Symmetric distribution of crossings and nestings in permutations of type B

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Abstract

This note contains two results on the distribution of crossing numbers and nesting numbers in permutations of type B. More precisely, we prove a B_n-analogue of the symmetric distribution of crossings and nestings of permutations due to Corteel [Adv. Appl. Math. 38(2)(2007), 149–163] as well as the symmetric distribution of k-crossings and k-nestings of permutations due to Burrill et al. [DMTCS proc. AN (2010), 593–600].

1 Introduction

In the last years, many results on symmetric distributions of some statistics "crossing" and "nesting" have appeared in several combinatorial structures. At the heart of these results, on the set of matchings and partitions, there are Chen et al's theorem on the symmetric distribution of k-crossing numbers and k-nestings numbers [3] and Kasraoui and Zeng's theorem on the symmetric distribution of crossing numbers and nestings numbers of two edges [6]. Then, some extensions of type B and C have been given by Rubey and Stump [9], and Krattenthaler [7] and de Mier [8] on the relation between increasing and decreasing chains in partitions and linked partitions, and fillings of Ferrers shapes. On the set of permutations, Corteel [4] has introduced the notion of crossings and nestings of permutations and proved that for any fixed number of weak exceedances, the distribution of crossing numbers and nestings numbers of permutations is symmetric. Recently, Burrill et al [2] have proved a similar result for k-crossings and k-nestings of permutations. The purpose of this paper is to extend the last two results to their analogue of type B.

2 Definitions and main results

For a positive integer n, let $[n] := \{1, 2, \ldots, n\}$. A type B permutation of rank n is an integer sequence $\sigma := (\sigma(1), \sigma(2), \ldots, \sigma(n))$ such that $\{|\sigma(1)|, ..., |\sigma(n)|\} = [n]$. In this paper, we shall identify σ with a permutation of $[-n,\ n] := \{-n, \ldots, -2, -1,\ 1, \ldots, n\}$ by $\sigma(-i) = -\sigma(i)$ for each $i \in [n]$. Let $\mathrm{neg}(\sigma)$ be the number of *negative numbers* in $\{\sigma(1), \ldots, \sigma(n)\}$, and B_n the set of type B permutations of rank n.

In the sequel, we use the natural order of integers in $\mathbb{Z}$.

As in [4], it is convenient to represent a permutation $\sigma \in B_n$ by a *permutation diagram* $G = (V, E)$, where $V = [-n,\ n]$ is the vertex set, and E is the set of edges $(i, \sigma(i))$ for $i \in [-n,\ n]$ such that the vertices $-n$, ..., -2, -1, 1, 2, ..., n are arranged from left to right on a straight line. We draw an arc from i to $\sigma(i)$ above (resp., under) the line if $i \leqslant \sigma(i)$ (resp., otherwise) such that two arcs cross at most once. A permutation diagram is given in Fig. 1.

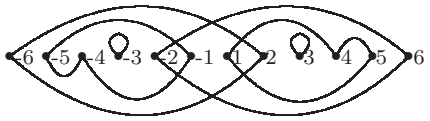

Fig. 1. The permutation diagram of $\sigma = (4, -6, 3, 5, 1, -2)$.

We call the set of arcs that are above (resp., under) the line the *upper* (resp., *under*) *permutation diagram* and denoted $Upp(\sigma)$ (resp., $Und(\sigma)$).

We start with an easy lemma that follows immediately from the definition of the permutation diagram since there is an easy bijective between upper and under diagrams.

Lemma 2.1 *Let $\sigma \in B_n$. The diagram of σ is completely determined by the Upp(σ).*

Note that there are five geometric patterns for two arcs above the line as illustrated in Fig. 2.

Fig. 2. Five patterns between two arcs above the line.

These patterns are called: (i) a proper crossing, (ii) a skew crossing, (iii) a proper nesting, (iv) a skew nesting and (v) an alignment. In another sense, one can recover these geometric patterns as in the two following definitions.

The first is the notion of crossings of type B given by Corteel et al. in [5] as follows.

Definition 2.2 *Let $\sigma \in B_n$. The number of weak exceedances of σ, denoted by $wex_B(\sigma)$, is the cardinality of the set $\{j \in [n]; \sigma(j) \geq j\}$. For two integers i and j in $[n]$, two arcs $(i,\ \sigma(i))$ and $(j,\ \sigma(j))$ form a crossing of σ if they satisfy either the relation $i < j \leqslant \sigma(i) <$*

$\sigma(j)$ (*upper crossing*), *or* $-i < j \leqslant \sigma(-i) < \sigma(j)$ (*upper crossing*) *or* $\sigma(i) < \sigma(j) < i < j$ (*lower crossing*).

Similarly, in the second, we can define the notion of nesting of type B.

Definition 2.3 *Let* $\sigma \in B_n$. *A pair of arcs* $(i, \sigma(i))$ *and* $(j, \sigma(j))$, *with* i *and* j *in* $[n]$, *is a nesting of* σ *if they satisfy either the relation* $i < j \leqslant \sigma(j) < \sigma(i)$ *(upper nesting), or* $-i < j \leqslant \sigma(j) < \sigma(-i)$ *(upper nesting) or* $\sigma(j) < \sigma(i) < i < j$ *(lower nesting). The number of crossings (resp., nestings) of* σ *is denoted by* $cro_B(\sigma)$ (*resp.,* $nes_B(\sigma)$).

Example 1. Let $\sigma = (4, -6, 3, 5, 1, -2) \in B_6$. Then the nestings in σ are $\{(-2, \sigma(-2)), (1, \sigma(1))\}$, $\{(3, \sigma(3)), (1, \sigma(1))\}$, $\{(3, \sigma(3)), (-2, \sigma(-2))\}$, $\{(4, \sigma(4)), (-2, \sigma(-2))\}$ and $\{(6, \sigma(6)), (5, \sigma(5))\}$. The crossings are $\{(-6, \sigma(-6)), (1, \sigma(1))\}$, $\{(1, \sigma(1)), (4, \sigma(4))\}$, $\{(5, \sigma(5)), (2, \sigma(2))\}$ and $\{(6, \sigma(6)), (2, \sigma(2))\}$ (see Fig. 1). Hence $nes_B(\sigma) = 5$ and $cro_B(\sigma) = 4$.

The following is our B_n-analogue of Corteel's result for permutations of type A [4, Proposition 4], which corresponds to the $a = 0$ case.

Theorem 2.4 *The number of permutations in* B_n *with* k *weak exceedances,* l *minus signs,* i *crossings and* j *nestings is equal to the number of permutations in* B_n *with* k *weak exceedances,* l *minus signs,* i *nestings and* j *crossings. In other words, we have*

$$\sum_{\sigma \in B_n} p^{nes_B(\sigma)} q^{cro_B(\sigma)} y^{wex_B(\sigma)} a^{neg(\sigma)} = \sum_{\sigma \in B_n} p^{cro_B(\sigma)} q^{nes_B(\sigma)} y^{wex_B(\sigma)} a^{neg(\sigma)}. \tag{1}$$

Now, we extend the definition of k-crossings and k-nestings for permutations of type A in [2] to permutations of type B.

Definition 2.5 *Let* $\sigma \in B_n$. *A set* $\{a_1, a_2, \ldots, a_k\}$ *of* k *integers in* $[n]$ *is a* k*-crossing of* σ *if they satisfy either the relation* $a_1 < a_2 < \ldots < a_k \leq \sigma(a_1) < \sigma(a_2) < \ldots < \sigma(a_k)$ (*upper* k*-crossing*), *or* $-a_1 < a_2 < \ldots < a_k \leq -\sigma(a_1) < \sigma(a_2) < \ldots < \sigma(a_k)$ (*upper* k*-crossing*) *or* $\sigma(a_k) < \sigma(a_{k-1}) < \ldots < \sigma(a_1) < a_k < a_{k-1} < \ldots < a_1$ (*lower* k*-crossing*).

Definition 2.6 *Let* $\sigma \in B_n$. *A set* $\{a_1, a_2, \ldots, a_k\}$ *of* k *integers in* $[n]$ *is a* k*-nesting of* σ *if they satisfy either the relation* $a_1 < a_2 < ... < a_k \leq \sigma(a_k) < \sigma(a_{k-1}) < \ldots < \sigma(a_1)$ (*upper* k*-nesting*), *or* $-a_1 < a_2 < \ldots < a_k \leq -\sigma(a_k) < \sigma(a_{k-1}) < \ldots < \sigma(a_1)$ (*upper* k*-nesting*) *or* $\sigma(a_k) < \sigma(a_{k-1}) < \ldots < \sigma(a_1) < a_1 < a_2 < \ldots < a_k$ (*lower* k*-nesting*).

As in [2], the *k-crossing number* (resp., *k-nesting number*) of a permutation σ of type B, denoted by $cro_B^*(\sigma)$ (resp., $nes_B^*(\sigma)$) is the size of the largest k such that σ contains a k-crossing (resp., k-nesting).

Example 2. Let $\sigma = (4, 5, 6, 2, -3, -1) \in B_6$. Then we have $cro_B^*(\sigma) = 4$ and $nes_B^*(\sigma) = 2$ that are illustrated respectively, in Fig. 3, by $\{5, 1, 2, 3\}$ and $\{4, 5\}$ or $\{4, 6\}$ since $-5 < 1 < 2 < 3 \leqslant -\sigma(5) < \sigma(1) < \sigma(2) < \sigma(3)$, $\sigma(5) < \sigma(4) < 4 < 5$ and $\sigma(6) < \sigma(4) < 4 < 6$.

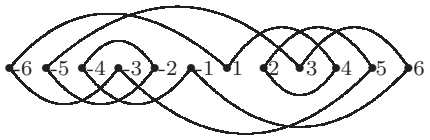

Fig. 3. The permutation diagram of $\sigma = (4, 5, 6, 2, -3, -1)$.

We also recall the following definition from [2]. Let $\sigma \in B_n$. The *degree sequence* of the upper permutation diagram $Upp(\sigma)$ is the sequence $(indegree_\sigma(i), outdegree_\sigma(i))_{i\in[-n,n]}$, where $indegree_\sigma(i)$ (resp., $outdegree_\sigma(i)$) is the left (resp., right) degree of the vertex i, *i.e.*, the number of arcs joining i to a vertex j with $j < i$ (resp., $j > i$). If an upper permutation diagram $Upp(\sigma)$ has d as its degree sequence (some other sources call this left-right degree sequence), we say that $Upp(\sigma)$ is a diagram on d.
But there is a straightforward difference that we do not put a loop (an arc if $\sigma(i) = i$) on the isolated vertex i with negative index in $Upp(\sigma)$, *i.e.*, we put (0,0) as a degree. By Lemma 2.1, we limit ourselves to study the upper permutation diagram of type B. The vertices with degree (0,1) (resp., (1,0), (1,1)) are called *openers* (resp., *closers*, *closer-opener* or *transient*). For instance, if we let $\sigma = (4, 5, 6, 2, -3, -1)$, then the degree sequence of the upper permutation diagram of σ is

$$d := d(\sigma) = (0,1)(0,1)(0,1)(0,0)(1,0)(0,0)(1,1)(0,1)(1,1)(1,0)(1,0)(1,0).$$

Let B_n^d be the set of the permutations in B_n that has the degree sequence d.

The following is our B_n-analogue of Burrill et al's result for permutations of type A [2, Theorem 1], which corresponds to the $z = 0$ case.

Theorem 2.7 *Let $NC_{B_n^d}(i, j, m)$ be the number of permutations in B_n with i-crossings, j-nestings, m minus signs and degree sequence specified by d. Then*

$$NC_{B_n^d}(i, j, m) = NC_{B_n^d}(j, i, m). \tag{2}$$

In other words, we have

$$\sum_{\sigma\in B_n^d} x^{nes_B^*(\sigma)} y^{cro_B^*(\sigma)} z^{neg(\sigma)} = \sum_{\sigma\in B_n^d} p^{cro_B^*(\sigma)} q^{nes_B^*(\sigma)} z^{neg(\sigma)}. \tag{3}$$

Now, we sketch the opener, closer and transient vertices of the upper (resp., under) permutation diagram with degree (0,1), (1,0) and (1,1) (resp., (1,0), (0,1), (1,1)) respectively. Then a vertex is said to be:

(i) an opener if it is illustrated by  (resp., )
(ii) a closer if it is illustrated by  (resp., )
(iii) a transient if it is illustrated by  (resp.,).

We shall prove Theorem 2.4 in Section 3 by constructing an explicit involution on B_n that interchanges the number of crossings and number of nestings. In fact, it is an extension of the involution defined in [6]. To prove the Theorem 2.7 in Section 4, we shall adapt the map defined by de Mier in [8] to B_n.

3 Proof of Theorem 2.4

First, for each $\sigma \in B_n$, the number of crossings of σ is equal, in $Upp(\sigma)$, to the number of proper crossings plus the number of transient vertices i with i in $[n]$. Similar, the number of nestings of σ is equal, in $Upp(\sigma)$, to the number of proper nestings plus the number of arcs (i, k) with i and k in $[-n, n]$ such that there exist two fixed vertices j and $-j$ satisfy $i < |j| < k$. For instance, in the left diagram of Fig. 4, we can count the 4 crossings and the 5 nestings of σ given in Fig. 1.

Now, we introduce some notations. For a positive integer n, let Λ_n be the set of the subsets of $[n]$ and $\overline{B}_n$ the set of $Upp(\sigma)$ for each $\sigma \in B_n$. Let also F and T be two maps defined by: for each $\sigma \in B_n$, $F(\sigma) := \{j \in [n]; \sigma(j) = j\}$ and $T(\sigma) := \{j \in [n];\ j \textit{ is an upper transient vertex of } \sigma\}$. We notice that, in [9], Rubey and Stump have studied the symmetry distribution of the number of crossings and number of nestings in a kind of set partitions of type B. Then, we study a similar result in B_n where we count the transient vertex (resp., an arc covers a fixed vertex) as a crossing (resp., nesting) and our arrangement of the set $[-n, n]$ is different to theirs.

Proof of Theorem 2.4. There are two steps.

First step. Let $\sigma \in B_n$. We define a map ψ by: let $(Upp(\sigma), F(\sigma), T(\sigma)) \in \overline{B}_n \times \Lambda_n^2$. Then ψ transforms each element i of $F(\sigma)$ to an arc (i, i') (see (a)) and each element j of $T(\sigma)$ to a proper crossing (see (b)), which in the two cases, we have $i < i'$ (resp., $j < j'$) and no vertex between i and i' (resp., j and j'). We adapt from [2] the following graphs

(a) (b)

For instance, the permutation diagram of σ in Example 1 has a fixed vertex indexed by 3 and a transient vertex indexed by 4, see the left diagram of Fig. 4. Then, its inverse ψ^{-1} reduces each two vertices i and i' introduced by ψ into a one vertex i.

Fig. 4. The left diagram is the upper permutation diagram of σ = (4, -6, 3, 5, 1, -2) and the right is its image by ψ.

Second step. We give an outline of the involution φ of [6]. Let π be a partition of type A and G be its partition diagram defined as the upper permutation diagram. For each two vertices k and j of G, we adapt that j is a vacant vertex for the kth position if $j < k$ and its corresponding closer vertex l satisfies $l > k$. Then for each arc (i, j) of G, let $\delta(i, j)$ (resp., $\gamma(i, j)$) be the number of vacant vertex k such that $k < i$ (resp., $k > i$) for the jth position. The algorithm describing the involution φ is to construct a partition diagram G$'$ from G, vertex by vertex and from left to right in the following paragraph.

For each vertex k of G from 1 to the rank of π, if k is a fixed (resp., opener) vertex then we conserve its form; fixed (resp., opener) vertex, at the position k in G$'$ and if k is a closer (resp., transient) vertex, we also conserve its form; closer (resp., transient) vertex,

at the same position in G′, but we exchange the arc $(s,\ k)$ where s is the corresponding opener of k in G into an arc $(t,\ k)$ in G′ with t is the $\gamma(s,k)$th vacant vertex, from left to right, for the position k. So, φ is a proper crossings and nestings interchanging map.

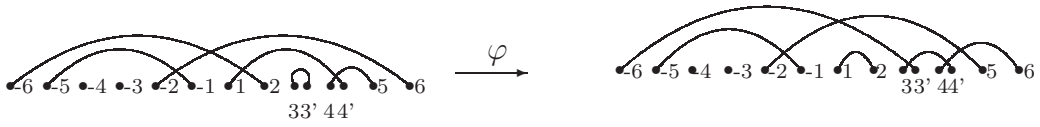

Fig. 5. The left diagram is the diagram of $\psi(\sigma)$ and the right is its image by φ.

It remains to prove that the number of the minus signs is unchanged. Since, for each $\sigma \in B_n$, the number of minus signs is equal to the number of arcs, in the upper permutation diagram $Upp(\sigma)$, joining a vertex with negative index and a vertex with positive index. But the number is unchanged since the map φ preserve the number of openers (resp., closers) vertices $Upp(\sigma)$.

Finally, for instance, by the map $\psi^{-1} \circ \varphi \circ \psi$ and Lemma 2. 1, we illustrate, in the following figure, the corresponding permutation $\sigma' = (2,\ \text{-}5,\ 4,\ 6,\ 1,\ \text{-}3)$ of the permutation $\sigma = (4,\ -6,\ 3,\ 5,\ 1,\ -2)$ in Example 1.

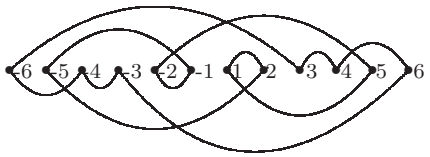

Fig. 6. The permutation diagram of σ' = (2, -5, 4, 6, 1, -3). □

4 Proof of Theorem 2.7

Our proof is based on an extension of de Mier's bijection in [8, Section 4] to B_n. First, we adapt the basic tool in [8] with a slight modification, that is the construction of a bijection between link partitions of type A and fillings of Young diagrams into a bijection, denoted by ξ, between upper permutation diagrams and fillings of Young diagrams on $\overline{B}_n$. For each σ in B_n, let i_1, ..., i_c be the closers vertices of $Upp(\sigma)$ and j_1, ..., j_o the openers ones. Let $p(i)$, for each closer vertex i, be the number of vertices j with $j < i$ that are openers such that for each transient we associate a closer before an opener. We consider a Young diagram T of shape $(p(i_c),\ ...,\ p(i_1))$, and if there is an arc going from the opener j_s to the closer i_r, we fill the cell in column s and row $c - r + 1$ with 1. For instance, in the following figure, we illustrate the upper permutation diagram of σ of Example 2 and its corresponding filling of Young diagram.

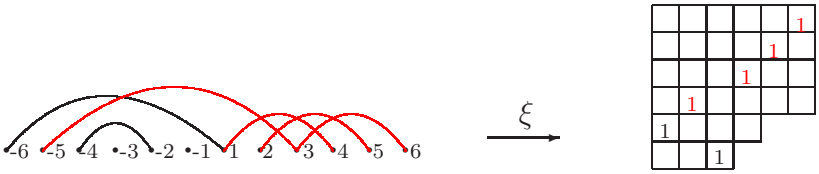

Fig. 7. The upper permutation diagram of $\sigma = (4, 5, 6, 2, -3, -1)$ and its corresponding filling of Young diagram.

Thus, we see that the $k-$nesting (one can also say $k+1$-nonnesting) (resp., $k-$crossing) in the upper permutation diagram corresponds to an identity matrix I_k (resp., anti-identity matrix J_k called also the anti-diagonal of I_k) in a largest rectangle in the corresponding Young diagram.

We apply the following result due to de Mier [8, Theorem 3.5].

Lemma 4.1 [*de Mier*] *For all diagrams T with prescribed row and column sums, the number of fillings T that avoid I_k equals the number of fillings of T that avoid J_k.*

So, we give an outline of the involution Ψ of [8], in a quite straightforward way, that interchanges I_k and J_k. First, if there are many anti-identities or identities matrices of rank k, we choose the one more to the right and the topmost. Second, we can divide the map Ψ into two maps. The first is φ. For each $\sigma \in B_n$, we see the largest k such that the filling of Young diagram corresponding to $Upp(\sigma)$ contains a largest rectangle which contains an anti-identity matrix J_k. Thus, the 1's of J_k, from left and bottom to right and top, correspond to (l_1, c_1), (l_2, c_2), $\ldots$, (l_k, c_k) cells in the diagram, *i.e.*, (l_i, c_i) is the intersection cell of the l_ith line and c_ith column, for each $1 \leqslant i \leqslant k$. Thus φ changes the places of the 1's of J_k in the diagram to new places define by: (l_2, c_1), (l_3, c_2), $\ldots$, (l_k, c_{k-1}) and (l_1, c_k), and we obtain in the first time the following matrix

$$\varphi(J_k) = \begin{pmatrix} J_{k-1} & 0 \\ 0 & 1 \end{pmatrix}$$

where J_{k-1} is the anti-identity matrix of rank $k-1$.

So on, we apply φ to J_{k-1}, J_{k-2}, $\ldots$, until we get I_k.

The second is ϕ (the inverse of φ). Let k be the largest integer such that the Young diagram contains a matrix I_k. The 1's of I_k have (a_1, b_1), (a_2, b_2), $\ldots$, (a_k, b_k) as the cells in the diagram from top to bottom and left to right with (a_i, b_i) is the intersection cell between the a_ith line and the b_i column. So, the image of I_k by ϕ is (a_2, b_1), (a_1, b_2), (a_3, b_3), $\ldots$, (a_k, b_k), *i.e.*,

$$\phi(I_k) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{k-2} \end{pmatrix}$$

where I_{k-2} is the identity matrix of rank $k-2$. The image of $\phi(I_k)$ by ϕ is (a_3, b_1), (a_2, b_2), (a_1, b_3), (a_4, b_4), $\ldots$, (a_k, b_k). So on, we apply ϕ to $\phi^i(I_k)$, for i from 0 to $k-1$, until we get J_k.

Then, by the two processes, φ changes J_k into I_k and ϕ changes I_k into J_k.

It remains to prove that the number of minus signs is unchanged by the above transformations ξ and Ψ. But we know that the number of minus signs is preserved as it can be computed from the degree sequence which is fixed by the maps .

For instance, the following diagram is the permutation diagram of $\vartheta(Upp(\sigma))$ of the permutation σ in Example 2. It is easy to see that $cro^*(\vartheta(Upp(\sigma))) = 2$ and $nes^*(\vartheta(Upp(\sigma))) = 4$.

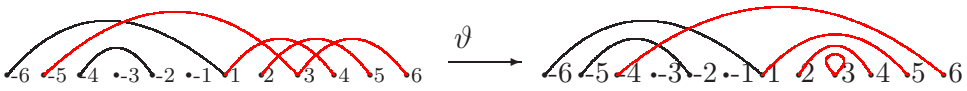

Fig. 8. The upper permutation diagram of σ = (4, 5, 6, 2, -3, -1) and and its corresponding $\vartheta(Upp(\sigma))$. □

5 Concluding remarks

5.1 Extending to type D permutations

Aside from the enumerative, is the symmetric distribution of crossings and nestings (resp., k-crossings and k-nestings) preserved in set permutations of classical type D? The type D permutations, denoted by D_n and called the *even-signed permutation group*, is the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\sigma \in B_n \mid neg(\sigma) \equiv 0 (mod 2)\}.$$

It is well known (see, e.g., [1, $\mathcal{S}$8.2]) that D_n is a Coxeter group with respect to the generating set $S := \{s_0, s_1, \ldots, s_{n-1}\}$ where

$$s_0 := (-2, -1, 3, \ldots, n) \text{ and } s_i := (1, 2, \ldots, i-1, i, i+2, \ldots, n)$$

for $i = 1, \ldots, n-1$.

As the number of minus signs, in each $\sigma \in B_n$, is unchanged by the two involutions in the two proofs of our theorems 2.4 and 2.7. Then, we have the property of the symmetric distribution of crossings and nestings (resp., k-crossings and k-nestings) over D_n, *i.e.*,

Theorem 5.1 *The number of permutations in D_n with k weak exceedances, l minus signs, i crossings and j nestings (resp., i-crossings, j-nestings, m minus signs and degree sequence specified by d) is equal to the number of permutations in D_n with k weak exceedances, l minus signs, i nestings and j crossings (resp., j-crossings, i-nestings, m minus signs and degree sequence specified by d).*

In other words,

$$\sum_{\sigma\in D_n} p^{nes_D(\sigma)}q^{cro_D(\sigma)}y^{wex_D(\sigma)}a^{neg(\sigma)} = \sum_{\sigma\in D_n} p^{cro_D(\sigma)}q^{nes_D(\sigma)}y^{wex_D(\sigma)}a^{neg(\sigma)}. \tag{4}$$

and

$$\sum_{\sigma\in D_n^d} p^{nes^*_D(\sigma)}q^{cro^*_D(\sigma)}a^{neg(\sigma)} = \sum_{\sigma\in D_n^d} p^{cro^*_D(\sigma)}q^{nes^*_D(\sigma)}a^{neg(\sigma)}. \tag{5}$$

where, for each $\sigma \in D$, $cros_D(\sigma)$ (resp., $nes_D(\sigma)$, $wex_D(\sigma)$) is the number of crossings (resp., nestings, weak exceedances) in σ and $cro^*_D(\sigma)$ (resp., $nes^*_D(\sigma)$) is the size of the largest k such that σ contains a k-crossing (resp., k-nesting).

5.2 Open question

It would be interesting to find a direct permutation description of our involution, *i.e.*, a description avoiding the passage through tableaux or fillings of Ferrers diagrams.

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References

[1] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, G.T.M. 231, Springer-Verlag, New York, (2005).

[2] S. Burrill, M. Mishna and J. Post, On k-crossings and k-nestings of permutations, Proc. 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), DMTCS proc. AN, (2010), 593–600.

[3] W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley and C. H. Yan, Crossings and nestings of matchings and partitions, Trans. Amer. Math. Soc. 359 (2007), 1555-1575.

[4] S. Corteel, Crossings and alignnments of permutations, Adv. in Appl. Math., 38 (2007), 149–163.

[5] S. Corteel, M. Josuat-Vergès and L. Williams, The matrix ansatz, orthogonal polynomials and permutations, Adv. in Appl. Math. 46 (2011), 209–225.

[6] A. Kasraoui and J. Zeng, Distribution of crosings, nestings and alignments of two edges in matchings and partitions, Electron. J. Combin. 13 (2006), R33.

[7] C. Krattenthaler, Growth diagrams, and increasing and decrasing chains in fillings of Ferrers sharps, Adv. in Appl. Math. 37 (2006), 404–431.

[8] A. De Mier, k-noncrossing and k-nonnesting graphs and fillings of Ferrers diagrams, Combinatorica 27 (2007), 699–720.

[9] M. Rubey and C. Stump, Crossings and nestings in set partitions of classical types, Electron. J. Combin. 17 (2010), R120.