On the Orchard crossing number of the complete bipartite graphs $K_{n,n}$

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Abstract

We compute the Orchard crossing number, which is defined in a similar way to the rectilinear crossing number, for the complete bipartite graphs $K_{n,n}$.

1 Introduction

Let $G = (V, E)$ be a graph. A *rectilinear drawing* $R(G)$ of G is a mapping of its vertices into distinct points in the plane in general position (i.e., no three points are collinear), and a mapping of the edges into straight line segments between the corresponding points. An intersection of an edge of $R(G)$ with a straight line through a pair of the points of $R(G)$ is interpreted as an *Orchard crossing* (see the Orchard relation introduced in [3, 4]).

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If $c(s,t)$ counts these Orchard crossings for an edge $(s,t) \in E$, then the number $c(R(G))$ of Orchard crossings of $R(G)$ is:

$$c(R(G)) = \sum_{(s,t)\in E} c(s,t);$$

note that the sum is taken only over the edges of the graph, whence $c(s,t)$ counts *all* the lines generated by pairs of points in $R(G)$.

The *Orchard crossing number* $\mathrm{OCN}(G)$ is the minimum over all rectilinear drawings $R(G)$:

$$\mathrm{OCN}(G) = \min_{R(G)}\{c(R(G))\}.$$

Note that the maximum Orchard crossing number $\mathrm{MOCN}(G)$ is also interesting due to Proposition 2.7 in [5] which states that $\mathrm{MOCN}(K_n)$ and the *rectilinear crossing number* $\overline{\mathrm{cr}}(K_n)$ (see [1, 6, 7]) are attained by the same $R(K_n)$. Therefore, the determination of $\mathrm{MOCN}(G)$ might be easier than the determination of $\overline{\mathrm{cr}}(G)$. Furthermore, the concept of Orchard crossing number can be considered in higher dimensions too (see [3]).

In this paper, we determine the Orchard crossing number for the complete bipartite graphs $K_{n,n}$.

Theorem 1.1

$$\mathrm{OCN}(K_{n,n}) = 4n\binom{n}{3}$$

This value is attained where all the $2n$ points are in a convex position and alternate in color (see Figure 1 for an example for $n = 4$).

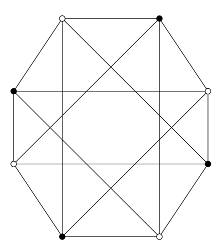

Figure 1: The optimal rectilinear drawing for $\mathrm{OCN}(K_{4,4})$

The ideas of the proof are quite similar to those of [2], where the maximal value of the maximum rectilinear crossing number has been computed for some families of graphs, but still are not straightforward from them. This again shows the tight connection between the Orchard crossing number and the rectilinear crossing number.

The proof is based on two parts: in the first part, we show that the optimal drawing of $K_{n,n}$, presented in Figure 1 for $n = 4$, has indeed $4n\binom{n}{3}$ Orchard crossings, and hence $\mathrm{OCN}(K_{n,n}) \leq 4n\binom{n}{3}$ (see Section 2). In the second part, we show that any drawing of $K_{n,n}$ has at least $4n\binom{n}{3}$ Orchard crossings, so $\mathrm{OCN}(K_{n,n}) \geq 4n\binom{n}{3}$ (see Section 3).

2 An upper bound for $\mathrm{OCN}(K_{n,n})$

In this section, we show the easy part of Theorem 1.1 by proving that the mentioned value $4n\binom{n}{3}$ is indeed attained by the rectilinear drawing $R_0(K_{n,n})$, where all the $2n$ points are in a convex position and alternate in color (see Figure 1 for an example for $n = 4$).

Lemma 2.1 $\mathrm{OCN}(K_{n,n}) \le 4n\binom{n}{3}$.

Proof. Each edge corresponds to a diagonal of length $2i+1$ (which means that there are $2i$ vertex points between its endpoints) and contributes $2i(2n-2i-2)$ Orchard crossings to $c(R_0(K_{n,n}))$, for all $1 \le i \le \frac{n-1}{2}$. In general, there are $2n$ diagonals of length $2i+1$, except for the case of odd n and $i = \frac{n-1}{2}$, where there are only n such diagonals. Thus, it follows that for even n:

$$c(R_0(K_{n,n})) = 2n \sum_{i=1}^{\frac{n-2}{2}} 4i(n-i-1),$$

and for odd n:

$$c(R_0(K_{n,n})) = 2n(n-1)(n - \frac{n-1}{2} - 1) + 2n \sum_{i=1}^{\frac{n-3}{2}} 4i(n-i-1).$$

Both sums equal to $4n\binom{n}{3}$, so we have $c(R_0(K_{n,n})) = 4n\binom{n}{3}$ as needed. □

3 A lower bound for $\mathrm{OCN}(K_{n,n})$

In this section, we show the difficult part of Theorem 1.1 by proving that for any rectilinear drawing of $K_{n,n}$ there are at least $4n\binom{n}{3}$ Orchard crossings. This will show that $4n\binom{n}{3}$ is indeed a lower bound for $\mathrm{OCN}(K_{n,n})$.

The idea of the proof is counting separately the Orchard crossings induced by pairs of points with different colors and by pairs of points with the same color (see Sections 3.1 and 3.2 respectively).

Together with the result of the previous section, this will prove Theorem 1.1 (see Section 3.3).

We start with one notation. A **bw**-*pair* is a pair of points consisting of a black point and a white point.

3.1 Orchard crossings induced by lines generated by pairs of points with different colors

Let D be a configuration of n white points and n black points. For each $k = 1, 2, \dots, n^2$, let ℓ_k be a line determined by a white point and a black point. For each ℓ_k, let a_k be the number of **bw**-pairs with one point in one halfplane determined by ℓ_k, and the other point in the other halfplane. Let $A = \sum_{k=1}^{n^2} a_k$.

Proposition 3.1 $A \geq 2n\binom{n}{3}$.

Proof. For each ℓ_k, a black (resp. white) endvertex will be of *type* i if the edge incident to ℓ_k divides the graph into two halfplanes, one contains i black (resp. white) vertices, and the other contains $n-i-1$ black (resp. white) vertices. By symmetry, we only have to consider $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor = N$. Let y_i be the number of endvertices of type i. Thus, we have: $y_0 + y_1 + \cdots + y_N = 2n^2$, since we have n^2 such edges and each one is counted twice (for the white and black vertices).

An edge which connects black and white vertices is of *type* i, j if one halfplane determined by that edge has i vertices of one color and j vertices of the other color. Let $x_{i,j}$ be the number of edges of type i, j. By symmetry, we can assume that $0 \leq i \leq j \leq N$ (we will justify this assumption later on). Note that $x_{i,j} = x_{j,i}$.

Thus, y_i is related to $x_{i,j}$ by the following equation:

$$y_i = 2x_{i,i} + \sum_{j=0}^{i-1} x_{j,i} + \sum_{j=i+1}^{N} x_{i,j}, \tag{1}$$

since for being counted in y_i, an edge should have i vertices of one color in one of the halfplanes it determines. The only case which is counted twice is when $i = j$, where it is counted for both colors.

For an edge of type i, j, there are $i(n-j-1) + j(n-i-1)$ **bw**-pairs in opposite halfplanes of that edge. Summing over all edges of a drawing, we obtain:

$$M = \sum_{i=0}^{N} \sum_{j=i}^{N} \left[i(n-j-1) + j(n-i-1) \right] x_{i,j}$$

bw-pairs in opposite halfplanes. We look for a drawing which minimizes M.

We now justify our assumption that $0 \leq i \leq j \leq N$, where i and j are the number of vertices of the two colors in the same halfplane of an edge. Assume that for a given type i, j edge, the i vertices of one color and the j vertices of the other color are in different halfplanes. This yields $ij + (n-j-1)(n-i-1)$ **bw**-pairs. However, $i(n-j-1) + j(n-i-1) \leq ij + (n-j-1)(n-i-1)$ for $0 \leq i \leq j \leq N$. Hence our assumption reduces the number of **bw**-pairs.

In order to minimize M, we start by multiplying Equation (1) by $i(n-i-1)$, and subtracting it from M for all values of i, yielding:

$$M - \sum_{i=1}^{N} i(n-i-1) y_i$$

$$
\begin{aligned}
&= \sum_{i=0}^{N} \sum_{j=i}^{N} \left[i(n-j-1) + j(n-i-1) \right] x_{i,j} \\
&\quad - \sum_{i=1}^{N} i(n-i-1) \left(2x_{i,i} + \sum_{j=0}^{i-1} x_{j,i} + \sum_{j=i+1}^{N} x_{i,j} \right) \\
&= \sum_{i=0}^{N} \sum_{j=i+1}^{N} \left[i(n-j-1) + j(n-i-1) \right] x_{i,j} - \sum_{i=1}^{N} i(n-i-1) \left(\sum_{j=0}^{i-1} x_{j,i} + \sum_{j=i+1}^{N} x_{i,j} \right) \\
&= \sum_{i=0}^{N} \sum_{j=i+1}^{N} \left[i(i-j) + j(n-i-1) \right] x_{i,j} - \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} j(n-j-1) x_{i,j} \\
&= \sum_{i=0}^{N} \sum_{j=i+1}^{N} \left[i(i-j) + j(j-i) \right] x_{i,j} = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} (j-i)^2 x_{i,j}.
\end{aligned}
$$

Hence, we have:

$$
M = \sum_{i=1}^{N} i(n-i-1) y_i + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} (j-i)^2 x_{i,j}. \tag{2}
$$

For the next step, we introduce a new notation $p_{s,t}$. We first motivate it. Note that every white (resp. black) vertex v_i serves as an endvertex for n edges of the graph. For each of these n edges, let $c_{i,j}$ $(1 \leq j \leq n)$ be the number of black (resp. white) points in the halfplane with a smaller number of black (resp. white) points determined by this edge. Thus, for each v_i, we have a sequence of n numbers $(c_{i,1}, \ldots, c_{i,n})$ representing the types which v_i is for the n edges connected to v_i.

For example, for a white point v on the convex hull of an alternating $2n$-gon (see Figure 2 for $n = 4$, where only some of the edges are drawn), the two edges going to the adjacent black points (the edges e_1, e_4) are of type 0. The next two edges (edges e_2, e_3) will be of type 1, the next two edges (do not exist in this configuration) will be of type 2, etc. Note that in this case we will have the same sequence of types for any point on the convex hull.

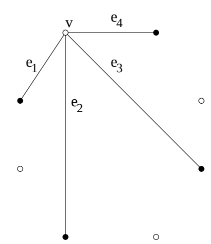

Figure 2: An example for computing $p_{s,t}$

Let $p_{s,t}$ be the number of white (resp. black) vertices having $s = \min\{c_{i,1}, \ldots, c_{i,n}\}$, and the index t is the number of distinct sorted sequences generated by all the vertices.

For example, for the $2n$-gon configuration, we have $p_{0,1} = 2n$, since all $2n$ vertices have 0 as the lowest term in the sequence $0, 0, 1, 1, 2, 2, \ldots$. Note that there is a unique sequence for all the vertices. Therefore, $p_{0,t} = 0$ for $t \geq 2$, since there is no vertex whose lowest type is 0 which has a different sequence. Moreover, $p_{s,t} = 0$ for $s \geq 1$, since there is no sequence whose lowest term is greater than 0.

Since $p_{s,t}$ counts number of vertices, and in total there are $2n$ vertices, we have:

$$\sum_{s=0}^{N} \sum_{t \geq 1} p_{s,t} = 2n.$$

Denote by $z_{s,t,i}$ the number of appearances of i in the tth sequence whose lowest term is s. For example, assume that the first sequence is $0, 1, 2, 1, 0$. Then: $z_{0,1,0} = 2$ because 0 appears twice in the sequence. Similarly, $z_{0,1,1} = 2$ since 1 appears twice and $z_{0,1,2} = 1$ since 2 appears once.

It follows that:

$$y_i = \sum_{t \geq 1} \sum_{s=0}^{i} z_{s,t,i} p_{s,t}. \tag{3}$$

Additionally, since every vertex has n edges, for fixed s and t we have that

$$\sum_{i=s}^{N} z_{s,t,i} = n. \tag{4}$$

Since $\sum_{t \geq 1} \sum_{s=0}^{N} p_{s,t} = 2n$, we obtain by Equation (3):

$$y_i = 4n + \sum_{t \geq 1} \left[\sum_{s=0}^{i} (z_{s,t,i} - 2) p_{s,t} - 2 \sum_{s=i+1}^{N} p_{s,t} \right]. \tag{5}$$

Respectively, for odd n, we have:

$$y_N = 2n + \sum_{t \geq 1} \sum_{s=0}^{N} (z_{s,t,N} - 1) p_{s,t}. \tag{6}$$

We continue for even n. Using Equation (5), we can rewrite the first part of the expression for M (Equation (2)) as follows:

$$\begin{aligned} \sum_{i=1}^{N} i(n-i-1) y_i &= 4n \sum_{i=1}^{N} i(n-i-1) + \\ &+ \sum_{t \geq 1} \sum_{i=1}^{N} i(n-i-1) \left[\sum_{s=0}^{i} (z_{s,t,i} - 2) p_{s,t} - 2 \sum_{s=i+1}^{N} p_{s,t} \right]. \end{aligned}$$

Following a change in the indices of the sums, this can be rewritten as:

$$4n\sum_{i=1}^{N} i(n-i-1) + \sum_{t\geq 1}\sum_{s=0}^{N} p_{s,t}\left[\sum_{i=s}^{N} i(n-i-1)(z_{s,t,i}-2) - 2\sum_{i=1}^{s-1} i(n-i-1)\right].$$

This can again be rewritten as:

$$\begin{aligned}
& 4n\sum_{i=1}^{N} i(n-i-1) + \sum_{t\geq 1}\sum_{s=0}^{N} p_{s,t}\Bigg[s(n-s-1)(z_{s,t,s}-2) + \\
& \quad + \sum_{i=s+1}^{N} i(n-i-1)(z_{s,t,i}-2) - 2\sum_{i=1}^{s-1} i(n-i-1)\Bigg] \\
= \; & 4n\sum_{i=1}^{N} i(n-i-1) + \sum_{t\geq 1}\sum_{s=0}^{N} p_{s,t}\Bigg[s(n-s-1)\sum_{i=s}^{N}(z_{s,t,i}-2) + \\
& + \sum_{i=s+1}^{N} [i(n-i-1) - s(n-s-1)](z_{s,t,i}-2) - 2\sum_{i=1}^{s-1} i(n-i-1)\Bigg].
\end{aligned}$$

Using Equation (4), it follows that this is also equal to:

$$\begin{aligned}
& 4n\sum_{i=1}^{N} i(n-i-1) \\
& \quad + \sum_{t\geq 1}\sum_{s=0}^{N} p_{s,t}\left[C(s,n) + \sum_{i=s+1}^{N} [i(n-i-1) - s(n-s-1)](z_{s,t,i}-2)\right],
\end{aligned}$$

where

$$\begin{aligned}
C(s,n) &= s(n-s-1)(n - \sum_{i=s}^{N} 2) - 2\sum_{i=1}^{s-1} i(n-i-1) \\
&= s(n-s-1)(n-2(N-s+1)) - 2\sum_{i=1}^{s-1} i(n-i-1).
\end{aligned}$$

Now, we show that $C(s,n)$ is non-negative for all $0 \leq s \leq N$ and n. For the proof, we will use the following observation:

Observation 3.2 *Let a, b be such that: $0 < a < b < \frac{n}{2}$. Then:*

$$a(n-a) < b(n-b).$$

Lemma 3.3 *For all $0 \leq s \leq N$ and n, $C(s,n) \geq 0$.*

Proof. By the definition of $N = \lfloor \frac{n-1}{2} \rfloor$, we have:

$$s(n-s-1)(n-2(N-s+1)) \geq s(n-s-1)(2s-1).$$

Moreover, by Observation 3.2,

$$2\sum_{i=1}^{s-1} i(n-i-1) \leq 2\sum_{i=1}^{s-1}(s-1)(n-s) = 2(s-1)^2(n-s) < s(n-s-1)(2s-2).$$

Therefore,

$$C(s,n) > s(n-s-1)(2s-1) - s(n-s-1)(2s-2) = s(n-s-1) \geq 0.$$

$\square$

We now show that $z_{s,t,i} - 2 \geq 0$. Since the term $z_{s,t,i} - 1$ must be carried throughout this summation, the expression for odd n is also minimized for $p_{s,t} = 0$ for all $s \geq 1$, provided $z_{s,t,N} - 1 \geq 0$.

Lemma 3.4 *$z_{s,t,i} \geq 2$ for all s, t, i such that $i > s$ and $z_{s,t,N} \geq 1$ for odd n.*

Proof. Without loss of generality, consider a given black vertex. We start by proving that there is at least one endvertex of type $\frac{n-1}{2}$ for odd n, and at least two endvertices of type $\frac{n-2}{2}$ for even n. This statement can be proved by induction on n. This statement is obvious for $n = 2$ and $n = 3$, so we start with the inductive step. Also, note that in traversing the n edges incident to the given vertex in a clockwise or counterclockwise manner, in moving from edge to edge, the number of white vertices in the clockwise following halfplane may be changed by at most 1. This fact will be used numerous times throughout the proof. We illustrate it by an example.

Example 3.5 *Given a rectilinear drawing of $K_{4,4}$ in Figure 3 (where only some of edges are drawn). If we are traversing clockwise the 4 edges starting from the lowest black point, we have that the type of the edge e_1 is 0, since there is no white point in the left halfplane defined by this edge. Next, the type of e_2 is 1, since there is only one white point in the left halfplane defined by this edge. The type of e_3 is again 1, since there is only one white point in the* right *halfplane defined by this edge. Finally, the type of e_4 is 0, since there is no white point in the* right *halfplane defined by this edge.*

Hence, we have that while moving from edge to edge, the number of white vertices in the clockwise following halfplane may be changed by at most 1. Consequently, the types of the corresponding edges may be changed by at most 1.

Case I: Passing from odd n to $n + 1$.
Consider the edge for which this endvertex is of type $\frac{n-1}{2}$ in the configuration of n white vertices and n black vertices. When the $(n+1)$st pair of vertices is added, this original endvertex will be the first endvertex of type $\frac{(n+1)-2}{2}$. If the $(n+1)$st white vertex is

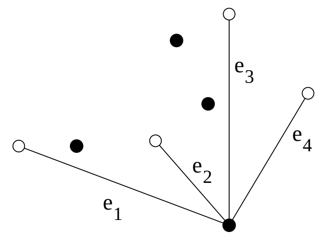

Figure 3: An example

added in this edge's clockwise following halfplane, then an immediately following edge or edge extension has an endvertex of type $\frac{(n+1)-2}{2}$. Thus, either this edge or the edge corresponding to this extension will have the second endvertex of type $\frac{n-1}{2}$.

Case II: Passing from even n to $n+1$.
Consider an edge with an endvertex of type $\frac{n-2}{2}$ which has $\frac{n}{2}$ white vertices in one of its halfplanes and $\frac{n-2}{2}$ in the other. If the $(n+1)$st white vertex is added in the halfplane with $\frac{n-2}{2}$ vertices, then the considered edge is now of type $\frac{(n+1)-1}{2}$. If the $(n+1)$st vertex is added in the halfplane with $\frac{n}{2}$ white vertices, then there are $\frac{(n+1)+1}{2}$ white vertices in this halfplane and $\frac{(n+1)-3}{2}$ white vertices in the clockwise following halfplane of this edge's extension. Since the number of white vertices in the clockwise following halfplane can be changed by at most 1 when moving from edge to edge (edge ray and edge extension), we find that traversing the graph from the edge with $\frac{(n+1)+1}{2}$ white vertices in the clockwise following halfplane to the extension with $\frac{(n+1)-3}{2}$, there must occur an edge or extension with $\frac{(n+1)-1}{2}$ white vertices in the clockwise following halfplane. Thus, this edge or the edge corresponding to the extension has an endvertex of type $\frac{(n+1)-1}{2}$. This completes the proof for the maximal values.

Using this result and the fact that in moving from edge to adjacent edge, the number of white vertices in the clockwise following halfplane may be changed by at most 1, we can prove that there are two endvertices of each type from the type $s+1$ to the maximal type N.

We split the proof according to the parity of n.

- For odd n, we have one endvertex of maximal type $N = \frac{n-1}{2}$. Traversing the n edges starting and ending with the edge with an endvertex of type N, from edge to edge we must go down to an edge or an extension with s vertices in the clockwise following halfplane, and then back up to an edge with N. Thus, we find there are at least two edges or extensions with endvertices of each type from $s+1$ to N.

- For even n, we have two edges with endvertices of maximal type $N = \frac{n-2}{2}$. Traversing the n edges from one of the edges of type N to the other must go down to an edge or an extension with s edges in the clockwise following halfplane and back up to an edge with N. Thus again, there are at least two edges or extensions with endvertices of each type from $s+1$ to N.

Hence, it follows that $z_{s,t,i} \geq 2$ for all s, t, i such that $i > s$, and $z_{s,t,N} \geq 1$ for odd n as needed. $\square$

Additionally, by Observation 3.2, $\sum_{i=s+1}^{N} (i(n-i-1)-s(n-s-1)) \geq 0$ for $i, s \leq N = \frac{n-1}{2}$.

Going back to the final expression for Equation (2), we have:

$$\begin{aligned} M &= 4n \sum_{i=1}^{N} i(n-i-1) + \sum_{t\geq 1} \sum_{s=0}^{N} p_{s,t} \Bigg[C(s,n) + \\ &\quad + \sum_{i=s+1}^{N} (i(n-i-1) - s(n-s-1))(z_{s,t,i} - 2) \Bigg] + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} (j-i)^2 x_{i,j}. \end{aligned}$$

Since $C(s,n)$, $z_{s,t,i} - 2$ and $(j-i)^2$ are non-negative, we find that this expression is minimized when $p_{s,t} = 0$ for $s \geq 1$ and $x_{i,j} = 0$ for $i < j$. For $s = 0$, the expression is minimized when $z_{s,t,i} = 2$ for all i. Evaluating the sum for these conditions, we have:

$$M = 4n \sum_{i=1}^{N} i(n-i-1) = 2n \binom{n}{3}$$

bw-pairs in opposite halfplanes determined by lines connecting two vertices of opposite colors.

By similar arguments, in the case of odd n, in the drawing for which M is minimized, we have the following expression:

$$M = 4n \sum_{i=1}^{N-1} i(n-i-1) + 2n(N(n-N-1)) = 2n \binom{n}{3}.$$

$\square$

3.2 Orchard crossings induced by lines generated by pairs of points of the same color

In the previous section, we dealt with lines determined by pairs of points of different colors. In this section, we are going to deal with lines determined by pairs of points of the same color.

For $k = 1, 2, \ldots, \binom{n}{2}$, let ℓ_k be a line determined by two white points (for a pair of black points, the computations are the same). Note that there is no edge in $K_{n,n}$ based on this line, but still this line is counted in the separating lines. Let b_k be the number of **bw**-pairs with one point in one halfplane determined by ℓ_k and the other point in the other halfplane. Let $B = \sum_{k=1}^{\binom{n}{2}} b_k$.

Proposition 3.6 $B \geq n\binom{n}{3}$.

Proof. The proof of this proposition is similar to that of Proposition 3.1, so we will omit most of the details. Each ℓ_k is determined by two white vertices. For each ℓ_k, let an endvertex be of *type* i if the line ℓ_k divides the graph into two halfplanes, one containing i white vertices, and the other containing $n-i-2$ white vertices. By symmetry, we have to consider only $0 \leq i \leq \lfloor \frac{n-2}{2} \rfloor = N$. Let y_i be the number of endvertices of type i. Thus, we have $y_0 + y_1 + \cdots + y_N = 2\binom{n}{2}$.

We call a line a *type* i, j *line* if one halfplane determined by that line has i white vertices and j black vertices. In the other halfplane, there are $n-i-2$ white vertices and $n-j$ black vertices. Let $x_{i,j}$ be the number of type i,j lines. Again, by symmetry, we can assume that $0 \leq j \leq \lfloor \frac{n}{2} \rfloor = N+1$, and that $i < j$. y_i is related to $x_{i,j}$ by the equation: $y_i = x_{i,i+1} + \sum_{j=i+2}^{N+1} x_{i,j}$.

Now, for a type i,j line, there are $i(n-j)+j(n-i-2)$ **bw**-pairs of vertices in opposite halfplanes of that edge. Summing this quantity over all lines of a drawing, we obtain:

$$M = \sum_{i=0}^{N} \sum_{j=i+1}^{N+1} [i(n-j) + j(n-i-2)]x_{i,j}$$

bw-pairs in opposite halfplanes. As in the previous proof, we are looking for a drawing which minimizes M. We start by multiplying the equation for y_i by $i(n-i-1)+(i+1)(n-i-2)$, and subtracting it from M for all values of i, yielding:

$$M = \sum_{i=0}^{N} [i(n-i-1) + (i+1)(n-i-2)]y_i + \sum_{i=0}^{N} \sum_{j=i+2}^{N+1} [(j-(i+1))(n-2(i+1))]x_{i,j} \quad (7)$$

Let $p_{s,t}$ be the number of white vertices having white endvertices of type s as the smallest type $(0 \leq s \leq N)$ and the index t is the number of distinct sorted sequences generated by all the vertices. We have: $n = \sum_{s=0}^{N} \sum_{t \geq 1} p_{s,t}$. Also, denote by $z_{s,t,i}$ the number of appearances of i in the tth sequence whose lowest term is s. We have that:

$$y_i = 2n + \sum_{t \geq 1} \left[\sum_{s=0}^{i} (z_{s,t,i} - 2)p_{s,t} - 2 \sum_{s=i+1}^{N} p_{s,t} \right],$$

and for even n, we have: $y_N = n + \sum_{t \geq 1} \sum_{s=0}^{N} (z_{s,t,N} - 1)p_{s,t}$.

Note that if $s = 0$, the corresponding vertex is on the convex hull generated by the white points. Hence, we have that $z_{0,t,i} = 2$, and for even n, we have: $z_{0,t,N} = 1$. Therefore:

$$y_i = 2n + \sum_{t \geq 1} \left[\sum_{s=1}^{i} (z_{s,t,i} - 2)p_{s,t} - 2 \sum_{s=i+1}^{N} p_{s,t} \right], \quad (8)$$

and for even n, we have: $y_N = n + \sum_{t\geq 1}\sum_{s=1}^{N}(z_{s,t,N}-1)p_{s,t}$.

We proceed for odd n. Similar computations to those we have done in the previous part yield that the first part of the expression for M (Equation (7)) is:

$$2n\sum_{i=0}^{N}[n(1+2i)-2(i+1)^2] + \sum_{t\geq 1}\sum_{s=1}^{N} p_{s,t}\Bigg[C(s,n) + \\ + \sum_{i=s+1}^{N}\left(\left[n(1+2i)-2(i+1)^2\right]-\left[n(1+2s)-2(s+1)^2\right]\right)(z_{s,t,i}-2)\Bigg],$$

where

$$\begin{aligned} C(s,n) &= \left[n(1+2s)-2(s+1)^2\right]((n-1)-2(N-s+1)) \\ &\quad -2\sum_{i=1}^{s-1}\left[n(1+2i)-2(i+1)^2\right]-2(n-2). \end{aligned}$$

Hence, we have by Equation (7):

$$\begin{aligned} M &= 2n\sum_{i=0}^{N}\left[n(1+2i)-2(i+1)^2\right] + \sum_{t\geq 1}\sum_{s=1}^{N} p_{s,t}\Bigg[C(s,n) + \\ &\quad + \sum_{i=s+1}^{N}\left(\left[n(1+2i)-2(i+1)^2\right]-\left[n(1+2s)-2(s+1)^2\right]\right)(z_{s,t,i}-2)\Bigg] \\ &\quad + \sum_{i=0}^{N}\sum_{j=i+2}^{N+1}[(j-(i+1))(n-2(i+1))]x_{i,j} \end{aligned}$$

Since $C(s,n)$, $z_{s,t,i}-2$ and $(j-(i+1))(n-2(i+1))$ are non-negative (similar to the previous case), this expression is minimized when $p_{s,t}=0$ for $s\geq 1$ and $x_{i,j}=0$ for $i<j$. Evaluating the sum for these conditions, we have:

$$M = 2n\sum_{i=0}^{N}\left[n(1+2i)-2(i+1)^2\right] = 2n\binom{n}{3}.$$

Since each line determined by two white points was counted twice, once for each endvertex, we have $n\binom{n}{3}$ **bw**-pairs in opposite halfplanes, as claimed.

As in the previous part, if we perform similar computations for even n, we get the same result. □

3.3 Final step of the proof

Here, we finish proving Theorem 1.1.

Proof of Theorem 1.1. For a given rectilinear drawing $R(K_{n,n})$, any Orchard crossing is determined by a **bw**-pair, where one point is in one halfplane of line ℓ_k and the other point is in the other halfplane of ℓ_k, where the type of the line ℓ_k is one of the following three types:

(a) a line determined by a **bw**-pair.

(b) a line determined by two white points.

(c) a line determined by two black points.

Let A, B, C be the numbers of **bw**-pairs determined by lines of types (a),(b),(c), respectively. Then, $c(R(K_{n,n})) = A + B + C$. By Propositions 3.1 and 3.6, we have $A \geq 2n\binom{n}{3}$ and $B, C \geq n\binom{n}{3}$. Hence:

$$c(R(K_{n,n})) \geq 2n\binom{n}{3} + n\binom{n}{3} + n\binom{n}{3} = 4n\binom{n}{3}$$

On the other hand, by Lemma 2.1, we have a drawing $R_0(K_{n,n})$ of $K_{n,n}$ which satisfies $c(R_0(K_{n,n})) = 4n\binom{n}{3}$. So finally we have $\mathrm{OCN}(K_{n,n}) = 4n\binom{n}{3}$ as claimed. □

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References

[1] O. Aichholzer, F. Aurenhammer and H. Krasser, *On the crossing number of complete graphs*, in: Proc. 18th Ann. ACM Symp. Computational Geometry, 19–24, Barcelona, Spain, 2002.

[2] M. Alpert, E. Feder and H. Harborth, *The maximum of the maximum rectilinear crossing numbers of d-regular graphs of order n*, Electron. J. Combin. **16**(1) (2009), Research Paper 54, 16 pp.

[3] R. Bacher, *Le cocycle du verger*, C. R. Acad. Sci. Paris, Ser. I, **338**(3) (2004), 187–190.

[4] R. Bacher and D. Garber, *The Orchard relation of planar configurations of points I*, Geombinatorics **15**(2) (2005), 54–68.

[5] E. Feder and D. Garber, *The Orchard crossing number of an abstract graph*, Proceedings of the 40th Southeastern International Conference on Combinatorics, Graph Theory and Computing, Cong. Numer. **197** (2009), 3–19.

[6] S. Felsner, *Geometric graphs and arrangements*, Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Wiesbaden, 2004.

[7] J. Pach and G. Tóth, *Thirteen problems on crossing numbers*, Geombinatorics **9** (2000), 194–207.