pair by Lemma 2.1. Hence B has no attachment at vertex y_3.

Similarly, paths $x_1x_2x_3x_4$ and $x_1x_2x_6x_5x_4$ serve to show that x_1 and x_4 are well-connected and hence $\{x_1, x_4\}$ is a non-co-bridge pair as well. Hence there is no attachment for B at y_4. By symmetry, there is no attachment for B at y_7 or y_8 either. Thus there is no attachment at y_1.

So B must have attachments at at least three of the four vertices y_3, y_4, y_7 and y_8. By symmetry, it is enough to consider the possibilities of attachments at y_3, y_4 and y_7 or at y_3, y_4 and y_8. But in the former case, induced paths $x_3x_2x_6x_7$ and $x_3x_4x_5x_6x_7$ serve to show that x_3 and x_7 are well-connected and since they are not adjacent, by Lemma 2.1 $\{x_3, x_7\}$ is a non-co-bridge pair. Similarly, in the latter case, induced paths $x_3x_4x_5x_6x_7x_8$ and $x_3x_2x_1x_9x_8$ suffice to show that x_3 and x_8 are also well-connected and hence $\{x_3, x_8\}$ is a non-co-bridge pair as well. Thus we have a contradiction and the proof of Lemma 3.5 is complete. ∎

Finally, we treat the case when there is a single edge in $N'(C)$.

Lemma 3.6: Suppose G is a cubic 3-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. Suppose G contains the graph L_3 shown in Figure 3.5 as a subgraph. Then $G \cong P_{10}$.

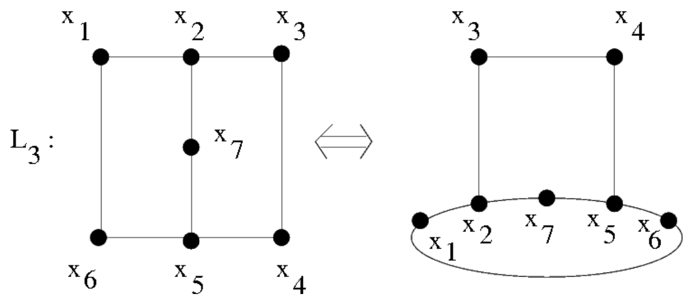

Figure 3.5

Proof: Suppose $G \ncong P_{10}$, but suppose G does contain the graph L_3 as a subgraph.

Claim 1: L_3 is induced.

This follows immediately from the girth hypothesis, since the diameter of L_3 is only 3.

Since G is cubic, $N_2' = N_5' = \emptyset$ and each of N_1', N_3', N_4', N_6' and N_7' consists of a single vertex. Let $N_i' = \{y_i\}$, for $i = 1, 3, 4, 6, 7$ and let $L_3' = \{y_1, y_3, y_4, y_6, y_7\}$.

Claim 2: y_1, y_3, y_4, y_6 and y_7 are all distinct.

By the girth hypothesis, we need only check pairs at distance at least 3 and hence we need only check $\{x_1, x_4\}$.

Suppose $y_1 = y_4$. Then consider the 5-cycle $C = x_1x_8x_4x_5x_6x_1$ and note that $N'(C)$ contains the 2-path $x_3x_2x_7$ and moreover, this 2-path is induced by the girth hypothesis. Hence by Lemma 3.3, $G \cong P_{10}$, a contradiction. Hence $y_1 \neq y_4$ and Claim 2 is proved.

Claim 3: For $i, j \in \{1, 3, 4, 6, 7\}$, there is no edge joining y_i and y_j, that is, L_3' is independent.

By symmetry, we need only check the pairs $\{i, j\} = \{1, 3\}, \{1, 4\}$ and $\{1, 7\}$. But if there is an edge joining y_1 and y_3, using induced path $x_1x_6x_5x_4x_3$ we obtain a chordless 7-cycle, a contradiction. Similarly, for the pair $\{1, 4\}$, using induced path $x_1x_2x_7x_5x_4$ we obtain a chordless 7-cycle as well.

Suppose, then, that $y_1 \sim y_7$ and consider $L_3 \cup \{x_1y_1, y_1y_7, y_7x_7\}$. The 5-cycle $C = x_1x_2x_7x_5x_6x_1$ has the property that $N'(C)$ contains the independent edges y_1y_7 and x_3x_4. But then by Lemma 3.5, $G \cong P_{10}$, a contradiction, and Claim 3 is proved.

Claim 4: The pairs $\{x_i, x_j\}$, for $\{i, j\} = \{1, 4\}, \{1, 7\}, \{3, 6\}, \{3, 7\}, \{4, 7\}$ and $\{6, 7\}$ are well-connected.

By symmetry, it suffices to treat only the two pairs $\{1, 4\}$ and $\{1, 7\}$. But induced paths $x_1x_2x_3x_4$ and $x_1x_2x_7x_5x_4$ show that $\{x_1, x_4\}$ is well-connected, while for $\{x_1, x_7\}$, paths $x_1x_2x_7$ and $x_1x_6x_5x_7$ guarantee well-connectedness.

Claim 5: Each of $\{x_1, x_4\}, \{x_1, x_7\}, \{x_3, x_6\}, \{x_3 x_7\}, \{x_4, x_7\}$ and $\{x_6, x_7\}$ is a non-co-bridge pair.

These are all pairs of non-adjacent vertices and hence by Lemma 2.1, the Claim follows.

Since G is cubic, there must be a bridge B in $V(G) - (L_3 \cup L'_3)$. Since G is 3-connected, bridge B must then have three vertices of attachment among the set $\{y_1, y_3, y_4, y_6, y_7\}$. Vertex y_7 cannot be one of the three, since $\{x_i, x_7\}$ is a non-co-bridge pair, for $i = 1, 3, 4$ and 6, by Claim 5. It then follows that either $\{y_1, y_4\}$ or $\{y_3, y_6\}$ are attachment sets for B. But these pairs are non-co-bridge pairs by Claim 5 and we have a contradiction. The Lemma follows. ∎

We are now prepared for our main result for cubic graphs.

Theorem 3.7: Suppose G is a cubic 3-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. Then $G \cong P_{10}$.

Proof: Let C be a 5-cycle in G. By Lemmas 3.1, 3.3, 3.5 and 3.6, we may assume that $N'(C)$ is independent. So once again, since G is cubic, there must be a bridge in $G - (V(C) \cup N'(C))$ with vertices of attachments in at least three of $N'_1, \ldots, N'_5$. Hence there must be two *non-adjacent* x_i's which are a co-bridge pair. But this contradicts Lemma 2.1 and the Theorem is proved. ∎

The original conjecture of Robertson did not include the assumption that the graphs are internally-4-connected. However, without this assumption, the conclusion does not follow as is shown by the following counterexample.

Let G_1 be the bipartite graph on twenty-six vertices and G_2, the graph on twelve vertices shown in Figure 3.6.

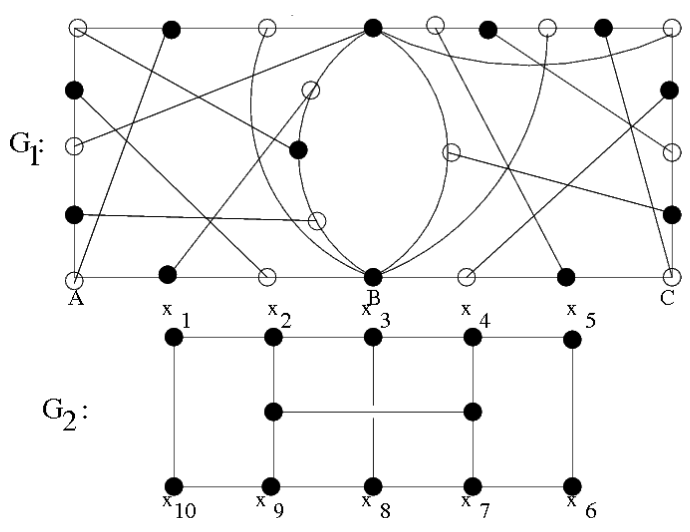

Figure 3.6

Join a copy of G_1 to one central copy of G_2 by joining A to x_1, B to x_2 and C to x_3 via a matching, a second copy of G_1 to G_2 by joining A to x_3, B to x_4 and C to x_5 via a matching, a third copy of G_1 to G_2 by joining A to x_6, B to x_7 and C to x_8 via a matching, and a fourth copy of G_1 to G_2 by joining A to x_8, B to x_9 and C to x_{10} via a matching. The resulting graph on 116 vertices is 3-connected, has girth 5 and every odd cycle of length greater than 5 has a chord. Clearly, it is not internally 4-connected.

Note that we may obtain infinitely many more counterexamples by attaching additional copies of the graph G_1 to each other along their common path $A \cdots B \cdots C$ shown in Figure 3.6.

4. $N'(C)$ is not independent.

Beginning in this section we turn our attention to the original conjecture in which we drop the assumption that G is cubic, but add the assumption that G is internally-4-connected. In these next three sections, we will follow, as far as we can, the general approach of Section 3 in that we will begin with a 5-cycle C and analyze the structure of the subgraph induced by $N'(C)$. In doing so, we will see that a number of claims follow just as they did in the cubic case. But not all.

Lemma 4.1: Let G be a 3-connected internally 4-connected graph of girth 5 in which every odd cycle of length greater than 5 has a chord. Then if C is a 5-cycle in G, $N'(C)$ is not an independent set.

Proof. By way of contradiction, let us suppose that $C = x_1x_2x_3x_4x_5x_1$ is a 5-cycle in G such that $N'(C)$ is independent. Note that by the girth 5 hypothesis, $N'_i \cap N'_j = \emptyset$, for $1 \leq i < j \leq 5$. Note also that since $N'(C)$ is independent, there exist no edge-bridges of $C \cup N'(C)$.

For any vertex pair $\{x_i, x_{i+2}\}$ in $V(C)$, (where i is read modulo 5), consider the set $V(C) - \{x_i, x_{i+2}\}$ which separates the vertices x_i and x_{i+2} on C. Since x_i and x_{i+2} are well-connected in C, by Lemma 2.1 they form a non-co-bridge pair. Similarly, any non-adjacent pair of vertices x_i and x_j on C are a non-co-bridge pair. In fact, then, all non-edge bridges are either monobridges or bibridges with one attachment in some N'_i and another in N'_{i+1}, for some $i, (\bmod 5)$.

Since G is internally 4-connected, there must be a third x_i-x_{i+2} (induced) path $P_{i,i+2}$ containing none of the vertices $V(C) - \{x_i, x_{i+2}\}$. Such a path must visit N'_i and N'_{i+2}. More particularly, this path can be assumed to visit, in turn, a sequence of sets of the form $\{x_i\}, N'_i, B_{i,i+1}, N'_{i+1}, B_{i+1,i+2}, N'_{i+2}, \{x_{i+2}\}$, or else, going around cycle C in the opposite direction, a sequence of sets of the form $\{x_i\}, B_{i,i+4}, N'_{i+4}, B_{i+4,i+3}, N'_{i+3}, B_{i+3,i+2}, N'_{i+2}, \{x_{i+2}\}$. In the first instance above, the path is called a *short overpath* and in the second, a *long overpath.* Note that both long and short overpaths may use monobridges.

(In Figure 4.1, $P_{1,3}$ denotes a short $\{x_1, x_3\}$ overpath and in Figure 4.2, $Q_{3,1}$ is a long $\{x_3, x_1\}$ overpath.)

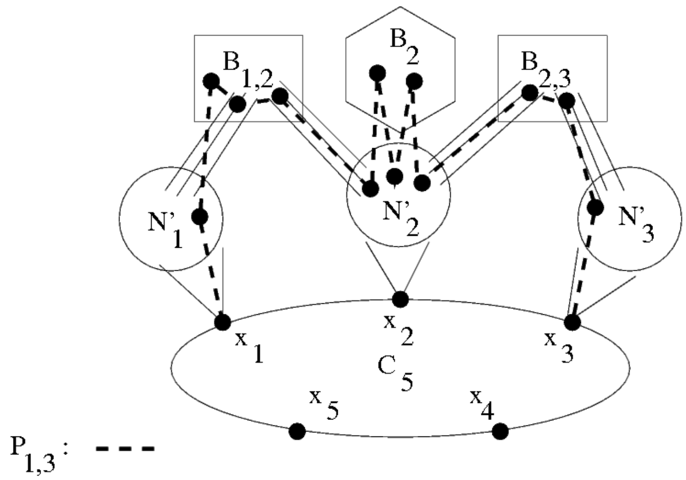

Figure 4.1. A short overpath.

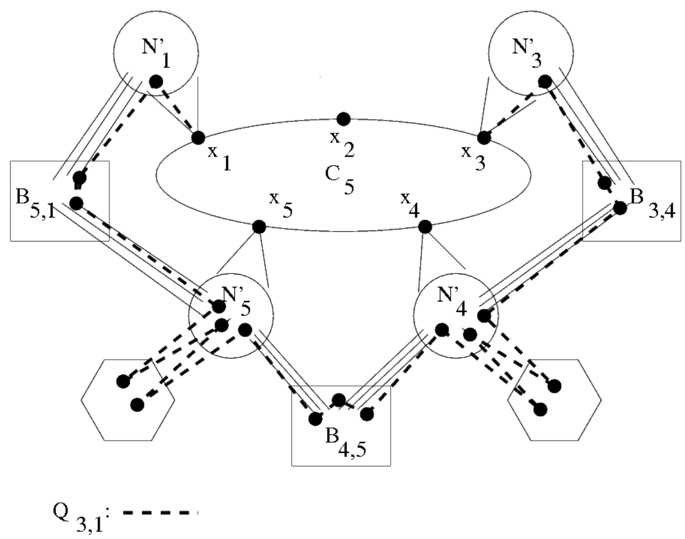

Figure 4.2. A long overpath.

Note that, if there is a short overpath of even length, or a long overpath of odd length, it can be taken together with a subpath of C of suitable parity to form a chordless odd cycle of length greater than 5, contrary to hypothesis. So we have the next observation.

(1) Every short overpath is of odd length and every long overpath is of even length.

We may also assume the following.

(2) Any pair of type $\{x_i, x_{i+2}\}(\bmod 5)$ cannot be joined by *both* a short and a long overpath.

To see this, suppose, to the contrary, that some pair is joined by both a short overpath P and a long overpath Q. Then $P \cup Q$ would contain an odd cycle of length greater than 5 (in fact, at least 9), containing no chords, contradicting an hypothesis of this lemma.

Henceforth, therefore, we shall say the pair $\{x_i, x_{i+2}\}$ is *short* (respectively, *long*) if it is joined by a short (respectively, long) overpath.

We also claim the following is true.

(3) There cannot exist simultaneously a short $\{x_i, x_{i+2}\}$ overpath and a short $\{x_{i+2}, x_{i+4}\}$ overpath.

To see this, suppose that both short overpaths exist. Since by (1), both are of odd length, together with the single edge $x_{i+4}x_i$, their union contains a chordless odd cycle of length greater than 5, again a contradiction.

The next is an obvious observation.

(4) Any long $\{x_i, x_{i+3}\}$ overpath gives rise to both a short $\{x_i, x_{i+2}\}$ overpath and a short $\{x_{i+1}, x_{i+3}\}$ overpath via a suitable selection of an edge joining x_{i+1} and N'_{i+1} and an edge joining x_{i+2} and N'_{i+2}.

(5) For some choice of i, there must exist a short $\{x_i, x_{i+2}\}$ overpath.

For suppose, to the contrary, that, for all $i = 1, \ldots, 5$, no short $\{x_i, x_{i+2}\}$ overpath exists. Fix i. Then by internal-4-connectivity, a long $\{x_{i+2}, x_i\}$ overpath P must exist. But then we are done by (4).

Without loss of generality, then, let us suppose that $\{x_1, x_3\}$ is short. Then by (3) $\{x_3, x_5\}$ cannot be short and hence must be long. Again by (3) and symmetry, $\{x_4, x_1\}$ cannot be short, and hence must be long as well. Thus by (4), $\{x_2, x_5\}$ must be short. Hence by (3), $\{x_2, x_4\}$ is long. By (4), then, $\{x_1, x_4\}$ is short and we have a contradiction. This proves the lemma. ∎

5. Forbidden induced paths of length at least 3.

Again let us suppose that G is a graph which is 3-connected, has girth 5 and that G has the further property that every odd cycle of length greater than 5 has a chord. This entire section is devoted to showing that if C is a 5-cycle in G and $N'(C)$ contains an induced path of length at least 3, then $G \cong P_{10}$. Note that we do not use the internal-4-connected assumption in this section. Hence Lemma 3.1 for cubic graphs is a corollary of Lemma 5.4. However, we have chosen to include the direct proof of Lemma 3.1 given in Section 3 to give the reader more appreciation as to how much the cubic assumption simplifies matters.

Lemma 5.1: Suppose G is 3-connected, has girth 5 and all odd cycles of length greater than 5 have a chord. Let $C_1 = x_1x_2x_3x_4x_5x_1$ be a cycle of length 5 in G and let $C_2 = y_1y_2\cdots y_5\cdots$ be any cycle in $N'(C_1)$. Then $|C_2| = 5$, $G[C_1 \cup C_2] \cong P_{10}$ and $N'(C_1)\backslash V(C_2)$ is independent.

Proof: Let $C_1 = x_1x_2x_3x_4x_5x_1$ and let $C_2 = y_1y_2\cdots y_ky_1$. Suppose $k > 5$. Without loss of generality, assume that $x_1 \sim y_1$. Then by symmetry, we may assume that $y_2 \sim x_3$. But then it follows that $y_3 \sim x_5, y_4 \sim x_2$ and $y_5 \sim x_4$. If y_6 is adjacent to any of $x_2, \ldots, x_5$, the girth 5 hypothesis is contradicted. But then $y_6 \sim x_1$. So $x_1y_1y_2x_3x_4y_5y_6x_1$ is a 7-cycle in G, a contradiction of Lemma 2.3. Therefore C_2 is a 5-cycle. Thus $G[C_1 \cup C_2] \cong P_{10}$ as claimed.

Now by way of contradiction, let us suppose that $N'(C_1)\backslash V(C_2)$ contains an edge y_6y_7. By the symmetry of the Petersen graph, we may then suppose, without loss of generality, that $y_6x_5 \in E(G)$. Then since the girth of G is 5, $y_7 \not\sim x_5$, $y_7 \not\sim x_4$ and $y_7 \not\sim x_1$. If $y_7 \sim x_2$, then $x_2y_4y_5x_4x_5y_6y_7x_2$ is a 7-cycle, contradicting Lemma 2.3. So $y_7 \not\sim x_2$. Similarly, if $y_7 \sim x_3$, then we get a 7-cycle $y_7y_6x_5y_3y_4x_2x_3y_7$, a contradiction, so $y_7 \not\sim x_3$. Thus y_7 is adjacent to no vertex of cycle C_1, a contradiction. ∎

Lemma 5.2: Let G be a 3-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Let C_1 be a 5-cycle in G. If $N'(C_1)$ contains a cycle C_2, then $G \cong P_{10}$.

Proof: Denote $G[C_1 \cup C_2]$ by H. By Lemma 5.1, C_2 is a 5-cycle and H is isomorphic to P_{10}. So let us adopt the notation that $C_1 = x_1x_2x_3x_4x_5x_1$, $C_2 = x_6x_8x_{10}x_7x_9x_6$ and each vertex $x_i \in V(C_1)$ is adjacent to the vertex $x_{i+5} \in V(C_2)$ where the subscripts are read modulo 5.

Suppose $G \neq H$.

Claim 1: For $1 \leq i < j \leq 10$, $N'_i \cap N'_j = \emptyset$.

This follows by Lemma 2.4(i) and the girth hypothesis.

Claim 2: Let B be a bridge of $H \cup N'(H)$. Then B cannot be an edge-bridge.

This follows by Lemma 2.4(iii) and Lemma 2.3.

Claim 3: For any non-edge-bridge B, either B is a monobridge B_i or there exist $i, j \in [1, 10]$ such that $B \cap N'(H) \subseteq N'_i \cup N'_j$ where the corresponding x_i and x_j are two *adjacent* vertices in H. (That is to say, $B = B_{i,j}$ is a bibridge where x_i and x_j are adjacent.)

To prove Claim 3, first note that by Lemma 2.4(ii) and (iii) as well as Claim 2, every pair of non-adjacent vertices in H are well-connected and hence form a non-co-bridge pair. Any three vertices of P_{10} must be such that at most two pair of them are adjacent. Therefore, no non-edge-bridge can have attachments in more than two N'_i's, and if it has attachments in two N'_i's, say in N'_i and in N'_j, then x_i and x_j are adjacent. This proves Claim 3.

Since G is 3-connected, not all bridges of H are monobridges. In fact, if a monobridge is joined only to N'_i, then there must be a bibridge $B_{i,j}$ for some $j \neq i$, since x_i is not a cutvertex in G. So let $B_{i,j}$ be a bibridge of H. Thus by Claim 3, $x_i \sim x_j$.

Since $B_{i,j}$ is not an edge-bridge, there must exist an interior vertex v of $B_{i,j}$. By Menger's theorem, there exist three internally disjoint paths from v to three distinct vertices of H. Since $B_{i,j}$ is a bibridge, all attachments of $B_{i,j}$ in $H \cup N'(H)$ are contained in $N'_i \cup N'_j$. Therefore, at least one of these three paths has to pass through a bridge different from $B_{i,j}$ which is either a $B_{i,k}$ bibridge or a $B_{j,k}$ bibridge, for some $k \neq i, j$. Without loss of generality, suppose one of the three paths passes through a $B_{j,k}$ bibridge. Hence, there must exist a path P in $G \backslash V(H)$ joining v to some vertex w_k in $N'_k, k \neq i, j$. Let w_i be a neighbor of v in N'_i. $P + vw_i$ is then a path joining w_i to w_k in $G \backslash V(H)$. Now choose any *shortest* path $P_{i,k}$ joining a vertex w_i of N'_i to a vertex $w_k \in N'_k$. By Claim 3, x_i is adjacent to x_j and x_j is adjacent to x_k. Therefore, by the girth hypothesis, x_i is not adjacent to x_k. (Note that $P_{i,k}$ may pass through monobridges of type B_j. See Figure 5.1.)

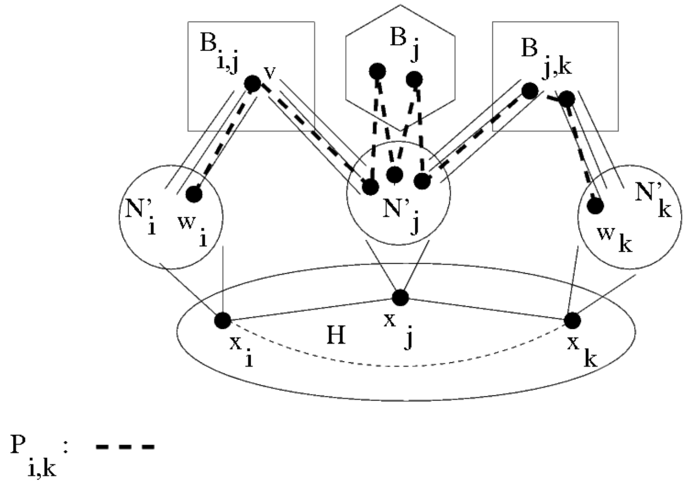

Figure 5.1

So by Lemma 2.5(i) and (iii), with $H \backslash x_j = P_{10} \backslash x_j$, there must exist an induced path P_1 of length 3 and an induced path P_2 of length 4 joining x_i and x_k in H, each avoiding vertex x_j. Let $Q_3 = P_{i,k} \cup w_i x_i \cup w_k x_k \cup P_1$ and $Q_4 = P_{i,k} \cup w_i x_i \cup w_k x_k \cup P_2$. Then Q_3 and Q_4 are chordless cycles since vertices in $G \backslash (H \cup N'(H))$ are not adjacent to vertices in H, $N'(H)$ is independent, $N'_a \cap N'_b = \emptyset$, for $1 \leq a < b \leq 10$, and $P_{i,k}$ is chordless. But one of Q_3 and Q_4 is odd with length at least 9, a contradiction.

Therefore no such vertex v exists; i.e., $B_{i,j} = \emptyset$. Hence, since G is 3-connected, no N'_i's exist either and it follows that $G = H \cong P_{10}$. ∎

Lemma 5.3: Let G be a 3-connected graph of girth five such that all odd cycles of length greater than 5 have a chord. Let C_1 be a 5-cycle in G. Then if $N'(C_1)$ contains an induced path P of length at least 4, $G \cong P_{10}$.

Proof: Let $C = x_1x_2x_3x_4x_5x_1$. By Lemma 5.2, we may assume $N'(C)$ contains no cycle. By hypothesis, on the other hand, $N'(C)$ contains an induced path P of length at least 4. Let $H = G[C \cup P]$. Then it is easy to see that H is isomorphic to $P_{10} \backslash uv$ for some pair of adjacent vertices u and v.

Without loss of generality, we may assume that H is labeled as in Figure 5.2.

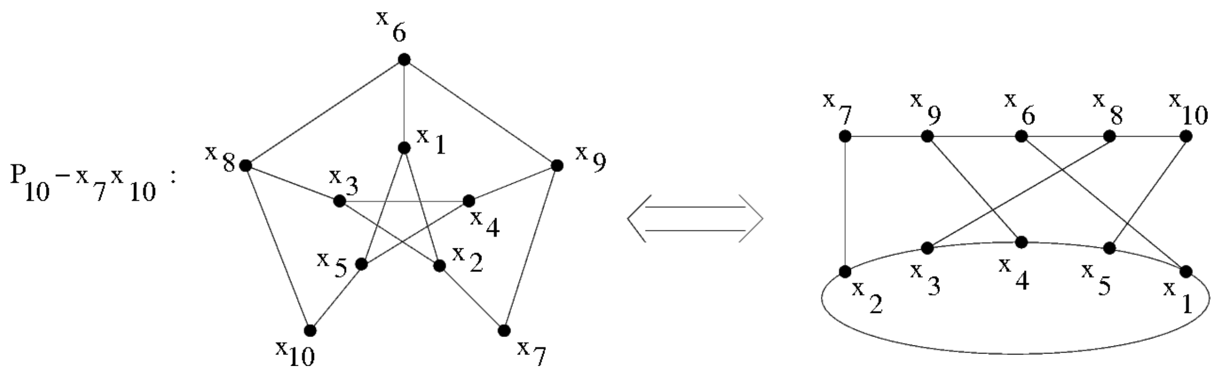

Figure 5.2

Claim 1: $N'_i \cap N'_j = \emptyset$, for $1 \leq i < j \leq 10$.

If the corresponding vertices x_i and x_j are at distance at most two in H, then $N'_i \cap N'_j = \emptyset$ by the girth hypothesis. There are five pairs of vertices at distance at least 3, namely $\{x_7, x_{10}\}, \{x_2, x_{10}\}, \{x_9, x_{10}\}, \{x_5, x_7\}$ and $\{x_7, x_8\}$. (See Figure 5.2.) Due to symmetry, we need only show that $N'_2 \cap N'_{10}, N'_7 \cap N'_{10}$ and $N'_9 \cap N'_{10}$ are empty.

First, let us assume that $N'_2 \cap N'_{10} \neq \emptyset$. Choose $w \in N'_2 \cap N'_{10}$. Then w is adjacent to both x_2 and x_{10}. But then $wx_2x_7x_9x_6x_8x_{10}w$ is a 7-cycle, a contradiction. Similarly, if $N'_7 \cap N'_{10} \neq \emptyset$ and $w \in N'_7 \cap N'_{10}$, then $wx_7x_2x_1x_6x_8x_{10}w$ is a 7-cycle and if $N'_9 \cap N'_{10} \neq \emptyset$ and $w \in N'_9 \cap N'_{10}$, then $wx_9x_7x_2x_3x_8x_{10}w$ is a 7-cycle, a contradiction in each case. Thus Claim 1 is proved.

Claim 2: There exists no edge-bridge.

This follows from Lemma 2.5(ii) and Lemma 2.3.

Claim 3: The only possible bridges are either monobridges or bibridges $B_{i,j}$ where $x_i \sim x_j$.

This is proved just as was Claim 3 of Lemma 5.2, except here we use Lemma 2.5(ii), instead of Lemma 2.4(ii) and (iii). So Claim 3 follows.

As noted in the proof of Lemma 5.2, not all bridges of H are monobridges, so let $B_{i,j}$ be a bibridge. Hence by Claim 3, vertices x_i and x_j are adjacent. By Claim 2, $B_{i,j}$ is not an edge-bridge, so there must exist an interior vertex v in $B_{i,j}$. Again arguing as in the proof of Lemma 5.2, there must exist a path P in $G\backslash V(H)$ joining v to a vertex w_k in some N'_k, $k \neq i, j$ and passing through a bibridge $B_{i,k}$ or a bibridge $B_{j,k}$. Say, without loss of generality, P passes through a bibridge of type $B_{j,k}$. If $w_i \in N'_i$, $P + vw_i$ is a path joining w_i and w_k in $G\backslash V(H)$. By Claim 3, $x_i \sim x_j$ and $x_j \sim x_k$. But then by our girth hypothesis, $x_i \not\sim x_k$. Therefore $\{x_i, x_k\}$ is a well-connected pair and by Lemma 2.5(iii) there exist induced paths of length 3 and 4 joining x_i and x_k, both avoiding x_j.

Now arguing just as in the proof of Lemma 5.2, again we conclude that $B_{i,j} = \emptyset$ and the lemma follows. ∎

Lemma 5.4: Let G be a 3-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Let C be a 5-cycle in G. Then if $N'(C)$ contains an induced path P of length 3, $G \cong P_{10}$.

Proof: Let $C = x_1x_2x_3x_4x_5x_1$ be a 5-cycle in G. By Lemma 5.3, $N'(C)$ contains no induced path of length 4. Suppose, on the other hand, that $N'(C)$ does contain an induced path P of length 3.

Suppose $G \ncong P_{10}$.

Let $H = G[C \cup P]$. Then it is easy to see that $H = G[C \cup P]$ is isomorphic to $P_{10}\backslash v$ for some vertex v.

Let us suppose that H is labeled as shown in Figure 5.3.

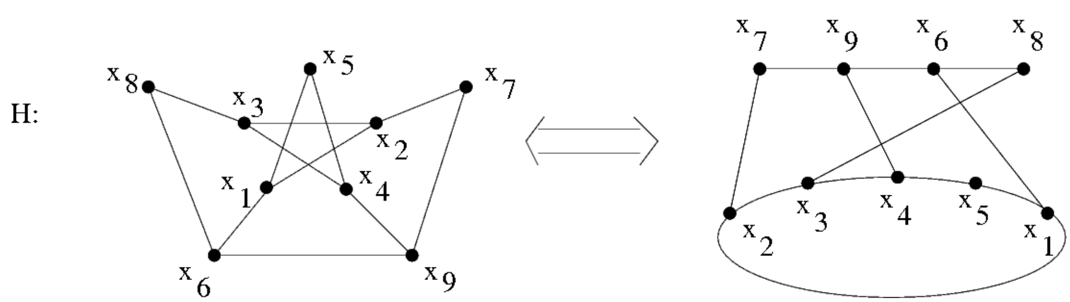

Figure 5.3

Claim 1: $N'_i \cap N'_j = \emptyset$, for $1 \leq i < j \leq 9$.

It is easy to check that every pair of vertices are at distance either 1 or 2 when one of the pair has degree 3. This implies that $N'_i \cap N'_j = \emptyset$ if one of i and j is not $5, 7$ or 8.

Suppose that, say, $N'_5 \cap N'_7 \neq \emptyset$ and we choose $w \in N'_5 \cap N'_7$. Then w is adjacent to both x_5 and x_7 and $H \cup w$ is a graph containing 5-cycle $C_1 = x_1x_5x_4x_9x_6x_1$ together with a path $P_{C_1} = wx_7x_2x_3x_8$ in $N'(C_1)$. But P_{C_1} has length 4 and so by Lemma 5.3, $G \cong P_{10}$, a contradiction. So $N'_5 \cap N'_7 = \emptyset$.

By symmetry, the same argument may be used to show that if $G \ncong P_{10}$, then $N'_5 \cap N'_8 \neq \emptyset$ and $N'_7 \cap N'_8 \neq \emptyset$.

Note that $H \cup w$ can viewed as follows. Add a vertex $w = x_{10}$ and join it to vertices x_5, x_7 and x_8 in Figure 5.3 and then delete either edge x_5x_{10}, x_7x_{10} or x_8x_{10}. But these three graphs are each isomorphic to $P_{10} \backslash e$, where e is any edge of P_{10}, since P_{10} is edge-transitive.

Therefore Claim 1 is true.

Claim 2: There exists no edge-bridge.

This follows from the existence of a path of length 4 as pointed out in Lemmas 2.5(i) and 2.3.

Claim 3: The only possible bridges are either monobridges or bibridges $B_{i,j}$ where $x_i \sim x_j$.

This is proved using Lemma 2.5(i) just as in the proof of Claim 3 of Lemma 5.3.

As noted in the proof of Lemma 5.2, not all bridges of H are monobridges, so let $B_{i,j}$ be a bibridge. Hence by Claim 3, vertices x_i and x_j are adjacent. By Claim 2, $B_{i,j}$ is not an edge-bridge, so there must exist an interior vertex v in $B_{i,j}$. Again arguing as in the proof of Lemma 5.2, there must exist a path P in $G \backslash V(H)$ joining v to a vertex w_k in some N'_k, $k \neq i, j$ and passing through a bibridge $B_{i,k}$ or a bibridge $B_{j,k}$. Say, without loss of generality, P passes through a bibridge of type $B_{j,k}$. If $w_i \in N'_i$, $P + vw_i$ is a path joining w_i and w_k in $G \backslash V(H)$. By Claim 3, $x_i \sim x_j$ and $x_j \sim x_k$. But then by our girth hypothesis, $x_i \not\sim x_k$ and therefore x_i and x_k are well-connected and by Lemma 2.5(iii) there exist induced paths of length 3 and 4 joining x_i and x_k, both avoiding vertex x_j.

Now arguing just as in the proof of Lemma 5.2, again we conclude that $B_{i,j} = \emptyset$ and the lemma follows. ∎

6. Forbidden induced paths of length 2.

In this section, we will show the following. Suppose G is 3-connected, internally 4-connected, has girth 5 and every odd cycle of length greater than 5 has a chord. Then if C is a 5-cycle in G such that $N'(C)$ contains an induced path of length 2, $G \cong P_{10}$.

So let us suppose that C is a 5-cycle in G and $N'(C)$ does contain such an induced 2-path $x_3x_4x_5$. Then by symmetry and the girth 5 hypothesis as well as 3-connectivity, we may assume that G contains the subgraph J_2 shown in Figure 3.2. (Note that J_2 must be induced by the girth 5 hypothesis.)

Our goal, then, is to show that if G contains J_2 as a subgraph, then $G \cong P_{10}$. Let us point out again that we have not yet invoked the hypothesis of internal-4-connectivity.

Lemma 6.1: Suppose G is 3-connected with girth 5 and every odd cycle of length greater than 5 has a chord. Suppose also that G contains a subgraph isomorphic to J_2. Let the vertices of this J_2 be labeled as in Figure 3.2. Then if there exists an edge joining N'_i and N'_j, $\{i, j\} = \{1, 3\}, \{1, 7\}, \{3, 5\}$, or $\{5, 7\}$.

Proof: Suppose that there is an edge joining $u \in N'_1$ to a vertex $v \in N'_i$. Then $i \neq 1$, since N'_1 is independent by the girth 5 hypothesis. For the same reason $i \neq 2, 8$. Suppose $v \in N'_4$. Then $uvx_4x_5x_6x_2x_1u$ is a 7-cycle, a contradiction of Lemma 2.3. Hence $v \notin N'_4$ and, by symmetry, $v \notin N'_6$. Similarly, $v \notin N'_5$, since if it were, $uvx_5x_4x_3x_2x_1u$ would be a 7-cycle. So $\{1, j\} = \{1, 3\}$ or $\{1, 7\}$.

Now consider possible edges joining N'_2 to N'_i, $i \neq 2$. Since no edge joins N'_1 and N'_4, by symmetry, no edge joins N'_2 and N'_5. Suppose there is an edge joining $u \in N'_2$ and $v \in N'_4$. Then $uvx_4x_8x_7x_6x_2u$ is a 7-cycle, a contradiction. Hence by the girth 5 hypothesis and symmetry, there is no edge joining N'_2 to N'_i, where $i \neq 2$. By symmetry, then, the proof of the Lemma is complete. ∎

The preceding Lemma shows that the only possible "ears" attached to the subgraph J_2 belong to one of four classes, namely $\{i, j\} = \{1, 3\}, \{1, 7\}, \{3, 5\}$ or $\{5, 7\}$. We now proceed as follows:

(1) In Lemma 6.2, we show that J_2 cannot have ears from three (or more) of these four classes. That is to say, G cannot contain configuration J_3 shown in Figure 6.1 as a subgraph.

(2) Using (1), we show that J_2 cannot possess ears from exactly two of the four classes. Here there are, up to isomorphism, two separate cases to treat. (See configuration J_4 in Figure 6.2 and configuration J_5 in Figure 6.3.) That G cannot contain J_4 or J_5 as subgraphs is shown in Lemmas 6.3 and 6.4 respectively.

Note that in the proof of Lemma 6.4, we will use the assumption of internal-4-connectivity for the first time.

(3) Using (2), we show in Lemma 6.5 that if G contains J_1 shown in Figure 3.1 as a subgraph, then $G \cong P_{10}$.

(4) And finally, using (3), we show by means of Lemma 6.6 that if G contains the graph J_2 shown in Figure 3.2 as a subgraph, then $G \cong P_{10}$.

We have then shown that if $N'(C)$ contains an induced path of length exactly 2, then $G \cong P_{10}$.

Lemma 6.2: Let G be a 3-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Then G does not contain as a subgraph the graph J_3 shown in Figure 6.1.

Proof: Suppose, by way of contradiction, that G does contain J_3 as a subgraph.

Cycle $C = x_1x_2 \cdots x_{11}x_1$ has length 11 and hence contains a chord. Since the girth of G is five and by Lemma 2.3 there are no 7-cycles, the only possible chords of C are of the form x_ix_{i+4} or x_ix_{i+7} (modulo 11). Therefore, C can only have eleven possible chords: $x_1x_5, x_2x_6, x_3x_7, x_4x_8, x_5x_9, x_6x_{10}, x_7x_{11}, x_1x_8, x_2x_9, x_3x_{10}$ and x_4x_{11}. However, if $x_1x_5 \in E(G)$, then $x_1x_5x_6x_{14}x_9x_{10}x_{11}x_1$ is a 7-cycle, if $x_1x_8 \in E(G)$, then $x_1x_8x_7x_6x_{13}x_3x_2x_1$ is a 7-cycle, if $x_2x_6 \in E(G)$, then $x_2x_6x_{13}x_3x_2$ is a 4-cycle, if $x_2x_9 \in E(G)$, then $x_2x_9x_{14}x_6x_5x_4x_3x_2$ is a 7-cycle, if $x_3x_7 \in E(G)$, then $x_3x_7x_6x_{13}x_3$ is a 4-cycle, if $x_3x_{10} \in E(G)$, then $x_3x_{10}x_{13}x_3$ is a 3-cycle, if $x_4x_8 \in E(G)$, then $x_4x_8x_9x_{10}x_{11}x_{12}x_3x_4$ is a 7-cycle, if $x_4x_{11} \in E(G)$, then $x_4x_{11}x_{12}x_3x_4$ is a 4-cycle, if $x_5x_9 \in E(G)$, then $x_5x_9x_{14}x_6x_5$ is a 4-cycle, if $x_6x_{10} \in E(G)$, then $x_6x_{10}x_{13}x_6$ is a 3-cycle, and if $x_7x_{11} \in E(G)$, then $x_7x_{11}x_{12}x_3x_4x_5x_6x_7$ is a 7-cycle, so each of these possibilities also leads to a contradiction of either Lemma 2.3 or the girth hypothesis. ∎

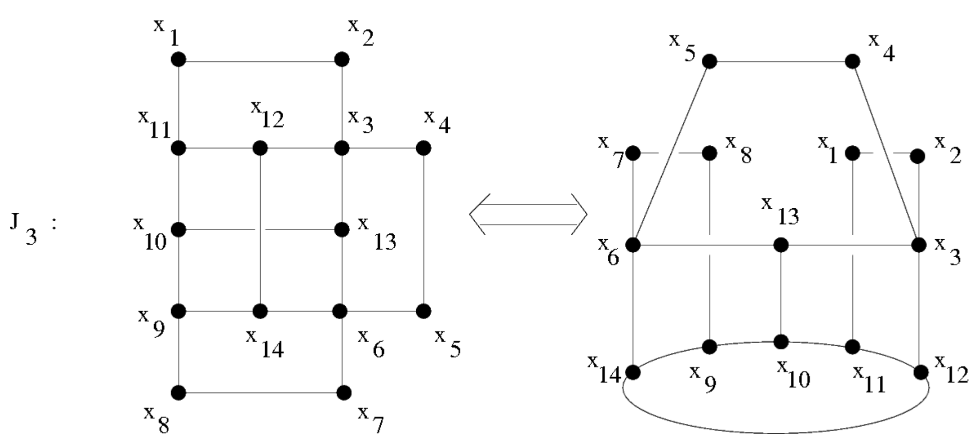

Figure 6.1

Lemma 6.3: Let G be a 3-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Then G does not contain a subgraph isomorphic to the graph J_4 shown in Figure 6.2.

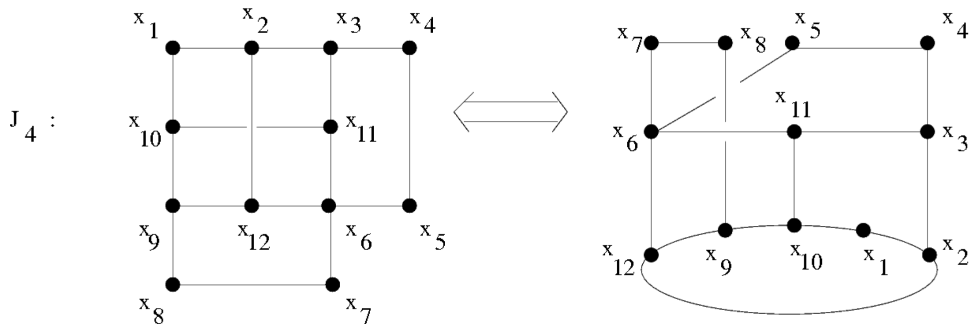

Figure 6.2

Proof: Claim 1: If J_4 is a subgraph of G, then it is an induced subgraph.

By the girth 5 assumption and symmetry, we need only check the pairs $\{x_1, x_5\}$, $\{x_3, x_8\}$ and $\{x_4, x_8\}$ for adjacency. But if x_1 and x_5 are adjacent, $x_1x_5x_6x_7x_8x_9x_{10}x_1$ is a 7-cycle, contradicting Lemma 2.3. Similarly, if x_3 and x_8 are adjacent, $x_3x_8x_9x_{12}x_6x_5$ x_4x_3 is a 7-cycle, a contradicton, and if x_4 and x_8 are adjacent, $x_4x_8x_9x_{12}x_6x_{11}x_3x_4$ is a 7-cycle, a contradiction. This proves Claim 1.

Claim 2: Let x_i and x_j be two nonadjacent vertices of J_4 and suppose that $x_i \in \{x_1, x_2, x_{10}\}$. Then:
(i) there exists an induced even path of length at least 4 joining x_i and x_j and
(ii) there exists an induced odd path of length at least 3 joining x_i and x_j with two exceptions: $\{i, j\} = \{2, 4\}, \{8, 10\}$.
(iii) If $x_i \not\sim x_j$ and $\{x_i, x_j\} \neq \{x_2, x_4\}, \{x_8, x_{10}\}$, then $\{x_i, x_j\}$ is a non-co-bridge pair.

Proof: This is easily checked.

Claim 3: For $1 \leq i < j \leq 12$, $N_i' \cap N_j' = \emptyset$.

Proof: Choose $v \in N_i'$, $i = 1, \ldots, 12$. By the girth hypothesis, v is not incident to any x_j with $d(x_i, x_j) \leq 2$. Since by Lemma 2.3 there is no 7-cycle in G, v is not adjacent to x_j if J_4 contains a path of length 5 joining x_i and x_j. By symmetry, we need only check $x_i = x_1, x_2, x_3, x_4, x_5, x_6, x_{11}$ versus x_j when $j > i$. This is easily done.

Claim 4: There is no edge-bridge joining N_i' to N_j', for $1 \leq i < j \leq 12$.

Proof: Suppose, to the contrary, that $N'(J_4)$ is dependent and hence there is an edge uv joining the two vertices $u, v \in N'(J_4)$ such that u is adjacent to x_i and v is adjacent to x_j. Clearly, x_i and x_j are not adjacent by the girth hypothesis. By Claim 2, graph J_4 contains an induced even path P_{ij} of length at least 4 joining x_i and x_j. Then $Q = P_{ij} \cup \{uv, ux_i, vx_j\}$ is an odd cycle of length at least 7. But then Q must contain a chord. However, by Claim 3, neither u nor v can be an endvertex of this chord. But then P_{uv} is not induced, a contradiction. Therefore Claim 4 is true.

Let $H = G[J_4 \cup N'(J_4)]$. We now consider the possible bridges of H. Let $\mathcal{B}_i$ denote the collection of all bridges B of H such that $B \cap N'_i \neq \emptyset$; that is, $\mathcal{B}_i$ is the collection of bridges of H having attachments in N'_i.

Since x_1 has degree two in J_4, $N'_1 \neq \emptyset$. By Claims 3 and 4, each vertex in N'_1 has degree one in H and hence $\mathcal{B}_1 \neq \emptyset$.

Claim 5: Suppose $B \in \mathcal{B}_1$. Then one of the following three cases must hold:

(i) B has all attachments contained in N'_1; that is, $B = B_1$ is a monobridge of H;

(ii) B has all attachments contained in $N'_1 \cup N'_2$; that is, B is a bibridge of type $B_{1,2}$; or

(iii) B has all attachments contained in $N'_1 \cup N'_{10}$; that is B is a bibridge of type $B_{1,10}$.

Proof of Claim 5: By Claims 2, 3 and 4, x_1 and x_i are non-co-bridge pairs, for $i \neq 1, 2$ and 10. Therefore, Claim 5 is true.

Claim 6: Suppose $B \in \mathcal{B}_2$. Then one of the following must occur:

(i) B has all attachments in N'_2; that is, $B = B_2$ is a monobridge of H;

(ii) B has all attachments in $N'_2 \cup N'_1$, $N'_2 \cup N'_3$, $N'_2 \cup N'_4$ or in $N'_2 \cup N'_{12}$; that is, B is a bibridge of type $B_{2,1}$, type $B_{2,3}$, type $B_{2,4}$ or type $B_{2,12}$ respectively; or

(iii) B has all attachments in $N'_2 \cup N'_3 \cup N'_4$, and each of N'_2, N'_3 and N'_4 contains at least one such attachment.

Proof of Claim 6: If $j \neq 4$ and x_2 and x_j are not adjacent, then by Claim 2(iii), $\{x_2, x_j\}$ is a non-co-bridge pair. Therefore, all attachments of B are contained in $N'_1 \cup N'_2 \cup N'_3 \cup N'_4 \cup N'_{12}$. Also by Claim 2(iii), $\{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_{12}\}, \{x_3, x_{12}\}$ and $\{x_4, x_{12}\}$ are non-co-bridge pairs, so B contains attachments in both N'_2 and N'_1 alone, in both N'_2 and N'_3 alone, in both N'_2 and N'_4 alone, in both N'_2 and N'_{12} alone, or in each of N'_2, N'_3 and N'_4. Thus Claim 6 is true.

We now proceed to complete the proof of the lemma.

Since $\{x_2, x_{10}\}$ is a 2-vertex cut in J_4 and G is 3-connected, there must exist paths from x_1 which pass through some vertices in N'_1, then (by Claim 5) through some bridge of type $B_{1,2}$ (or type $B_{1,10}$), and then, perhaps, through some other bridges in $\mathcal{B}_2$ (or $\mathcal{B}_{10}$) to vertices in N'_j, for each $j \in S = \{3, 4, 5, 6, 7, 8, 9, 11, 12\}$. Let Q be a shortest of these paths joining x_1 to some $x_k \in S$. By the minimality of the length of Q, Q contains only one vertex in N'_1 and hence Q does not pass through both a bibridge of type $B_{1,2}$ and one of type $B_{1,10}$. By symmetry, we may assume Q passes through a bridge of type $B_{1,2}$. Then by Claim 6, Q must pass some bridge of type $B_{2,3}$ to reach x_3, one of type $B_{2,4}$ to reach x_4, one of type $B_{2,12}$ to reach x_{12}, or some bridge B having all attachments in $N'_2 \cup N'_3 \cup N'_4$ to reach either x_3 or x_4.

First we suppose that Q passes through a bridge of type $B_{2,3}$ to reach x_3. Since x_1 and x_3 are connected by the odd path $P_1 = x_1x_{10}x_{11}x_3$ in J_5 and the even path $P_2 = x_1x_{10}x_9x_8x_7x_6x_5x_4x_3$, both of which avoid x_2, both $Q \cup P_1$ and $Q \cup P_2$ are chordless cycles of opposite parity and each has length greater than 5. So once more we get a chordless odd cycle of length greater than 5, a contradiction. (Note that since both P_1 and P_2 avoid x_2, this prevents possible chords joining x_2 and a vertex in N_2'.)

Now we suppose that Q passes through a bridge of type $B_{2,12}$ to reach x_{12}. Using the same argument with the odd path $P_1 = x_1x_{10}x_9x_{12}$ and the even path $P_2 = x_1x_{10}x_{11}x_6x_{12}$, both of which avoid x_2, we obtain a chordless odd cycle of length greater than 5, a contradiction.

For the case of a bridge of type $B_{2,4}$, we use the same argument with the induced odd path $P_1 = x_1x_{10}x_9x_8x_7x_6x_5x_4$ and the induced even path $P_2 = x_1x_{10}x_9x_{12}x_6x_5x_4$, both of which avoid x_2, to obtain a chordless odd cycle of length greater than 5, a contradiction.

The last case occurs when Q passes through a bridge B having all attachments in $N_2' \cup N_3' \cup N_4'$ and, moreover, has attachments in each of the three. Since Q is a shortest path, Q contains only one vertex w in $N_3' \cup N_4'$. If $w \in N_3'$, then $x_3 \in V(Q)$ and $Q \cap N_4' = \emptyset$ and if $w \in N_4'$, $x_4 \in V(Q)$ and $Q \cap N_3' = \emptyset$.

Suppose $x_3 \in V(Q)$. Then $Q[x_1, x_3] \cup x_1x_{10}x_{11}x_3$ and $Q[x_1, x_3] \cup x_1x_{10}x_9x_8x_7x_6x_5x_4x_3$ are of opposite parity. Moreover, both avoid vertex x_2 and hence are chordless. So x_1 and x_3 lie on a chordless odd cycle of length greater than 5. Suppose, then, that $x_4 \in V(Q)$. Then $Q[x_1, x_4] \cup x_1x_{10}x_9x_{12}x_6x_5x_4$ and $Q[x_1, x_4] \cup x_1x_{10}x_9x_8x_7x_6x_5x_4$ are of opposite parity, both fail to contain both x_2 and x_3 and hence are chordless. Hence x_1 and x_4 must lie on a chordless odd cycle of length greater than 5. This completes the proof of Lemma 6.3. ∎

In Lemma 6.4, we assume internal-4-connectivity for the first time in eliminating the subgraph J_5. In fact, we point out that the counterexamples presented at the end of Section 3 are constructed by beginning with a subgraph isomorphic to J_5.

Lemma 6.4: Let G be a 3-connected internally-4-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Then G contains no subgraph isomorphic to graph J_5 shown in Figure 6.3.

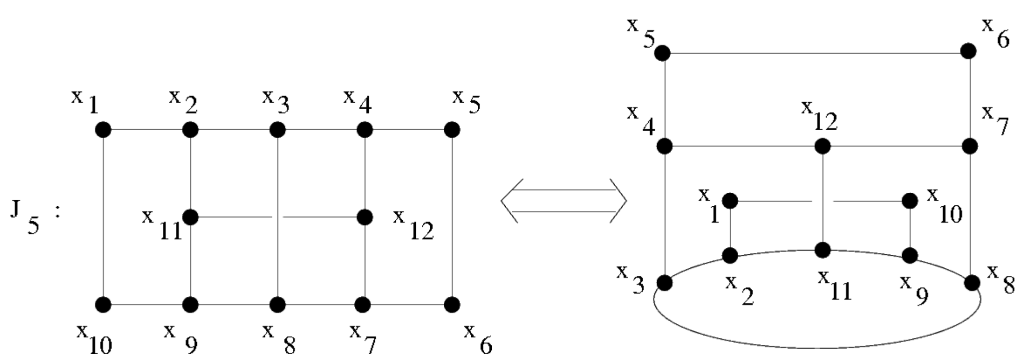

Figure 6.3

Proof: Suppose G does contain J_5 as a subgraph.

Claim 1: J_5 is an induced subgraph.

Proof of Claim 1: If x_1 and x_5 are adjacent, then $x_1x_5x_6x_7x_8x_9x_{10}x_1$ is a 7-cycle, if x_1 and x_6 are adjacent, then $x_1x_6x_7x_8x_9x_{11}x_2x_1$ is a 7-cycle and if x_1 and x_7 are adjacent, then $x_1x_2x_3x_4x_5x_6x_7x_1$ is a 7-cycle. But by Lemma 2.3 there are no 7-cycles.

Any other possible chord in J_5 would violate the girth 5 hypothesis.

Claim 2: For $1 \le i < j \le 12$, $N'_i \cap N'_j = \emptyset$.

By the girth 5 hypothesis, $N'_1 \cap N'_2 = N'_1 \cap N'_3 = N'_1 \cap N'_9 = N'_1 \cap N'_{10} = N'_1 \cap N'_{11} = N'_2 \cap N'_3 = N'_2 \cap N'_4 = N'_2 \cap N'_8 = N'_2 \cap N'_9 = N'_2 \cap N'_{10} = N'_2 \cap N'_{11} = N'_2 \cap N'_{12} = N'_3 \cap N'_4 = N'_3 \cap N'_5 = N'_3 \cap N'_7 = N'_3 \cap N'_8 = N'_3 \cap N'_9 = N'_3 \cap N'_{11} = N'_3 \cap N'_{12} = N'_{11} \cap N'_{12} = \emptyset$.

The following pairs of vertices $\{x_i, x_j\}$ are joined by the indicated paths of length 5. Hence $N'_i \cap N'_j = \emptyset$ for these pairs since G has no 7-cycle.

$\{x_1, x_4\}$: $x_1x_{10}x_9x_8x_3x_4$; $\{x_1, x_5\}$: $x_1x_2x_{11}x_{12}x_4x_5$; $\{x_1, x_6\}$: $x_1x_2x_3x_4x_5x_6$; $\{x_1, x_7\}$: $x_1x_2x_3x_4x_{12}x_7$; $\{x_1, x_8\}$: $x_1x_2x_{11}x_{12}x_7x_8$; $\{x_1, x_{12}\}$: $x_1x_2x_3x_8x_7x_{12}$; $\{x_2, x_5\}$: $x_2x_3x_8x_7x_6x_5$; $\{x_2, x_6\}$: $x_2x_{11}x_9x_8x_7x_6$; $\{x_2, x_7\}$: $x_2x_1x_{10}x_9x_8x_7$; $\{x_3, x_6\}$: $x_3x_2x_{11}x_{12}x_7x_6$.

All remaining pairs are symmetric to one of those listed above and hence Claim 2 follows.

Claim 3: If $\{i, j\} \notin \{\{1, 11\}, \{5, 12\}, \{6, 12\}, \{10, 11\}, \{2, 9\}, \{4, 7\}\}$, then there is no edge joining N'_i and N'_j.

If $\{i, j\} = \{1, 2\}, \{1, 10\}, \{2, 3\}, \{2, 11\}, \{3, 4\}, \{3, 8\}, \{11, 12\}$, then no edge joins N'_i and N'_j by the girth five hypothesis.

For the following pairs $\{x_i, x_j\}$, we list either an induced 4-path, an induced 6-path or an induced 8-path joining x_i and x_j. If there is an edge joining N'_i and N'_j in these cases, we get either a 7-cycle, which contradicts Lemma 2.3, a chordless 9-cycle or a chordless 11-cycle which contradicts the girth hypothesis of the present Lemma.

$\{x_1, x_3\}$: $x_1x_{10}x_9x_8x_3$; $\{x_1, x_4\}$: $x_1x_2x_{11}x_{12}x_4$; $\{x_1, x_5\}$: $x_1x_2x_3x_4x_5$; $\{x_1, x_6\}$: $x_1x_2x_{11}x_9x_8x_7x_6$; $\{x_1, x_7\}$: $x_1x_2x_3x_8x_7$; $\{x_1, x_8\}$: $x_1x_2x_{11}x_9x_8$; $\{x_1, x_9\}$: $x_1x_2x_3x_8x_9$; $\{x_1, x_{12}\}$: $x_1x_2x_3x_4x_{12}$; $\{x_2, x_4\}$: $x_2x_1x_{10}x_9x_8x_7x_6x_5x_4$; $\{x_2, x_5\}$: $x_2x_{11}x_{12}x_4x_5$; $\{x_2, x_6\}$: $x_2x_{11}x_{12}x_7x_6$; $\{x_2, x_7\}$: $x_2x_3x_4x_{12}x_7$; $\{x_2, x_8\}$: $x_2x_1x_{10}x_9x_8$; $\{x_2x_{10}\}$: $x_2x_3x_8x_9x_{10}$; $\{x_2, x_{12}\}$: $x_2x_3x_8x_7x_{12}$; $\{x_3, x_5\}$: $x_3x_8x_7x_6x_5$; $\{x_3, x_6\}$: $x_3x_4x_{12}x_7x_6$; $\{x_3, x_7\}$: $x_3x_2x_{11}x_{12}x_7$; $\{x_3, x_{11}\}$: $x_3x_8x_7x_{12}x_{11}$.

All other pairs are symmetric to one of those listed above. This proves Claim 3.

Let $\mathcal{S}_1 = \{1, 2, 9, 10\}$ and $\mathcal{S}_2 = \{4, 5, 6, 7, 12\}$; that is, the vertex partition in J_5 induced by the 3-cut $\{x_3, x_8, x_{11}\}$.

Claim 4: The following pairs of vertices are well-connected in J_3 and, except for $\{x_1, x_{10}\}$ and $\{x_{10}, x_{11}\}$, each is a non-co-bridge pair.

(i) all $\{x_i, x_j\}$ where $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, and

(ii) $\{x_3, x_6\}, \{x_3, x_7\}, \{x_3, x_9\}, \{x_3, x_{10}\}, \{x_3, x_{11}\}, \{x_3, x_{12}\}, \{x_1, x_8\}, \{x_2, x_8\}, \{x_4, x_8\}, \{x_5, x_8\}, \{x_8, x_{11}\}, \{x_8, x_{12}\}, \{x_1, x_{11}\}, \{x_4, x_{11}\}, \{x_5, x_{11}\}, \{x_6, x_{11}\}, \{x_7, x_{11}\}, \{x_{10}, x_{11}\}$.

The reader may easily check that each of the pairs in (1) and (2) are well-connected.

By Claim 3, except for $\{x_1, x_{10}\}$ and $\{x_{10}, x_{11}\}$, there do not exist edges between N'_i and N'_j, for the pairs $\{x_i, x_j\}$ listed in (i) and (ii). Hence the Claim follows.

For each $i = 1, \ldots, 12$, define

$$\mathcal{B}_i = \{B | B \text{ is a non} - \text{edge bridge of } J_3 \cup N'(J_3) \text{ such that } B \text{ has attachments in } N'_i\}.$$

Since $\{x_3, x_8, x_{11}\}$ is a 3-vertex cut in J_3, and since G is internally 4-connected, there must exist paths from each vertex in $\mathcal{S}_1$ to each vertex in $\mathcal{S}_2$ which do not pass through any of x_3, x_8 or x_{11}. By Claim 3, there are no edges joining N'_i to N'_j, for all i, j, $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$. By Claim 4(i), for each $i \in \mathcal{S}_1$ and each $j \in \mathcal{S}_2$, $\{x_i, x_j\}$ is a non-co-bridge pair and hence any such path joining $\mathcal{S}_1$ and $\mathcal{S}_2$ must pass through some vertices in $N'_3 \cup N'_8 \cup N'_{11}$ and through some non-edge bridges in $\mathcal{B}_3 \cup \mathcal{B}_8 \cup \mathcal{B}_{11}$.

Now let Q be a shortest such path from $\{x_i | i \in \mathcal{S}_1\}$ to $\{x_j | j \in \mathcal{S}_2\}$, which does not pass through x_3, x_8 or x_{11}. By minimality of the length of Q, Q intersects only one N'_i, $i \in \mathcal{S}_1$, and that in only one vertex and Q intersects only one N'_j, $j \in \mathcal{S}_2$, and that in only one vertex. Since $\{x_3, x_{11}\}$ and $\{x_8, x_{11}\}$ are non-co-bridge pairs, $\mathcal{B}_3 \cap \mathcal{B}_{11} = \mathcal{B}_8 \cap \mathcal{B}_{11} = \emptyset$. But Q must meet $N'_3 \cup N'_8 \cup N'_{11}$. Therefore, Q can only meet N'_{11} or $N'_3 \cup N'_8$, but not both. So we have two cases.

Case 1. Suppose Q leaves $\mathcal{S}_1$ via one of x_1 or x_{10}, say without loss of generality by symmetry, via vertex x_1. Since $\{1, 8\}$ is a non-co-bridge pair, Q must use either a bibridge of type $B_{1,3}$ or of type $B_{1,11}$, but not both, by the minimality of the length of Q.

Case 1.1. Suppose Q uses a bibridge of type $B_{1,3}$. If Q then traverses a bridge of type $B_{3,8}$, the induced paths $x_1x_2x_{11}x_9x_8$ and $x_1x_{10}x_9x_8$ (both of which avoid vertex x_3) guarantee the existence of a chordless odd cycle of length greater than 5, a contradiction.

On the other hand, if Q traverses a bridge of type $B_{3,4}$ after traversing $B_{1,3}$, the induced paths $x_1x_{10}x_9x_8x_7x_{12}x_4$ and $x_1x_{10}x_9x_{11}x_{12}x_4$ (both of which avoid vertex x_3) guarantee a chordless odd cycle of length greater than 5, again a contradiction.

This completes the proof in Case 1.1, since the remaining possible pairs of type $\{x_3, x_j\}$, namely those for which $j = 5, 6, 7, 12$ are all non-co-bridge pairs.

Case 1.2. Suppose, on the other hand, that Q uses a bibridge of type $B_{1,11}$. Then, since $\{x_4, x_{11}\}, \{x_5, x_{11}\}, \{x_6, x_{11}\}$ and $\{x_7, x_{11}\}$ are non-co-bridge pairs, Q must next traverse a bibridge of type $B_{11,12}$. The induced paths $x_1x_2x_3x_4x_{12}$ and $x_1x_{10}x_9x_8x_7x_{12}$

(both of which avoid vertex x_{11}) then guarantee a chordless odd cycle of length greater than 5, a contradiction.

Case 2. Suppose now that Q leaves $\mathcal{S}_1$ via one of x_2 or x_9, say without loss of generality by symmetry, via x_2. Since $\{x_2, x_8\}$ is a non-co-bridge pair, Q must traverse a bibridge of type $B_{2,3}$ or one of type $B_{2,11}$ (but not both).

Case 2.1. Suppose first that Q uses a bibridge of type $B_{2,3}$.

Suppose Q next traverses a bridge of type $B_{3,j}$. Since $\{x_3, x_j\}$, for $j = 6, 7, 12$ is a non-co-bridge pair, there are just three cases to treat.

If $j = 4$, induced paths $x_2x_{11}x_{12}x_4$ and $x_2x_1x_{10}x_9x_8x_7x_6x_5x_4$ (both of which avoid vertex x_3) guarantee a chordless odd cycle of length greater than 5. If $j = 5$, induced paths $x_2x_{11}x_{12}x_4x_5$ and $x_2x_{11}x_{12}x_7x_6x_5$ (both of which avoid x_3) imply a chordless odd cycle of length greater than 5. Finally, if $j = 8$, induced paths $x_2x_{11}x_9x_8$ and $x_2x_1x_{10}x_9x_8$ (both of which avoid x_3) guarantee a chordless odd cycle of length greater than 5. In all three instances we obtain a contradiction.

Case 2.2. Suppose Q next traverses a bibridge of type $B_{2,11}$.

Since $\{x_4, x_{11}\}, \{x_5, x_{11}\}, \{x_6, x_{11}\}$ and $\{x_7, x_{11}\}$ are all non-co-bridge pairs, Q must next traverse a bibridge of type $B_{11,12}$. But in this instance induced paths $x_2x_3x_4x_{12}$ and $x_2x_3x_8x_7x_{12}$ (both of which avoid x_{11}) imply the existence of a chordless odd cycle of length greater than 5, a contradiction. ∎

Lemma 6.5: Let G be a 3-connected internally-4-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Then if G contains a subgraph isomorphic to graph J_1 shown in Figure 3.1, $G \cong P_{10}$.

Proof: Suppose G does contain a subgraph isomorphic to J_1. Let us assume the vertex labelling shown in Figure 3.1.

Suppose $G \ncong P_{10}$.

Claim 1: The subgraph J_1 must be induced.

It is easy to check that adding any edge different from x_1x_7 and x_4x_{10} results in the formation of a cycle of size less than five, contradicting the girth hypothesis.

So then let us assume x_1 is adjacent to x_7. Then if $C = x_2x_3x_8x_9x_{11}x_2$, $N'(C)$ contains the *induced* path $x_{10}x_1x_7x_{12}x_4$ of length 4, contradicting Lemma 5.3. By symmetry, if we add the edge x_4x_{10}, a similar contradiction is reached. This proves Claim 1.

Claim 2: For $1 \leq i < j \leq 12$, $N'_i \cap N'_j = \emptyset$.

It is routine to check that any possible non-empty intersection of two different N'_i's produces either a cycle of length less than 5, thus contradicting the girth hypothesis, or else a 7-cycle, thus contradicting Lemma 2.3. This proves Claim 2.

Claim 3: For all pairs $\{i,j\}, 1 \leq i < j \leq 12$, except $\{1,11\},\{10,11\}$, and $\{2,9\}$, there is no edge joining N'_i and N'_j.

For all pairs $\{i,j\}$ in the pair set under consideration, except $\{2,4\},\{4,7\}$, and $\{2,9\}$, if there is an edge joining N'_i and N'_j, there results either a cycle of length less than 5 (contradicting the girth hypothesis) or a 7-cycle (contradicting Lemma 2.3). On the other hand, if there is an edge joining N'_2 and N'_4, this results in a subgraph isomorphic to J_4 and hence by Lemma 6.3, $G \cong P_{10}$, a contradiction, while if there is an edge joining N'_4 and N'_7, we get a subgraph isomorphic to J_5 contradicting Lemma 6.4. This proves Claim 3.

Let $\mathcal{S}_1 = \{1,2,9,10\}$ and $\mathcal{S}_2 = \{4,7,12\}$; i.e., the partition in J_1 induced by the 3-cut $\{x_3,x_8,x_{11}\}$.

Claim 4: The following pairs of vertices are well-connected in J_5 and hence, except for $\{x_1,x_{11}\}$ and $\{x_{10},x_{11}\}$, each is a non-co-bridge-pair.

(i) all $\{i,j\}$ where $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, and
(ii) $\{1,8\},\{1,11\},\{2,8\},\{3,7\},\{3,9\},\{3,10\},\{3,11\},\{3,12\},\{4,8\},\{4,11\},\{7,11\},$ $\{8,11\},\{8,12\},\{10,11\}$.

Again, this is easily checked.

For each $i = 1,2,3,4,7,8,9,10,11,12$, define $\mathcal{B}_i$ as in the proof of Claim 4 of Lemma 6.4.

Since $\{x_3,x_8,x_{11}\}$ is a 3-vertex cut in J_5, and since G is internally 4-connected, there must exist paths from each vertex in $\mathcal{S}_1$ to each vertex in $\mathcal{S}_2$ which do not pass through any of x_3,x_8 or x_{11}. By Claim 3, there are no edges joining N'_i to N'_j, for all $i,j,\ i \in \mathcal{S}_1,\ j \in \mathcal{S}_2$. By Claim 4(i), for each $i \in \mathcal{S}_1$ and each $j \in \mathcal{S}_2$, $\{x_i,x_j\}$ is a non-co-bridge pair and hence any such path joining $\mathcal{S}_1$ and $\mathcal{S}_2$ must pass through some vertices in $N'_3 \cup N'_8 \cup N'_{11}$ and through some non-edge bridges in $\mathcal{B}_3 \cup \mathcal{B}_8 \cup \mathcal{B}_{11}$.

Now let Q be a shortest such path from $\{x_i | i \in \mathcal{S}_1\}$ to $\{x_j | j \in \mathcal{S}_2\}$, which does not pass through x_3,x_8 or x_{11}. By minimality of the length of Q, Q intersects only one $N'_i,\ i \in \mathcal{S}_1$, and that in only one vertex and Q intersects only one $N'_j,\ j \in \mathcal{S}_2$, and that in only one vertex. Since $\{x_3,x_{11}\}$ and $\{x_8,x_{11}\}$ are non-co-bridge pairs, $\mathcal{B}_3 \cap \mathcal{B}_{11} = \mathcal{B}_8 \cap \mathcal{B}_{11} = \emptyset$. But Q must meet $N'_3 \cup N'_8 \cup N'_{11}$. Therefore, Q can only meet N'_{11} or $N'_3 \cup N'_8$, but not both. So we have two cases.

Case 1: Suppose Q leaves $\mathcal{S}_1$ via one of x_2 or x_9, say without loss of generality by symmetry, via x_2. Since $\{x_2,x_8\}$ is a non-co-bridge pair, Q must use a bibridge of type $B_{2,3}$ or $B_{2,11}$ (but not both).

Case 1.1. Suppose Q traverses a bibridge of type $B_{2,3}$. Then Q must next traverse a bibridge of type $B_{3,j}$, where $j \neq 7,11,12$, since $\{x_3,x_7\}$, $\{x_3,x_{11}\}$ and $\{x_3 x_{12}\}$ are non-co-bridge pairs. There are then only two possibilities. If $B_{3,j} = B_{3,4}$, the induced paths

$x_2x_{11}x_{12}x_4$ and $x_2x_{11}x_9x_8x_7x_{12}x_4$ (both of which avoid vertex x_3) imply the existence of a chordless odd cycle of length greater than 5, a contradiction.

If on the other hand, $B_{3,j} = B_{3,8}$, the induced paths $x_2x_{11}x_9x_8$ and $x_2x_1x_{10}x_9x_8$ (both of which avoid x_3) imply the existence of a chordless odd cycle of length greater than 5, again a contradiction.

Case 1.2. Suppose then that Q traverses a bibridge of type $B_{2,11}$. If then Q traverses a bibridge of type $B_{11,j}$, since $\{x_{11}, x_j\}$ is a non-co-bridge pair for $j = 3, 4, 7, 8$, the only possibility is that Q traverses a bibridge of type $B_{11,12}$. However, in this case the induced paths $x_2x_3x_4x_{12}$ and $x_2x_3x_8x_7x_{12}$ (both of which avoid x_{11}) imply the existence of a chordless odd cycle of length greater than 5, a contradiction.

Case 2. So suppose Q leaves $\mathcal{S}_1$ one of x_1 or x_{10}, say without loss of generality by symmetry, via x_1. Since $\{x_1, x_8\}$ is a non-co-bridge pair, Q must next traverse a bibridge of type $B_{1,3}$ or one of type $B_{1,11}$ (but not both).

Case 2.1. Suppose Q traverses a bibridge of type $B_{1,3}$. There are then only two possibilities for the next bibridge encountered by Q. If this bibridge is of type $B_{3,4}$, the induced paths $x_1x_2x_{11}x_{12}x_4$ and $x_1x_{10}x_9x_{11}x_{12}x_4$ (both of which avoid vertex x_3) together imply the existence of a chordless odd cycle of length greater than 5, a contradiction.

On the other hand, if this bibridge is of type $B_{3,8}$, the induced paths $x_1x_2x_{11}x_9x_8$ and $x_1x_{10}x_9x_8$ (both of which avoid x_3) then guarantee the existence of a chordless odd cycle of length greater than 5, a contradiction.

Case 2.2. So Q must traverse a bibridge of type $B_{1,11}$. But then since $\{x_{11}, x_j\}$ is a non-co-bridge pair for $j = 3, 4, 7, 8$, Q must pass through a bibridge of type $B_{11,12}$. But in this instance, the induced paths $x_1x_2x_3x_4x_{12}$ and $x_1x_2x_3x_8x_7x_{12}$ (both of which avoid x_{11}) guarantee a chordless odd cycle of length greater than 5, a contradiction. ∎

Lemma 6.6: Let G be a 3-connected internally-4-connected graph of girth 5 such that all odd cycles of length greater than 5 have a chord. Then if G contains a subgraph isomorphic to graph J_2, $G \cong P_{10}$.

Proof: Suppose G does contain J_2 as a subgraph. We will assume the labeling shown in Figure 3.2.

Suppose $G \ncong P_{10}$.

Claim 1: The subgraph J_2 is induced.

This is immediate by the girth five hypothesis.

Claim 2: For all $1 \leq i < j \leq 8$, $N'_i \cap N'_j = \emptyset$.

For all pairs $\{i, j\} \notin \{\{1, 5\}, \{3, 7\}\}$, the result follows by the girth five hypothesis. Suppose $N'_1 \cap N'_5 \neq \emptyset$ and $u \in N'_1 \cap N'_5$. Let $C = x_1x_2x_6x_7x_8x_1$. Then $N'(C)$ contains the induced path $x_3x_4x_5u$ of length 3, thus contradicting Lemma 5.4. Thus $N'_1 \cap N'_5 = \emptyset$ and, by symmetry, $N'_3 \cap N'_7 = \emptyset$. The Claim follows.

Claim 3: There exists no $N'_i - N'_j$ edge, for all $i \neq j$.

If $\{i, j\} = \{1, 3\}$, we get the configuration J_1 as an induced subgraph, contradicting Lemma 6.5. If $\{i, j\} = \{1, 4\}$, then together with the path $x_1x_2x_6x_5x_4$ we get a 7-cycle, contradicting Lemma 2.3. Similarly, if $\{i, j\} = \{1, 5\}$, using path $x_1x_2x_3x_4x_5$ we get a 7-cycle, and if $\{i, j\} = \{2, 4\}$, using path $x_2x_6x_7x_8x_4$, we get a 7-cycle in violation of Lemma 2.3. The Claim follows for all other pairs $\{i, j\}$ by the girth five hypothesis and symmetry.

Claim 4: All non-adjacent pairs of vertices are well-connected and hence, by Claim 3 and Lemma 2.1, are non-co-bridge-pairs.

This is easily checked.

Claim 5: For all i, $i = 1, \ldots, 8$, no bridge in $\mathcal{B}_i$ has an attachment in three distinct N'_j's, $j \neq i$.

Suppose to the contrary, that $B \in \mathcal{B}_i$ has attachments in N_j, N_k and N_ℓ, where i, j, k and ℓ are distinct. Since there are no triangles in J_2, some two of x_j, x_k and x_ℓ must be non-adjacent. But then these two vertices form a co-bridge-pair, contradicting Claim 4 and proving Claim 5.

Now consider the set $\{x_2, x_8\}$ as a vertex cut in J_2. By the 3-connectivity of G, there exists a path joining x_1 to one of the vertices in $\{x_3, x_4, x_5, x_6, x_7\}$ which does not pass through x_2 or x_8. Let Q be such a path of minimum length.

Since x_1 and x_i, for $i = 3, 4, 5, 6, 7$, are non-co-bridge pairs, Q must pass through bridges in $\mathcal{B}_2 \cup \mathcal{B}_8$. Since $\{x_2, x_8\}$ is a non-co-bridge pair, P must pass through exactly one bibridge of type $B_{1,2}$ or exactly one bibridge of type $B_{1,8}$, but not both. Say without loss of generality by symmetry, Q traverses a bibridge of type $B_{1,2}$. Then Q must pass through a bibridge of type $B_{2,j}$, $j \neq 1$. But by Claim 4, the only possible bridges of this type are of types $B_{2,3}$ and $B_{2,6}$. In the former case, induced paths $x_1x_8x_4x_3$ and $x_1x_8x_7x_6x_5x_4x_3$ both avoid vertex x_2 and hence guarantee the existence of a chordless odd cycle of length greater than 5, while in the latter case, induced paths $x_1x_8x_7x_6$ and $x_1x_8x_4x_5x_6$ both avoid x_2 and therefore imply the existence of a chordless odd cycle of length greater than 5. Thus in each instance we arrive at a contradiction and the Lemma is proved. ∎

Thus as explained at the beginning of this section, we have proved the following result.

Lemma 6.7: Suppose G is 3-connected, internally 4-connected, has girth 5 and every odd cycle of length greater than 5 has a chord. Let C be a 5-cycle in G. Then if $N'(C)$ contains an induced path of length 2, $G \cong P_{10}$. ∎

7. Final Remarks.

Thus we have reduced the general conjecture to the case in which for every 5-cycle C in G, $N'(C)$ consists of a matching together with an independent set. We can say a bit more, however. Let $C = x_1x_2x_3x_4x_5x_1$ and suppose M is a matching in $N'(C)$. Suppose without loss of generality that a matching edge $y_1y_2 \in G[N'(C)]$ is such that $y_1 \sim x_1$ and $y_2 \sim x_3$. Then if there were a second edge of M with attachments at x_3 and x_5, we would have a 7-cycle, contradicting Lemma 2.3. Hence by the girth 5 hypothesis and symmetry we may assume that M can be partitioned $M = M_1 \cup M_2$ where all edges in M_1 attach to C only at x_1 and x_3, while those in M_2 attach only at x_2 and x_4 or else $M_2 = \emptyset$.

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