A new upper bound on the global defensive alliance number in trees*

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Abstract

A global defensive alliance in a graph $G = (V, E)$ is a dominating set S satisfying the condition that for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. In this note, a new upper bound on the global defensive alliance number of a tree is given in terms of its order and the number of support vertices. Moreover, we characterize trees attaining this upper bound.

Keywords: global defensive alliance number, tree, upper bound

1 Introduction

Graph theory terminology not presented here can be found in [2]. Let $G = (V, E)$ be a graph with $|V| = n$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph induced by $S \subseteq V$ is denoted by $G[S]$. An *endvertex* is a vertex which is only adjacent to one vertex. An endvertex in a tree T is also called a *leaf*, while a *support vertex* of T is a vertex adjacent to a leaf. Let $L(T)$ denote the set of leaves of T. A *double star* is a tree that contains exactly two vertices that are not endvertices. If

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one of these vertices is adjacent to r leaves and the other to s leaves, then we denote this double star by $S(r, s)$.

A set S is called a *dominating set* if every vertex in $V \setminus S$ has a neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A minimum dominating set of a graph G is called a $\gamma(G)$*-set*.

A set S is called a *total dominating set* if every vertex in V has a neighbor in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G.

In [1] Hedetniemi, Hedetniemi and Kristiansen introduced several types of alliances, including defensive alliance. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V \setminus S)|$.

A defensive alliance S is called *global* if it effects every vertex in $V \setminus S$, that is, every vertex in $V \setminus S$ is adjacent to at least one member of the defensive alliance S. In this case, S is a dominating set. The *global defensive alliance number* $\gamma_a(G)$ is the minimum cardinality of a defensive alliance of G that is also a dominating set of G. A minimum global defensive alliance of a graph G is called a $\gamma_a(G)$*-set*.

Haynes, Hedetniemi and Henning [2] studied global defensive alliance in graphs. They gave the following results.

Lemma 1.1 (Haynes *et. al* [2]). *For the star* $K_{1,r}$, $\gamma_a(K_{1,r}) = 1 + \lfloor \frac{r}{2} \rfloor$.

Lemma 1.2 (Haynes *et. al* [2]). *For the double star* $S(r, s)$, $\gamma_a(S(r, s)) = 2 + \lfloor \frac{r-1}{2} \rfloor + \lfloor \frac{s-1}{2} \rfloor$.

Let τ be the family of trees T, where $T = P_5$ or $T = K_{1,4}$ or T is the tree obtained from $tK_{1,4}$ (the disjoint union of t copies of $K_{1,4}$) by adding $t-1$ edges between leaves of these copies of $K_{1,4}$ in such a way that the center of each $K_{1,4}$ is adjacent to exactly three leaves in T. Haynes *et. al* established a sharp upper bound on the global defensive alliance number for trees of order greater than 3.

Lemma 1.3 (Haynes *et. al* [2]). *If T is a tree of order $n \geq 4$, then $\gamma_a(T) \leq \frac{3n}{5}$, with equality if and only if $T \in \tau$.*

Chellai and Haynes [3] gave an upper bound on total domination number of a tree in terms of its order and the number of support vertices.

Lemma 1.4 (Chellai and Haynes [3]). *If T is a tree of order $n \geq 3$ and with s support vertices, then $\gamma_t(T) \leq \frac{n+s}{2}$.*

Haynes, Hedetniemi and Henning [2] showed that the global defensive alliance and total domination numbers are the same for graphs with minimum degree at least two and maximum degree at most three.

Lemma 1.5 (Haynes *et. al* [2]). *For any graph G with $\delta(G) \geq 2$, $\gamma_t(G) \leq \gamma_a(G)$. Furthermore, if $\Delta(G) \leq 3$, then $\gamma_t(G) = \gamma_a(G)$.*

In this note, a new upper bound on the global defensive alliance number of a tree is given. We show that for a tree of order n and with s support vertices, $\gamma_a(G) \leq \frac{n+s}{2}$, and we characterize trees attaining this upper bound.

2 Main results

In order to establish a sharp upper bound on the global defensive alliance number of a tree and to characterize trees achieving this bound, we introduce more notation. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the sets of children and descendants, respectively, of v, and let $D[v] = D(v) \cup \{v\}$.

We introduce a family ξ of trees T, where T is a star of odd order or T is the tree obtained from $K_{1,2t_1}, K_{1,2t_2}, \ldots, K_{1,2t_s}$ and tP_4 (the disjoint union of t copies of P_4) by adding $s+t-1$ edges between leaves of these stars and paths in such a way that the center of each star $K_{1,2t_i}$ is adjacent to at least $1 + t_i$ leaves in T and each leaf of every copy of P_4 is incident to at least one new edge, where $t \geq 0$, $s \geq 2$ and $t_i \geq 2$ for $i = 1, 2, \ldots, s$. Note that each support vertex of each tree in ξ must be adjacent with at least 3 leaves.

Lemma 2.1. *Let T be a tree of order n and with s support vertices. If $T \in \xi$, then $\gamma_a(T) = \frac{n+s}{2}$.*

Proof: Suppose T is a star of odd order. In this case $s = 1$. So by Lemma 1.1, we have $\gamma_a(T) = \frac{n+s}{2}$. Hence we assume that T is not a star. Let $P_4^1, P_4^2, \ldots, P_4^t$ denote the t disjoint copies of P_4 when constructing T. Let u_i, w_i be the two support vertices of the path P_4^i. It is obvious that $n = s + 2 \sum\limits_{1 \leq i \leq s} t_i + 4t$. Let v_i be the center of the star $K_{1,2t_i}$ for $i = 1, 2, \ldots, s$. Let S be a $\gamma_a(T)$-set. Since v_i is adjacent to at least $1 + t_i$ leaves in T, it follows that $|S \cap V(K_{1,2t_i})| \geq 1 + t_i$ for $i = 1, 2, \ldots, s$. In order to dominate u_j, w_j, $|S \cap V(P_4^j)| \geq 2$. Hence, $\gamma_a(T) = |S| = \sum\limits_{1 \leq i \leq s} |S \cap V(K_{1,2t_i})| + \sum\limits_{1 \leq j \leq t} |S \cap V(P_4^j)| \geq s + \sum\limits_{1 \leq i \leq s} t_i + 2t = \frac{n+s}{2}$. Let L_i denote a set of t_i leaves of $K_{1,2t_i}$ in T for $i = 1, 2, \ldots, s$. Let $S' = \{v_i \mid 1 \leq i \leq s\} \cup \{u_j, w_j \mid 1 \leq j \leq t\} \bigcup\limits_{1 \leq j \leq s} L_j$. Then S' is a global defensive alliance of T. So, $\gamma_a(T) \leq |S'| \leq \frac{n+s}{2}$. Hence, $\gamma_a(T) = \frac{n+s}{2}$. $\square$

For each tree $T \in \xi$, by its construction, we have the following two simple lemmas.

Lemma 2.2. *Let $T \in \xi$. For any $v \in V(T) \setminus L(T)$, there exists a $\gamma_a(T)$-set S of T such that $v \in S$ and $|N[v] \cap S| > |N(v) \cap (V(T) \setminus S)|$.*

Lemma 2.3. *Let $T \in \xi$. For any $v \in V(T) \setminus \{v_1, v_2, \ldots, v_s\}$, there exists a $\gamma_a(T)$-set S of T such that $v \notin S$, where v_1, v_2, $\ldots$, v_s are defined in the proof of Lemma 2.1.*

Theorem 2.4. *Let T be a tree of order $n \geq 3$ and with s support vertices. Then $\gamma_a(T) \leq \frac{n+s}{2}$, with equality if and only if $T \in \xi$.*

Proof: We proceed by induction on $n \geq 3$. If $n = 3$, then $T = P_3$, and so $\gamma_a(T) = 2 = \frac{n+s}{2}$ and $T \in \xi$. Suppose, then, that for all trees T' of order n' and with s' support vertices, where $3 \leq n' < n$, $\gamma_a(T') \leq \frac{n'+s'}{2}$ and equality holds if and only if $T' \in \xi$. Let T be a tree of order n. If T is a star, then, by Lemma 1.1, $\gamma_a(K_{1,n-1}) = 1 + \lfloor \frac{n-1}{2} \rfloor \leq \frac{n+s}{2}$ with equality if and only if n is odd. Hence $T \in \xi$. If T is a double star, then it follows from Lemma 1.2 that $\gamma_a(T) < \frac{n+s}{2}$. Hence we may assume that $diam(T) \geq 4$.

Let $S(T)$ be the set of support vertices of T. Suppose that there exists $v \in S(T)$ such that $d(v) = 2$. Let $N(v) \cap L(T) = \{v'\}$ and $N(v) \setminus \{v'\} = \{u\}$. Choose $T_0 = T - \{v, v'\}$. Let T_0 have order n_0 and s_0 support vertices. Then $n = n_0 + 2$. Since $diam(T) \geq 4$, it follows that $n_0 \geq 3$. Applying the induction hypothesis to T_0, $\gamma_a(T_0) \leq \frac{n_0+s_0}{2}$. Let S_0 be a $\gamma_a(T_0)$-set. If $u \notin S_0$, let $S_0' = S_0 \cup \{v'\}$, while if $u \in S_0$, let $S_0' = S_0 \cup \{v\}$. Then, S_0' is a global defensive alliance of T. So $\gamma_a(T) \leq |S_0| + 1 \leq \frac{n_0+s_0}{2} + 1 = \frac{n+s_0}{2}$. Since $s_0 \leq s$, it follows that $\gamma_a(T) \leq \frac{n+s}{2}$. Furthermore, suppose $\gamma_a(T) = \frac{n+s}{2}$. Then $s_0 = s$ and $\gamma_a(T_0) = \frac{n_0+s_0}{2}$. By the induction hypothesis, $T_0 \in \xi$. Since $s_0 = s$, it follows that $d_T(u) = 2$ and $x \notin S(T)$, where $x \in N(u) \setminus \{v\}$. Hence, x is a support of T_0 and is adjacent to only one leaf in T_0, which is a contradiction. So, $\gamma_a(T) < \frac{n+s}{2}$. In the following, we may assume that $d_T(v) \geq 3$ for any $v \in S(T)$.

Choose v having the smallest degree among all support vertices of T of eccentricity $diam(T) - 1$. Let r be a vertex at distance $diam(T) - 1$ from v. View T as the rooted tree at r. Let u denote the parent of v, and x the parent of u. Let $|C(v)| = l$. Then $l \geq 2$.

Case 1: Suppose that $d_T(u) \geq 3$. Let $T_1 = T - D[v]$. Assume that T_1 is of order n_1 and has s_1 support vertices. Then $n = n_1 + l + 1$ and $n_1 \geq 3$. By the induction hypothesis we have $\gamma_a(T_1) \leq \frac{n_1+s_1}{2}$. Among all $\gamma_a(T_1)$-sets, let S_1 be chosen to contain the vertex u, if possible.

If $u \in S_1$, then by adding v and $\lfloor \frac{l-1}{2} \rfloor$ children of v to S_1 produces a global defensive alliance of T. So $\gamma_a(T) \leq |S_1| + 1 + \lfloor \frac{l-1}{2} \rfloor \leq \frac{n-1-l+s_1}{2} + \frac{l+1}{2} = \frac{n+s_1}{2}$. Since $s_1 < s$, it follows that $\gamma_a(T) < \frac{n+s}{2}$. Hence we may assume that $u \notin S_1$.

If u has a child v' different from v that is a support vertex, then $|C(v')| \geq 2$. Since $u \notin S_1$, either $C(v') \subset S_1$ or $v' \in S_1$. For both cases, we can choose another global defensive alliance of T containing u and v', contrary to our choice of S_1. Hence we assume that every child of u different from v is a leaf.

If u is adjacent to more than one leaf, then we can choose a $\gamma_a(T)$-set containing u, contrary to our choice of S_1. So we assume $d_T(u) = 3$ and the child v' of u different from v is a leaf. Thus, $v' \in S_1$. Delete v' from S_1, add u, v and $\lfloor \frac{l-1}{2} \rfloor$ children of v to S_1 to get a global defensive alliance of T. So $\gamma_a(T) \leq |S_1| + 1 + \lfloor \frac{l-1}{2} \rfloor \leq \frac{n-1-l+s_1}{2} + \frac{l+1}{2} = \frac{n+s_1}{2}$. Since $s_1 < s$, it follows that $\gamma_a(T) < \frac{n+s}{2}$.

Case 2: Suppose that $d_T(u) = 2$. Let $T_2 = T - D[u]$. Assume that T_2 has order n_2 and s_2 support vertices. Then $n = n_2 + l + 2$. Since $diam(T) \geq 4$, it follows from our choice of v that $n_2 \geq 3$. By the induction hypothesis we have $\gamma_a(T_2) \leq \frac{n_2+s_2}{2}$. Let S_2 be a $\gamma_a(T_2)$-set. Let w be the parent of x.

Case 2.1: Suppose that $d_T(x) \geq 3$ or $d_T(x) = 2$ and $w \in S(T)$. Then $s = s_2 + 1$. Add u, v and $\lfloor \frac{l-1}{2} \rfloor$ children of v to S_2 to obtain a global defensive alliance of T. So $\gamma_a(T) \leq |S_2| + 2 + \lfloor \frac{l-1}{2} \rfloor \leq \frac{n-2-l+s_2}{2} + \frac{l+3}{2} = \frac{n+s_2+1}{2} = \frac{n+s}{2}$.

Furthermore, suppose $\gamma_a(T) = \frac{n+s}{2}$. Then l is odd and $\gamma_a(T_2) = \frac{n_2+s_2}{2}$. By the induction hypothesis, $T_2 \in \xi$. Suppose that $T_2 = K_{1,2t}$, for some $t \geq 1$.

Suppose x is the center of $K_{1,2t}$. The set consisting of $\{u, v, x\}$, $t-1$ children of x and $\frac{l-1}{2}$ children of v forms a global defensive alliance of T. So $\gamma_a(T) \leq 3+(t-1)+\frac{l-1}{2} = \frac{2t+l+3}{2} = \frac{n}{2} < \frac{n+s}{2}$, which is a contradiction. Hence, x is a leaf of star $K_{1,2t}$. Since the degree of any support vertex of T is at least three, it follows that $t \geq 2$. Then T is a tree obtained from $K_{1,2t}$ and $K_{1,l+1}$ by adding an edge between one leaf of each star. Then $T \in \xi$.

So we may assume that T_2 is not a star. Choose S_2 to be a $\gamma_a(T_2)$-set that contains all centers of stars, all leaves of stars that are incident to added edges when constructing T_2 and support vertices of all copies of P_4. Since the center of each star is adjacent to at least $1+t_i$ leaves in T_2, S_2 contains at least one leaf of each star $K_{1,2t_i}$.

Suppose x is the center of some $K_{1,2t_i}$ for some i. Let $y \in N(x) \cap S_2 \cap L(T_2)$. Then add u, v and $\frac{l-1}{2}$ children of v to $S_2 \setminus \{y\}$ to get a global defensive alliance of T. We have $\gamma_a(T) \leq |S_2| + 1 + \frac{l-1}{2} = \frac{n_2+s_2}{2} + \frac{l+1}{2} < \frac{n+s}{2}$, which is a contradiction.

Suppose that x is a support vertex of $P_4^i = l_i u_i w_i m_i$, say $x = u_i$ for some i. Choose S^* to be a $\gamma_a(T_2)$-set that contains all centers of stars and all end vertices of added edges when constructing T_2.

Let T' be the subgraph of T_2 induced by the vertices of all copies of P_4 when constructing T_2. Let T'' be the component of $T' - l_i u_i$ that contains vertex l_i. For each subgraph P_4^k of T'', label P_4^k with $l_k u_k w_j m_k$ such that $d_{T_2}(m_k, l_i) < d_{T_2}(l_k, l_i)$. Let S^{**} be obtained from $S^* \setminus \{l_i\}$ by replacing l_k with w_k for each subgraph P_4^k of T''. Then add u, v and $\frac{l-1}{2}$ children of v to S^{**} to get a global defensive alliance of T. So $\gamma_a(T) \leq |S^{**}| + 2 + \frac{l-1}{2} = |S^*| + 1 + \frac{l-1}{2} = \frac{n_2+s_2}{2} + \frac{l+1}{2} < \frac{n+s}{2}$, which is a contradiction.

Suppose that x is a leaf of a star when constructing T_2. Let v_i be the center of the star $K_{1,2t_i}$ in T_2 that contains x for some i. Suppose $|N_T(v_i) \cap L(T)| < 1+t_i$. Let S^* be a $\gamma_a(T_2)$-set containing x, all centers of stars and all end vertices of added edges when constructing T_2. Let S be obtained from $(S^* \setminus V(K_{1,2t_i})) \cup (N_T(v_i) \cap L(T))$ by adding u, v and $\frac{l-1}{2}$ children of v. Then S is a global defensive alliance of T with cardinality less than $\frac{n+s}{2}$. It is a contradiction. So $|N_T(v_i) \cap L(T)| \geq 1 + t_i$ and hence $T \in \xi$.

Suppose x is a leaf of P_4^i for some i. Then clearly $T \in \xi$.

Case 2.2: Suppose $d_T(x) = 2$ and $w \notin S(T)$. Let $T_3 = T - D[x]$. Assume that T_3 has order n_3 and s_3 support vertices. Then $n = n_3 + l + 3$ and $n_3 \geq 3$. Applying the induction hypothesis to T_3, $\gamma_a(T_3) \leq \frac{n_3+s_3}{2}$. Let S_3 be a $\gamma_a(T_3)$-set.

Subcase 1: Suppose that $d_T(w) \geq 3$. Then $s_3 = s - 1$.

i) Suppose $w \notin S_3$, or $w \in S_3$ and $|N_{T_3}[w] \cap S_3| > |N_{T_3}(w) \cap (V(T_3) - S_3)|$. By adding u and v and $\lfloor \frac{l-1}{2} \rfloor$ children of v to S_3 produces a global defensive alliance of T. So $\gamma_a(T) \leq |S_3| + 2 + \lfloor\frac{l-1}{2}\rfloor \leq \frac{n-3-l+s_3}{2} + \frac{l+3}{2} = \frac{n+s_3}{2} < \frac{n+s}{2}$.

ii) Now we assume that $w \in S_3$ and $|N_{T_3}[w] \cap S_3| = |N_{T_3}(w) \cap (V(T_3) - S_3)|$ for all $\gamma_a(T_3)$-set S_3. Suppose there exists $v' \in S(T_3) \cap D[w]$ such that $d_T(w, v') = 3$. Let $wx'u'v'$ denote the $w - v'$ path in T. By a similar way as above, we may assume that $d_T(x') = d_T(u') = 2$. Let $|C(v')| = l'$. Since $|N_{T_3}[w] \cap S_3| = |N_{T_3}(w) \cap (V(T_3) - S_3)|$, it follows that $d_T(w) \geq 4$. Let $T_{31} = T - D[x] - D[x']$. Let T_{31} have order n_{31} and s_{31} support vertices. Then $n = n_{31} + l + l' + 6$ and $s_{31} = s - 2$. Since $n_{31} \geq 3$, by applying the induction hypothesis to T_{31}, $\gamma_a(T_{31}) \leq \frac{n_{31}+s_{31}}{2}$. Let S_{31} be a $\gamma_a(T_{31})$-set. Then by adding $\{u, v, u', v', x\}$, $\lfloor\frac{l-1}{2}\rfloor$ children of v and $\lfloor\frac{l'-1}{2}\rfloor$ children of v' to S_{31} produces a global defensive alliance of T. So $\gamma_a(T) \leq |S_{31}| + 5 + \lfloor\frac{l-1}{2}\rfloor + \lfloor\frac{l'-1}{2}\rfloor \leq \frac{n-6-l-l'+s_{31}}{2} + \frac{l+l'+8}{2} = \frac{n+s}{2}$. For the sake of contradiction, suppose we have equality throughout this inequality chain. In particular, $\gamma_a(T) = \frac{n+s}{2}$, l and l' are odd, and $\gamma_a(T_{31}) = \frac{n_{31}+s_{31}}{2}$. So, $T_{31} \in \xi$. Since $w \notin S(T)$, $w \notin \{v_1, v_2, \ldots, v_s\}$. By Lemma 2.3, we can choose a $\gamma_a(T_{31})$-set S^* such that $w \notin S^*$. By adding $\{u, v, u', v'\}$, $\frac{l-1}{2}$ children of v and $\frac{l'-1}{2}$ children of v' to S^* produces a global defensive alliance of T. Hence $\gamma_a(T) \leq |S^*| + 4 + \frac{l-1}{2} + \frac{l'-1}{2} < \frac{n+s}{2}$, which is a contradiction. So in this case $\gamma_a(T) < \frac{n+s}{2}$.

We may assume that $d_T(w, v') \leq 2$ for any $v' \in S(T_3) \cap D[w]$. Suppose $v' \notin S_3$. Then $N(v') \cap L(T_3) \subseteq S_3$. Choose a vertex $u' \in N(v') \cap L(T_3)$. Then $(S_3 \setminus \{u'\}) \cup \{v'\}$ is a global defensive alliance of T_3. So, we obtain a global defensive alliance of T_3, still say S_3, such that $S(T_3) \cap D[w] \subseteq S_3$. Let p be the parent of w. Suppose there is a vertex $q \in N(w) \setminus \{p\}$ and $q \notin S_3$. Then $q \notin S(T_3)$. Let $q' \in N(q) \cap S(T_3)$. Then $q' \in S_3$. Since q' is adjacent to at least two leaves, there exists a vertex $k \in S_3 \cap L(T_3) \cap C(q')$. Then $(S_3 \setminus \{k\}) \cup \{q'\}$ is a global defensive alliance of T. So, we can get another global defensive alliance of T_3, say S_3', such that $N(w) \setminus \{p\} \subseteq S_3'$. Then $w \in S_3'$ and $|N_{T_3}[w] \cap S_3'| > |N_{T_3}(w) \cap (V(T_3) - S_3')|$, contradicting our choice of w.

Subcase 2: Suppose that $d_T(w) = 2$. Let $T_4 = T - D[w]$. Let T_4 of order n_4 and s_4 support vertices. Then $n = n_4 + l + 4$. By the choice of v we have $n_4 \geq 3$. Applying the induction hypothesis to T_4, we have $\gamma_a(T_4) \leq \frac{n_4+s_4}{2}$. Let S_4 be a $\gamma_a(T_4)$-set. Let p be the parent of w. Suppose that $d_T(p) \geq 3$. Then $s_4 = s - 1$. If $p \in S_4$, then adding $\{w, u, v\}$ and $\lfloor\frac{l-1}{2}\rfloor$ children of v to S_4 produces a global defensive alliance of T. If $p \notin S_4$, then adding $\{x, u, v\}$ and $\lfloor\frac{l-1}{2}\rfloor$ children of v to S_4 produces a global defensive alliance of T. So $\gamma_a(T) \leq |S_4| + 3 + \lfloor\frac{l-1}{2}\rfloor \leq \frac{n-4-l+s_4}{2} + \frac{l+5}{2} = \frac{n+s}{2}$. Furthermore, suppose $\gamma_a(T) = \frac{n+s}{2}$. Then l is odd and $\gamma_a(T_4) = \frac{n_4+s_4}{2}$. So, $T_4 \in \xi$. By Lemma 2.2, we can choose a $\gamma_a(T_4)$-set S^* such that $p \in S^*$ with $|N_{T_4}[p] \cap S^*| > |N_{T_4}(p) \cap (V(T_4) - S^*)|$. Then adding $\{u, v\}$ and $\frac{l-1}{2}$ children of v to S^* produces a global defensive alliance of T. So $\gamma_a(T) \leq |S^*| + 2 + \frac{l-1}{2} \leq \frac{n+s-2}{2} < \frac{n+s}{2}$, which is impossible.

Hence, $\gamma_a(T) < \frac{n+s}{2}$.

Now we assume that $d_T(p) = 2$. Let q be the parent of p. Let T_5 be obtained from T by deleting vertices of $\{p, w, x, u\}$ and adding edge qv. Let T_5 have order n_5 and support vertices number s_5. It follows that $n_5 \geq 3$. Applying the inductive hypothesis to T_5, $\gamma_a(T_5) \leq \frac{n_5+s_5}{2}$. Let S_5 be a $\gamma_a(T_5)$-set. Then $n = n_5 + 4$ and $s = s_5$. Then $S_5 \cup \{p, u\}$, $S_5 \cup \{p, w\}$ or $S_5 \cup \{w, x\}$ is a global defensive alliance of T. Hence, $\gamma_a(T) \leq |S_5| + 2 \leq \frac{n-4+s_5}{2} + 2 = \frac{n+s}{2}$. Furthermore, suppose $\gamma_a(T) = \frac{n+s}{2}$. Then l is odd, $\gamma_a(T_5) = \frac{n_5+s_5}{2}$ and $T_5 \in \xi$. Hence $T \in \xi$. □

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