Binomial edge ideals with quadratic Gröbner bases

Marilena Crupi
Dipartimento di Matematica,
Università di Messina
Viale Ferdinando Stagno d'Alcontres, 31
98166 Messina, Italy
mcrupi@unime.it

Giancarlo Rinaldo
Dipartimento di Matematica,
Università di Messina
Viale Ferdinando Stagno d'Alcontres, 31
98166 Messina, Italy
giancarlo.rinaldo@tiscali.it

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Abstract

We prove that a binomial edge ideal of a graph G has a quadratic Gröbner basis with respect to some term order if and only if the graph G is closed with respect to a given labelling of the vertices. We also state some criteria for the closedness of a graph G that do not depend on the labelling of its vertex set.

1 Introduction

In this article a graph G means a simple graph without isolated vertices, loops and multiple edges. Let $V(G) = [n] = \{1, \dots, n\}$ denote the set of vertices and $E(G)$ the set of edges.

One of the main objects of study in combinatorial commutative algebra is the edge ideal of a graph G which is generated by the monomials $x_i x_j$, where $\{i, j\}$ is an edge of G, in the polynomial ring $K[x_1, \dots, x_n]$ over the field K. Edge ideals of a graph has been introduced by Villarreal in 1990 [16], where he studied the Cohen-Macaulay property of such ideals. Many authors have focused their attention on such ideals (see for example [15], [9],[7], [2]).

In 2010, binomial edge ideals were introduced in [10] and appear independently, but at the same time, also in [13]. Let $S = K[x_1, \cdots, x_n, y_1, \cdots, y_n]$ be the polynomial ring in $2n$ variables with coefficients in a field K. For $i < j$, set $f_{ij} = x_i y_j - x_j y_i$. The ideal J_G of S generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ such that $i < j$ and $\{i, j\}$ is an edge of G, is called *the binomial edge ideal* of G.

Such class of ideals is a natural generalization of the ideal of 2-minors of a $2 \times n$-matrix of indeterminates. Really, the ideal of 2-minors of a $2 \times n$-matrix may be considered as the binomial edge ideal of a complete graph on $[n]$. Moreover the binomial edge ideal of a line graph, which can be interpreted as an ideal of adjacent minors, has been examined in [3]. The importance of this class of binomial edge ideals for algebraic statistics is unquestionable [10]. Indeed these ideals arise naturally in the study of conditional independence statements [4]. Many algebraic properties of binomial edge ideals in terms of properties of the underlying graph were studied in [10] and [12].

In [10], Theorem 1.1, the authors proved the following:

Theorem 1.1. *Let G be a graph on the vertex set $[n]$, and let $<$ the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ on S. Then the following conditions are equivalent:*

(1) The generators f_{ij} of J_G form a quadratic Gröbner basis.

(2) For all edges $\{i,j\}$ and $\{k,\ell\}$ with $i < j$ and $k < \ell$ one has $\{j,\ell\} \in E(G)$ if $i = k$, and $\{i,k\} \in E(G)$ if $j = \ell$.

The authors in [10], called a graph G on $[n]$ *closed with respect to the given labelling of the vertices* if G satisfies condition (2). The term closed graph is not standard terminology in graph theory. Nevertheless this class of graphs is related to a well-known class of graphs: the chordal graphs. A closed graph is chordal ([10]) but the converse is not true. Indeed a closed graph is a claw-free chordal graph, where by a claw we mean a graph with three different edges e_1, e_2, e_3 such that $e_1 \cap e_2 \cap e_3 \neq \emptyset$.

In Theorem 1.1 the role of the lexicographic order on S is fundamental. In this article we are able to state that the existence of a quadratic Gröbner basis for J_G is not related to the lexicographic order on S. In fact, one of the main result in the paper implies that the closed graphs are the only graphs for which the binomial edge ideal J_G has a quadratic Gröbner basis with respect to some term order on S (Theorem 3.4). Our result underlines also the relation between binomial edge ideals and edge ideals. In fact as a consequence we obtain that J_G has a quadratic Gröbner basis with respect to some term order $\prec$ on S if and only if $\mathrm{in}(J_G)$ is the edge ideal of a bipartite graph with bipartition $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_n\}$. The strict relation between algebraic invariants of an ideal J and $\mathrm{in}(J)$ is well known (see for example [5], Chapter 15).

Furthermore Theorem 1.1 and Theorem 3.4 suggest that it would be interesting to state some criteria for the closedness of a simple graph G. Since the characterizations of closed graphs G (see [10], [12]) depend on the labelling of $V(G)$, our aim is to state some new criteria for the closedness of a graph that do not depend on the labelling of its vertex set (Theorem 5.5 and Corollary 5.7).

We believe that by an ordering on the vertices obtained by lexicographic breadth first search and an appropriate specialization of the algorithm on chordality test (see Algorithms 2, 3 of [8] or [14]), it is possible to test the closedness of a graph as a consequence of Theorem 5.5 in linear time. But this is not the aim of this paper.

The paper is organized as follows.

Section 2 contains some preliminaries and notions that we will use in the paper.

In Section 3, we state a fundamental result that gives the motivation of an intensive study of closed graphs: we prove that the only graphs having quadratic Gröbner basis with respect to a given monomial order are the closed ones (Theorem 3.4). The statement is obtained by the construction of a special oriented graph (Definition 3.1).

In Section 4, we introduce the notion of a linear quasi-tree simplicial complex (Definition 4.3) and we relate it with a closed graph (Proposition 4.6). Moreover we give a characterization of the closedness of a graph G in terms of particular cliques of G (Proposition 4.8). This result will be crucial in the sequel.

In Section 5, we analyze the behaviour of the set of facets $\mathcal{F}(\Delta(G))$ of the clique complex $\Delta(G)$ (Definition 2.1) of a graph G when $\Delta(G)$ is a linear quasi-tree (Proposition 5.1). We introduce a special subclass of the linear quasi-tree complexes: the class of *closed* complexes (Definition 5.2). The section contains the main results in the paper. We give a criterion for the closedness of a graph G that is independent from the labelling of $V(G)$ (Theorem 5.5). We show that a graph G is closed if and only if the clique complex $\Delta(G)$ is a closed complex (Corollary 5.7).

2 Preliminaries

In this section we recall some concepts and a notation on graphs and on simplicial complexes that we will use in the article.

Let G be a simple graph with vertex set $V(G)$ and the edge set $E(G)$. Let $v, w \in V(G)$. A path π from v to w is a sequence of vertices $v = v_0, v_1, \cdots, v_t = w$ such that $\{v_i, v_{i+1}\}$ is an edge of the underlying graph. A graph G is *connected* if for every pair of vertices v_1 and v_2 there is a path from v_1 to v_2. If G is *directed* (or *digraph*), that is, G consists of a finite nonempty set of vertices with a prescribed collection X of ordered pairs of distinct vertices, then the path is called *directed*, if either (v_i, v_{i+1}) is an arrow for all i, or (v_{i+1}, v_i) is an arrow for all i.

When we fix a given labelling on the vertices we say that G is a graph on $[n]$.
Let G be a graph with vertex set $[n]$. A subset C of $[n]$ is called a *clique* of G is for all i and j belonging to C with $i \neq j$ one has $\{i, j\} \in E(G)$.

Set $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$ and

(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a *face* of Δ. For $F \in \Delta$ we define the *dimension* of F by $\dim F = |F| - 1$, where $|F|$ is the cardinality of the set F. A maximal face of Δ with respect to inclusion is called a *facet* of Δ.

If Δ is a simplicial complex with facets $F_1, \ldots, F_q$, we call $\{F_1, \ldots, F_q\}$ the facet set of Δ and we denote it by $\mathcal{F}(\Delta)$. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_q\}$, we write $\Delta = \langle F_1, \ldots, F_q \rangle$.

Definition 2.1. The *clique complex* $\Delta(G)$ of G is the simplicial complex whose faces are the cliques of G.

Definition 2.2. Let Δ be a simplicial complex. A facet $F \in \mathcal{F}(\Delta)$ is said to be a *leaf* of Δ if either F is the only facet of Δ, or there exists a facet $B \in \mathcal{F}(\Delta)$, $B \neq F$, called a *branch* of F, such that $H \cap F \subseteq B \cap F$ for all $H \in \mathcal{F}(\Delta)$ with $H \neq B$.

Observe that for a leaf F the subcomplex Δ' with $\mathcal{F}(\Delta') = \mathcal{F}(\Delta) \setminus F$ coincides with the restriction $\Delta_{[n]\setminus(F\setminus(B\cap F))}$.

We finish this section by recalling the following definition from [11].

Definition 2.3. Let Δ be a simplicial complex. Δ is called a quasi-forest if there exists a labelling $F_1, \cdots, F_q$ of the facets of Δ, such that for every $1 < i \leq q$, the facet F_i is a leaf of the subcomplex $\langle F_1, \cdots, F_i \rangle$. The sequence $F_1, \ldots, F_q$ is called a leaf order of the quasi-tree. A connected quasi-forest is called a quasi-tree.

3 Quadratic Gröbner bases

In this section we observe that the only graphs having quadratic Gröbner bases with respect to a monomial order $\prec$ are the *closed graphs* with respect to a labelling induced by $\prec$.

Let G be a graph on the vertex set $V(G) = [n]$, $E(G)$ its edge set and $S = K[x_1, \cdots, x_n,$
$y_1, \cdots, y_n]$.

Definition 3.1. Let J_G be the binomial edge ideal of G and let $\prec$ a term order on S. We define an oriented graph $G_\prec$ with $V(G_\prec) = V(G)$ and edge set

$$E(G_\prec) = \{(i, j) : x_i y_j \in \mathrm{in}_\prec J_G\}.$$

Proposition 3.2. *$G_\prec$ is an acyclic directed graph.*

Proof. It is sufficient to show that every cycle in G is not a directed cycle in $G_\prec$. Let

$$\{i_1, i_2, \ldots, i_r\} \subseteq V(G)$$

be the vertices of a cycle and suppose that $(i_j, i_{j+1}) \in E(G_\prec)$ for $j = 1, \ldots, r-1$. We will show that $(i_r, i_1) \notin E(G_\prec)$.

By hypothesis we have that $x_{i_j} y_{i_{j+1}} \succ x_{i_{j+1}} y_{i_j}$ for $j = 1, \ldots, r-1$. Since $\prec$ is a term order, then $y_{i_3}(x_{i_1} y_{i_2}) \succ y_{i_3}(x_{i_2} y_{i_1})$ and $y_{i_1}(x_{i_2} y_{i_3}) \succ y_{i_1}(x_{i_3} y_{i_2})$. Therefore $y_{i_3}(x_{i_1} y_{i_2}) \succ y_{i_1}(x_{i_3} y_{i_2})$ and $x_{i_1} y_{i_3} \succ x_{i_3} y_{i_1}$.

By the same argument we have that $y_{i_4}(x_{i_1} y_{i_3}) \succ y_{i_4}(x_{i_3} y_{i_1})$ and $y_{i_1}(x_{i_3} y_{i_4}) \succ y_{i_1}(x_{i_4} y_{i_3})$. Hence $y_{i_4}(x_{i_1} y_{i_3}) \succ y_{i_1}(x_{i_4} y_{i_3})$ and $x_{i_1} y_{i_4} \succ x_{i_4} y_{i_1}$, and so on. Finally, we will have that $x_{i_1} y_{i_r} \succ x_{i_r} y_{i_1}$. □

Remark 3.3. We observe that the ideal J_G of S is multigraded if we assign the following multidegrees to the indeterminates of S:

$$\deg(x_i) = \deg(y_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n,$$

where the entry 1 is at the i-th position. Hence the only binomials of degree 2 in J_G are the generators of J_G up to scaling.

Theorem 3.4. *Let G be a graph. The following conditions are equivalent:*

(1) G is closed on $[n]$;

(2) J_G has a quadratic Gröbner basis with respect to some term order $\prec$ on S.

Proof. (1)$\Rightarrow$ (2). See [10], Theorem 1.1.
(2)$\Rightarrow$ (1). By Proposition 3.2 $G_\prec$ is a directed acyclic graph. Hence there exists a labelling

$$\omega : V(G_\prec) \to [n]$$

such that for all $(i, j) \in E(G_\prec)$ we have that $\omega(i) < \omega(j)$. This means that ω is compatible with the orientation of $G_\prec$ (see for example [1], Proposition 1.4.3).

We will show that the graph G is closed with respect to the labelling ω.

Let $i_1, i_2, i_3 \in V(G_\prec)$ such that $\omega(i_1) = i$, $\omega(i_2) = j$, $\omega(i_3) = k$ and let $\{i_1, i_2\}$, $\{i_1, i_3\} \in E(G)$. It follows that $\{i, j\}$, $\{i, k\}$ are edges of G with respect to the labelling ω. By condition (2) of Theorem 1.1, we have to analyze the following two cases:

(a) $i < j$, $i < k$;

(b) $i > j$, $i > k$.

Case (a). Since ω is compatible with the oriented graph $G_\prec$, we have the following inequalities

$$x_{i_1}y_{i_2} \succ x_{i_2}y_{i_1} \text{ and } x_{i_1}y_{i_3} \succ x_{i_3}y_{i_1}. \tag{3.1}$$

By hypothesis the S-polynomial

$$S(f_{i_1i_2}, f_{i_1i_3}) = y_{i_1}f_{i_2i_3} = y_{i_1}(x_{i_2}y_{i_3} - x_{i_3}y_{i_2})$$

reduces to 0. Therefore there exists a binomial $x_{i_s}y_{i_t} - x_{i_t}y_{i_s} \in J_G$ (see Remark 3.3) whose leading monomial divides the leading monomial of $y_{i_1}f_{i_2i_3}$. Suppose that $\mathrm{in}(f_{i_1i_2}) = x_{i_2}y_{i_1}$. This contradicts the first inequality in (3.1). By the same argument and the second inequality in (3.1), $\mathrm{in}(f_{i_1i_3})$ does not divide $\mathrm{in}(y_{i_1}f_{i_2i_3})$. Hence $f_{i_2i_3} \in J_G$ and $\{j, k\}$ is an edge of G with respect to the labelling ω. Case (b) follows by similar arguments.

4 Closed graphs and linear quasi-tree complexes

In this section we introduce the notion of a simplicial complex which is a linear quasi-tree. This class of simplicial complexes is a subclass of the quasi-forest complexes (Definition 2.3). Our aim is to underline the close link that there exists between the closed graphs and these simplicial complexes. First of all we recall the following definition ([12], Definition 2.1).

Definition 4.1. A graph G is *closed* if there exists a labelling for which it is closed.

We quote the next result from ([12], Theorem 2.2).

Theorem 4.2. *Let G be a graph on $[n]$. The following conditions are equivalent:*

(1) G is closed;

(2) there exists a labelling of G such that all facets of $\Delta(G)$ are intervals $[a, b] \subseteq [n]$.

Moreover, if the equivalent conditions hold and the facets $F_1, \ldots, F_r$ of $\Delta(G)$ are labeled such that $\min(F_1) < \min(F_r) < \cdots < \min(F_q)$, then $F_1, \ldots, F_r$ is a leaf order of $\Delta(G)$.

Since a graph is closed if and only if each connected component is closed we assume from now on that the graph G is connected.

Thanks to Theorem 4.2 if G is a closed graph on the vertex set $[n]$ and $\Delta(G)$ is the clique complex, then we may assume that

$$\Delta(G) = \langle [m_1, M_1], [m_2, M_2], \ldots, [m_r, M_r] \rangle, \tag{4.1}$$

with $1 = m_1 < m_2 < \ldots < m_r < n$, $1 < M_1 < M_2 < \ldots < M_r = n$ with $m_i < M_i$ and $m_{i+1} \leq M_i$, for $i \in [r]$.

Now we introduce a special subclass of the quasi-trees complexes.

Definition 4.3. A simplicial complex is a linear quasi-tree if there exists an order on the facets

$$F_1, \ldots, F_q$$

such that

(1) F_i is a leaf for the subcomplex $\langle F_i, \ldots, F_q \rangle$;

(2) F_{i+1} is the only branch of F_i for all $i < q$.

Remark 4.4. Let Δ be a simplicial complex and let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_q\}$ be the set of its facets. It is always possible to verify if Δ is a linear quasi tree and in the positive case it is possible to order $\mathcal{F}(\Delta)$ so that conditions (1) and (2) of Definition 4.3 are satisfied. In fact, if Δ is a linear quasi tree, then there exists a leaf F_i, that is a facet of Δ satisfying Definition 2.2. In order to determine F_i it is sufficient to intersect the facet F_i, $i = 1, \ldots, q$, with the other facets. Let F_{i_1} be such a facet and let F_{i_2} be its branch. It must be unique by (2) of Definition 4.3.
If F_{i_1} is a leaf and F_{i_2} is its unique branch, then we consider the subcomplex $\Delta' = \mathcal{F}(\Delta) \setminus \{F_{i_1}\}$ and we verify if F_{i_2} is a leaf of Δ' and if its branch is unique and so on. Proceeding in this way we will obtain a linear order $F_{i_1}, F_{i_2}, \ldots, F_{i_q}$ with respect to which Δ is a linear quasi tree.
We will show this process by the next example.

Example 4.5. Let $\Delta = \langle F_1, F_2, F_3, F_4 \rangle$, with $F_1 = \{a, b, f\}$, $F_2 = \{a, e, f\}$, $F_3 = \{b, c, f\}$ and $F_4 = \{d, e, f\}$. We want to determine a order on the facet set $\mathcal{F}(\Delta)$ so that Δ is a linear quasi tree.
Consider the facet F_1. We have:

$$F_1 \cap F_2 = \{a, f\}, \quad F_1 \cap F_3 = \{b, f\}, \quad F_1 \cap F_4 = \{f\}.$$

Since $F_1 \cap F_2$ and $F_1 \cap F_3$ are not comparable, then F_1 is not a leaf of Δ (Definition 2.2). Consider the facet F_2. We have:

$$F_2 \cap F_1 = \{a, f\}, \quad F_2 \cap F_3 = \{f\}, \quad F_2 \cap F_4 = \{e, f\}.$$

Since $F_2 \cap F_1$ and $F_2 \cap F_4$ are not comparable, then F_2 is not a leaf of Δ (Definition 2.2). Now consider the facet F_3. We have:

$$F_3 \cap F_1 = \{b, f\}, \quad F_3 \cap F_2 = \{f\}, \quad F_3 \cap F_4 = \{f\}.$$

Hence F_1 is the unique branch of F_3 and consequently F_3 is a leaf of Δ.
Now consider the subcomplex of Δ: $\Delta' = \langle F_1, F_2, F_4 \rangle$. We have:

$$F_1 \cap F_2 = \{a, f\}, \quad F_1 \cap F_4 = \{f\}.$$

It follows that F_2 is the unique branch of F_1 and F_1 is a leaf of Δ'. It is easy to observe that we can conclude that Δ is a linear quasi tree with respect to the following order on $\mathcal{F}(\Delta)$: F_3, F_1, F_2, F_4.

From now on when we consider a simplicial complex Δ that is a linear quasi-tree we write $\Delta = \langle F_1, \dots, F_q \rangle$ with leaf order $\{F_1, F_2, \dots, F_q\}$ on the facet set. We state the following.

Proposition 4.6. *Let G be a graph on [n]. If G is a closed graph, then $\Delta(G)$ is a linear quasi-tree.*

Proof. From (4.1), since G is closed, we may assume $\Delta(G) = \langle F_1, \dots, F_r \rangle$, where $F_i = [m_i, M_i]$, for $i = 1, \dots, r$.

We observe that $[m_i, M_i] \cap [m_{i+1}, M_{i+1}] = [m_{i+1}, M_i]$. Since $m_{i+d} > m_{i+1}$ for all $d \geq 2$, then

$$F_i \cap F_{i+d} = [m_{i+d}, M_i] \subsetneq [m_{i+1}, M_i].$$

Therefore F_i is a leaf and F_{i+1} is the unique branch for F_i. □

Example 4.7. The converse of Proposition 4.6 is not true. In fact there are linear quasi-trees that are not closed.

Let $V(G) = \{a, b, c, d, e, f\}$ and let $\Delta(G) = \langle F_1, F_2, F_3 \rangle$ be the facet set of its clique complex, where $F_1 = \{a, b, c\}$, $F_2 = \{b, c, d, e\}$ and $F_3 = \{b, e, f\}$. We can easily check that $\langle F_1, F_2, F_3 \rangle$ is a linear quasi-tree but the subgraph induced by the vertices $\{a, b, d, f\}$ is a claw, i.e. the complete bipartite graph $K_{1,3}$. Therefore by ([10], Proposition 1.2) G is not closed.

We finish this section giving a criterion for the closedness of a graph with respect to a given labelling that will be crucial in the sequel.

Let G be a graph on the vertex set $V(G) = [n]$. For each vertex $j \in V(G)$ we define a partition of its neighborhood $N_G(j) = \{i \in [n] : \{i, j\} \in E(G)\}$ into two sets as follows:

$$N_G(j) = N_G^{<}(j) \cup N_G^{>}(j),$$

where

$$N_G^<(j) = \{i : \{i,j\} \in E(G), i < j\},\ N_G^>(j) = \{k : \{j,k\} \in E(G), j < k\}.$$

Proposition 4.8. *Let G be a graph on $[n]$. The following conditions are equivalent:*

(1) G is closed with respect to the given order of the vertices;

(2) for all vertices $j \in V(G)$ the sets $N_G^<(j)$, $N_G^>(j)$ are cliques of G.

Proof. (1) $\Rightarrow$ (2). Let $j \in V(G)$. For all i_1, $i_2 \in N_G^<(j)$, by definition, we have that $\{i_1, j\}$, $\{i_2, j\} \in E(G)$ with $i_1 < j$ and $i_2 < j$. Since G is closed, then $\{i_1, i_2\} \in E(G)$. Hence $N_G^<(j)$ is a clique. Similarly for $N_G^>(j)$.
(2) $\Rightarrow$ (1). Let $\{j, k_1\}$, $\{j, k_2\} \in E(G)$ with $j < k_1$, $j < k_2$. This implies $k_1, k_2 \in N_G^>(j)$. Since $N_G^>(j)$ is a clique, then $\{k_1, k_2\} \in E(G)$. The other case follows by similar argument. □

5 Closed graphs with respect to any labelling

In this section we give a characterization of closed graphs which does not depend on the labelling of their vertex sets. For this reason we study the clique complex $\Delta(G)$ of the simple graph G.

Let $\Delta = \langle F_1, \ldots, F_r \rangle$ be a simplicial complex. We set

$$F_{i_1,i_2,\ldots,i_s} := F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_s}$$

with $1 \le i_1 < i_2 < \ldots < i_s \le r$ and $F_{i,i} := F_i$ for $i \in [r]$.

Proposition 5.1. *If $\Delta = \langle F_1, \ldots, F_r \rangle$ is a linear quasi-tree, then $F_{i,j} = F_{i,i+1,\ldots,j}$, $1 \le i < j \le r$. In particular, $F_{k,\ell} \supseteq F_{i,j}$ for all k, ℓ such that $i \le k \le \ell \le j$.*

Proof. We proceed by descending induction on i, for $i < j$. If $i = j - 1$ there is nothing to prove. Let $i \le j - 1$ and suppose $F_{i,j} = F_{i,i+1,\ldots,j}$. We have to prove that $F_{i-1,j} = F_{i-1,i,i+1,\ldots,j}$.

Since $F_{i-1,i,j} = F_{i-1} \cap F_{i,j} = F_{i-1,i,i+1,\ldots,j}$, we need to show that

$$F_{i-1} \cap F_{i,j} = F_{i-1,j}.$$

By definition $F_{i-1,i,j} \subseteq F_{i-1,j}$. Since F_i is a branch of F_{i-1}, then $F_{i-1,j} \subseteq F_{i-1,i}$. Hence $F_{i-1,j} \cap F_j \subseteq F_{i-1,i} \cap F_j$, that is, $F_{i-1,j} \subseteq F_{i-1,i,j}$ and the assertion follows. □

Denote by $\mathcal{P} = \{F_{i,j} : 1 \le i \le j \le r\}$ the poset whose order is given by the inclusion and set $F_{i,j} = \emptyset$ if either $i < 1$ or $j > r$. If $F, G \in \mathcal{P}$ are not comparable or $F \neq \emptyset$ or $G \neq \emptyset$, we write $F \not\sim G$.

Definition 5.2. Let $\Delta = \langle F_1, \dots, F_r \rangle$ be a linear quasi-tree. Δ is called closed if the following properties are satisfied:

(I) $F_{i,j} \not\sim F_{k,\ell}$ if $i < k$, $j < \ell$, $i, j, k, \ell \in [r]$ (incomparability);

(C) $F_{i+1,i+d} = F_{i,i+d} \cup F_{i+1,i+d+1}$ if $F_{i,i+d+1} \neq \emptyset$ with $d \geq 1$ and $i \in [r]$ (covering).

Theorem 5.3. *Let G be a graph on $[n]$. If G is closed, then $\Delta(G)$ is closed.*

Proof. Since, from Proposition 4.6, $\Delta(G)$ is a linear quasi-tree, we have only to prove that the facet set $\mathcal{F}(\Delta(G)) = \{F_1, \dots, F_r\}$ satisfies properties (I) and (C) in Definition 5.2.

(I). Since G is closed on $[n]$, if $F_{i,j} \neq \emptyset$ and $F_{k,\ell} \neq \emptyset$, from (4.1) we have:

$$F_{i,j} = F_i \cap F_j = [m_i, M_i] \cap [m_j, M_j] = [m_j, M_i],$$

$$F_{k,\ell} = F_k \cap F_\ell = [m_k, M_k] \cap [m_\ell, M_\ell] = [m_\ell, M_k],$$

with $i < j$ and $k < \ell$. We may assume $i < k$ and $j < \ell$. Hence by (4.1) $m_j < m_\ell$ and $M_i < M_k$. Therefore $M_k \in F_{k,\ell} \setminus F_{i,j}$ and $m_j \in F_{i,j} \setminus F_{k,\ell}$, that is $F_{i,j} \nsim F_{k,\ell}$.

(C). Since $F_{i,i+d+1} \neq \emptyset$ and G is closed, then

$$F_{i,i+d+1} = [m_{i+d+1}, M_i] \neq \emptyset.$$

Therefore $m_{i+d+1} \leq M_i$, and

$$F_{i,i+d} \cup F_{i+1,i+d+1} = [m_{i+d}, M_i] \cup [m_{i+d+1}, M_{i+1}] = [m_{i+d}, M_{i+1}] = F_{i+1,i+d}.$$

□

To prove that $\Delta(G)$ closed implies G closed we need a labelling on the vertices of G for which G is closed.

Lemma 5.4. *Let $\Delta(G) = \langle F_1, \dots, F_r \rangle$ be a linear quasi-tree. Set $n_i = \max\{j : F_{i,j} \neq \emptyset, j \in [r]\}$. Then $n_1 \leq n_2 \leq \cdots \leq n_r$ and every set $F_{i,j}$ in*

$$\mathcal{B} = \{F_{1,1}, \dots, F_{1,n_1}, F_{2,n_1}, \dots, F_{2,n_2}, \dots, F_{r,r}\}$$

is not empty.

Proof. Since $F_{i,n_i} \neq \emptyset$, then $F_{i+1,n_i} \neq \emptyset$ (Proposition 5.1). Hence $n_{i+1} \geq n_i$. Moreover, by Proposition 5.1, we can also state that every set in $\mathcal{B}$ is not empty. □

Now we are in position to state the main result in the paper.

Theorem 5.5. *Let G be a graph. Suppose that $\Delta(G)$ is closed. Let $F_1, \dots, F_r$ be the leaf order of $\Delta(G)$ and consider the family*

$$\mathcal{F} = \{F'_{i,j}\}_{F_{i,j} \in \mathcal{B}},$$

where $\mathcal{B}$ is defined as in Lemma 5.4 and $F'_{i,j} = F_{i,j} \setminus (F_{i-1,j} \cup F_{i,j+1})$. Then

(1) The family $\mathcal{F}$ is a partition of $V(G)$;

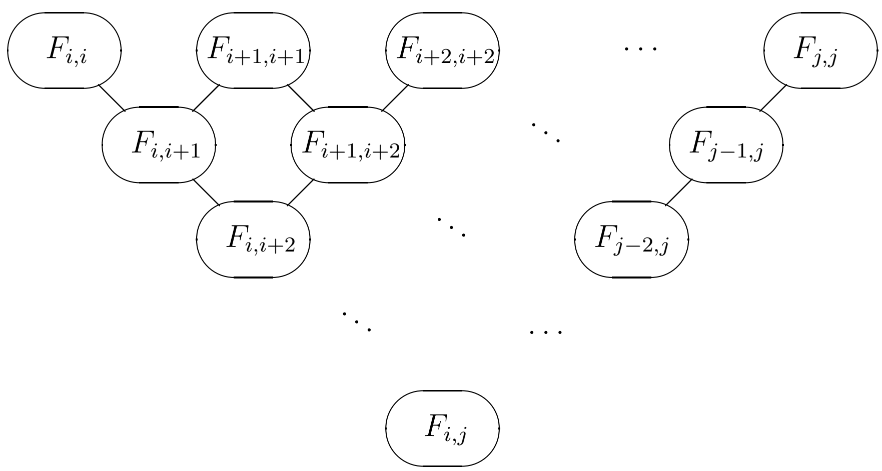
(2) G is closed with respect to the following total order on the vertices: For the vertices in each $F'_{i,j}$ we fix an arbitrary total order and set $u < v$, if $u \in F'_{i,j}$ and $v \in F'_{k,\ell}$ with $i < k$ or $i = k$ and $j < \ell$.

Proof. (1). First of all, we prove the following claim.

Claim 5.6. Let $F_{i,j} \neq \emptyset$ then

$$F_i \cup F_j = \bigcup_{k=i}^{j} F_k = \left(\bigcup_{k=i}^{j-1} F_{i,k}\right) \cup \left(\bigcup_{k=i+1}^{j} F_{k,j}\right) \cup F_{i,j}.$$

Proof of the Claim. Let $i \leq k \leq \ell \leq j$. Since, by assumption $F_{i,j} \neq \emptyset$, then Proposition 5.1 implies that $F_{k,\ell} \neq \emptyset$.
By condition (I) in Definition 5.2 and Proposition 5.1, $F_{k,\ell} \subseteq F_{i,j}$ if and only if $1 \leq i \leq k \leq \ell \leq j \leq r$. Hence the poset $\mathcal{P}_{ij} = \{F_{k,\ell} : i \leq k \leq \ell \leq j\}$, whose partial order is given by the inclusion, is the following:



We observe that $\bigcup_{k=i}^{j} F_k = \bigcup_{F \in \mathcal{P}_{i,j}} F$. Since $F_{k-1,k+1} \neq \emptyset$, for $k = i+1, \ldots, j-1$, then by condition (C) we have $F_{k,k} = F_{k-1,k} \cup F_{k,k+1}$, that is

$$\bigcup_{k=i}^{j} F_k = \bigcup_{F \in \mathcal{P}'_{i,j}} F$$

with $\mathcal{P}'_{i,j} = \mathcal{P}_{i,j} \setminus \{F_k : k = i+1, \ldots, j-1\}$. By similar argument we may subtract all the redundant elements $F_{k,\ell}$ with $i < k < \ell < j$. Hence

$$\bigcup_{k=i}^{j} F_k = \left(\bigcup_{k=i}^{j-1} F_{i,k}\right) \cup \left(\bigcup_{k=i+1}^{j} F_{k,j}\right) \cup F_{i,j} = F_i \cup F_j,$$

and Claim 5.6 is proved.

Let $\mathcal{P} = \{F_{i,j} : 1 \leq i \leq j \leq r\}$ be the poset induced by the inclusion. We say that an element $F_{i,j} \in \mathcal{P}$ is an *inner* element if $F_{i-1,j+1} \in \mathcal{P}$ and $F_{i-1,j+1} \neq \emptyset$. Otherwise an element of $\mathcal{P}$ is said to be *border* element.

We observe that the border elements are exactly the elements of $\mathcal{B}$ described in Lemma 5.4, and

$$V(G) = \bigcup_{F_{i,j} \in \mathcal{B}} F_{i,j}. \tag{5.1}$$

In fact if $v \in V(G)$, then $v \in F_{k,k} \in \mathcal{F}(\Delta(G))$. If $F_{k,k} \in \mathcal{B}$ we have nothing to prove. Suppose $F_{k,k} \notin \mathcal{B}$ then $F_{k-1,k+1} \neq \emptyset$ and, since $\Delta(G)$ is closed by property (C) $F_{k,k} = F_{k-1,k} \cup F_{k,k+1}$. We may assume $v \in F_{k-1,k}$. If $F_{k-1,k} \notin \mathcal{B}$ applying the same argument after a finite number of steps we obtain $v \in F_{i,j} \in \mathcal{B}$. If we remove the redundant elements in (5.1) we obtain

$$V(G) = \bigcup_{F_{i,j} \in \mathcal{B}} F'_{i,j},$$

where $F'_{i,j} = F_{i,j} \setminus (F_{i-1,j} \cup F_{i,j+1})$. We observe the following

$$\text{if } v \in F'_{i,j}, \text{ then } v \in F_k \text{ if and only if } k = i, \ldots, j. \tag{5.2}$$

This assertion can be deduced from the structure of the poset $\mathcal{P}$. For sake of completeness we give a direct proof. Since $v \in F'_{i,j}$ then $v \in F_{i,j}$ and by Proposition 5.1, $v \in F_k$ with $k = i, \ldots, j$. Suppose that $v \in F_\ell$, with $\ell > j$. Then $v \in F_i \cap F_\ell = F_{i,\ell}$. Therefore $v \in F_{i,\ell} \subsetneq F_{i,j}$ and this is a contradiction since $v \in F_{i,j} \setminus F_{i,j+1}$ and $F_{i,j+1} \supseteq F_{i,\ell}$.

By (5.2), it easily follows that

$$\mathcal{F} = \{F'_{i,j}\}_{F_{i,j} \in \mathcal{B}},$$

is a partition of $V(G)$.

(2). We prove that G is closed with respect to the labelling induced by the ordering defined in the statement. By Proposition 4.8 it is sufficient to prove that for every $v \in V(G)$, $N_G^<(v)$, $N_G^>(v) \in \Delta(G)$. Since $v \in V(G)$, then $v \in F'_{i,j} \in \mathcal{F}$. We claim that $N_G^>(v) \subseteq F_j$, $N_G^<(v) \subseteq F_i$.

Let $\{v, w\} \in E(G)$ with $v < w$, we want to prove that $\{v, w\} \subseteq F_j$. Since $v \in F'_{ij}$ by (5.2) the only cliques containing v are $F_i, \ldots, F_j$. Therefore, since $\{v, w\}$ is contained in a clique of G, then $\{v, w\} \subseteq F_i \cup F_{i+1} \cup \ldots \cup F_j$. By Claim 5.6 $\{v, w\} \subseteq F_i \cup F_j$. Since $v < w$, we have the following cases:

(a) $w \in F'_{i,j}$;

(b) $w \in F'_{k,\ell}$, with $k > i$;

(c) $w \in F'_{k,\ell}$, with $k = i$ and $\ell > j$.

(a). Obvious. (b). If $w \in F'_{k,\ell}$ with $k > i$, then we have that $w \notin F_i$, by (5.2). Hence $w \in F_j$. (c). If $w \in F'_{i,\ell}$ with $\ell > j$, then we have that $w \in F_j$, by (5.2).

By the same argument we prove that $N_G^<(v) \subset F_i$. $\square$

Corollary 5.7. *Let G be a graph. The following conditions are equivalent:*

(1) The graph G is closed on $[n]$;

(2) the clique complex $\Delta(G)$ is closed;

(3) the binomial edge ideal J_G has a quadratic Gröbner basis.

Proof. The equivalence follows from Theorems 5.3, 5.5 and 3.4. $\square$

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