Digraphs are 2-weight choosable

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Abstract

An edge-weighting vertex colouring of a graph is an edge-weight assignment such that the accumulated weights at the vertices yield a proper vertex colouring. If such an assignment from a set S exists, we say the graph is S-weight colourable.

We consider the S-weight colourability of digraphs by defining the accumulated weight at a vertex to be the sum of the inbound weights minus the sum of the outbound weights. Bartnicki et al. showed that every digraph is S-weight colourable for any set S of size 2 and asked whether one could show the same result using an algebraic approach. Using the Combinatorial Nullstellensatz and a classical theorem of Schur, we provide such a solution.

Let G be a simple graph with no isolated edge and S be a set of real numbers. An *S-edge-weighting* of G is an assignment $w : E(G) \to S$. The *weighted degree* of a vertex is the sum of weights of the edges incident with it. If the weighted degrees of adjacent vertices are different we say that w is an *S-weight colouring*, and we say that G is *S-weight colourable*. Karoński et al. [KŁT04] asked whether every graph is $\{1, 2, 3\}$-weight colourable; they showed this is true for 3-colourable graphs. Kalkowski et al. [KKP09] showed that general graphs are $\{1, 2, 3, 4, 5\}$-weight colourable. We generalize S-weight colourings to directed graphs by defining the *weighted degree of a vertex* of D to be the sum of the inbound weights minus the sum of the outbound weights of that vertex. We allow our digraphs to contain multiple parallel arcs in either direction, but no loops.

These problems have natural list-colouring variants. Suppose that each $e \in E(G)$ of a (directed) graph G is assigned a set of real numbers $\mathcal{L}_e$. Let $\mathcal{L} = \cup_{e\in E(G)}\mathcal{L}_e$. The

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(directed) graph G is *weight colourable from the lists* $\{\mathcal{L}_e\}_{e\in E(G)}$ if G admits an $\mathcal{L}$-edge-weighting w so that $w(e) \in \mathcal{L}_e$ for each e. A (directed) graph is *k-weight choosable* if it is weight colourable for any collection of lists of size k.

Problem 1. *Given a graph G, find the minimum k such that G is k-weight choosable.*

Problem 2. *Given a digraph D, find the minimum k such that D is k-weight choosable.*

Recently, the questions stated in Problems 1 and 2 were studied by Bartnicki, Grytczuk, and Niwczyk [BGN09]. Using a method relying on permanents of matrices and Combinatorial Nullstellensatz, they determined many families of undirected graphs which are 3-weight choosable and thus determined classes of graphs for which the solution to Problem 1 must be $k \leq 3$. Furthermore, they conjectured that every graph without an isolated edge is 3-weight choosable.

Regarding digraphs, it is shown in [BGN09] that all digraphs are 2-weight choosable, thus completely solving Problem 2. However, as their proof was algorithmic in nature, they asked whether it was possible for their Combinatorial Nullstellensatz approach to be extended to digraphs. We show that it is indeed possible by applying a classical theorem of Schur. We also note a few interesting corollaries of this result which relate to both graphs and digraphs.

We begin by presenting the two necessary tools for our proof – Combinatorial Nullstellensatz [Alo99] and a classical theorem of Schur [Sch18] (see [Min78, Theorem 2.5, p. 26] for a proof of Schur's theorem in English).

Theorem 1 (Combinatorial Nullstellensatz)**.** *Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the total degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. If $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$ then there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that*

$$f(s_1, \ldots, s_n) \neq 0.$$

Theorem 2 (Schur)**.** *If A is a positive semi-definite Hermitian matrix, then* $\mathrm{per}(A) \geq \det(A)$*, with equality if and only if A is diagonal or A has a zero row.*

We remind the reader that the *permanent* of an $n \times n$ matrix $A = (a_{i,j})$ is defined by

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n denotes the set of all permutations on $\{1, \ldots, n\}$.

Theorem 3. *Every digraph is 2-weight choosable.*

Proof. Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$. Let $|E(D)| = m$. We use e_{h} to denote the head of arc e, and e_{t} to denote the tail of arc e.

Assign a variable x_e and a list $\mathcal{L}_e$ of size 2 to each $e \in E(D)$. To every vertex v of D, we associate the term

$$P_v = \sum_{e_{\mathrm{h}}=v} x_e - \sum_{e_{\mathrm{t}}=v} x_e,$$

that is, the sum of the variables of the inbound arcs minus the sum of the variables of the outbound arcs. Define the real multivariate polynomial $P_D = \prod_{e\in E(D)}(P_{e_{\mathrm{h}}} - P_{e_{\mathrm{t}}})$. Then D is 2-weight choosable if and only if there exists an $s_e \in \mathcal{L}_e$ for every $e \in E(D)$ such that $P_D(s_{e_1}, \ldots, s_{e_m}) \neq 0$.

Each monomial in P_D has total degree m. We will show that the coefficient of $\prod_{e\in E(D)} x_e$ is nonzero and then the result follows from Theorem 1.

For every $e \in E(D)$, we may write $P_{e_{\mathrm{h}}} - P_{e_{\mathrm{t}}} = \sum_{e'\in E(D)} M_{e,e'} x_{e'}$, where M is an $m \times m$ matrix indexed by the arcs of D and defined by

$$M_{e,e'} = \begin{cases} 2 & \text{if } e_{\mathrm{t}} = e'_{\mathrm{t}}, e_{\mathrm{h}} = e'_{\mathrm{h}}, \\ -2 & \text{if } e_{\mathrm{t}} = e'_{\mathrm{h}}, e_{\mathrm{h}} = e'_{\mathrm{t}}, \\ 1 & \text{if } e_{\mathrm{t}} = e'_{\mathrm{t}}, e_{\mathrm{h}} \neq e'_{\mathrm{h}} \text{ or } e_{\mathrm{h}} = e'_{\mathrm{h}}, e_{\mathrm{t}} \neq e'_{\mathrm{t}}, \\ -1 & \text{if } e_{\mathrm{t}} = e'_{\mathrm{h}}, e_{\mathrm{h}} \neq e'_{\mathrm{t}} \text{ or } e_{\mathrm{h}} = e'_{\mathrm{t}}, e_{\mathrm{t}} \neq e'_{\mathrm{h}}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $P_D = \prod_{e\in E(D)} \sum_{e'\in E(D)} M_{e,e'} x_{e'}$. Therefore the coefficient of $\prod_{e\in E(D)} x_e$ is $\mathrm{per}(M)$.

Note that $M = XX^t$, where X is the $m \times n$ vertex-arc incidence matrix of D, that is $X_{e,w}$ equals 1 if $e_{\mathrm{t}} = w$, -1 if $e_{\mathrm{h}} = w$, and 0 otherwise. Furthermore, XX^t is diagonal if and only if no two edges are incident, in which case the result holds trivially. Otherwise, XX^t is a positive semi-definite matrix that is neither diagonal nor has a zero row, and so by Schur's theorem we have $\mathrm{per}(M) = \mathrm{per}(XX^t) > \det(XX^t) \geq 0$. Thus, the desired coefficient is nonzero. □

The 2-weight choosability of digraphs not only solves the problem of 2-weight colourability of digraphs (Problem 2), but it also yields the following interesting results which are not, at first glance, weight choosability problems.

Corollary 4. *Every graph G has an orientation D such that the weighted degrees of D give a proper vertex colouring of G.*

Proof. Let D' be any orientation of G. Let $\mathcal{L}_e = \{-1, 1\}$ for each $e \in E(D')$ in Theorem 3. The digraph D obtained from D' by swapping the direction of those arcs with weight -1 gives the desired result. □

Note that in the vertex colouring guaranteed by the above result, the parity of the colour assigned to a vertex will be the same as the parity of its degree. This implies that the number of colours used to colour a d-regular graph via Corollary 4 is at most $d + 1$, which is one more than the number of colours guaranteed by Brooks' Theorem in the case of non-complete graphs which are not odd cycles and precisely the chromatic number if $G = K_n$ or $G = C_{2n+1}$.

Corollary 5. *Every digraph D has a subgraph F such that the weighted degrees of F give a proper vertex colouring of D.*

Proof. Let $\mathcal{L}_e = \{0, 1\}$ for each $e \in E(D)$ in Theorem 3. $\square$

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