Distinct triangle areas in a planar point set over finite fields

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Submitted: May 24, 2011; Accepted: Oct 18, 2011; Published: Oct 31, 2011
Mathematics Subject Classification: 05C35, 05C38

Abstract

Let $\mathcal{P}$ be a set of n points in the finite plane $\mathbb{F}_q^2$ over the finite field $\mathbb{F}_q$ of q elements, where q is an odd prime power. For any $s \in \mathbb{F}_q$, denote by $A(\mathcal{P}; s)$ the number of ordered triangles whose vertices in $\mathcal{P}$ having area s. We show that if the cardinality of $\mathcal{P}$ is large enough then $A(\mathcal{P}; s)$ is close to the expected number $|\mathcal{P}|^3/q$.

1 Introduction

Let $\mathcal{P}$ be a set of n points in the plane $\mathbb{R}^2$. We consider all the triangles whose vertices are any three non-collinear points of $\mathcal{P}$. We regard these triangles as triangles determined by the set $\mathcal{P}$. Denote by $g(\mathcal{P})$ the number of distinct areas of triangles determined by $\mathcal{P}$. For every $n \in \mathbb{N}$, let $g(n)$ be the minimum of $g(\mathcal{P})$ over all sets $\mathcal{P}$ of n noncollinear points in the plane. The first estimates on $g(n)$ were given by Erdős and Purdy [7], who proved that

$$c_1 n^{3/4} \leq g(n) \leq c_2 n, \tag{1.1}$$

for some absolute constant $c_1, c_2 > 0$. The upper bound ([1]) can be archived by taking $\mathcal{P}$ to be a set of equally space $\lceil \frac{n}{2} \rceil$ points on a line l together with $\lfloor \frac{n}{2} \rfloor$ equally spaced points on a line l' parallel to l. It gives $\lfloor \frac{n-1}{2} \rfloor$ triangles of distinct areas. Motivated by this example, Erdős, Purdy, and Straus [8] conjectured the following:

Conjecture 1.1 ([8]) *For every n, $g(n) = \lfloor \frac{n-1}{2} \rfloor$.*

A linear lower bound was first established by Burton and Purdy [1]. More precisely, they showed that $g(n) \geq 0.32n$. Dumitrescu and Cs. Tóth [5] improved this bound to

$g(n) \geq \frac{17}{38}n - O(1) \approx 0.04473$. Pinchasi [18] settled Conjecture 1.1 using an ingenious counting argument. Recently, a stronger version of the result of Pinchasi is proved by Iosevich and Rudnev [13].

The remarkable results of Bourgain, Katz and Tao [2] on sum-product problem and its application in Erdős distance problem over finite fields have stimulated a series of studies of finite field analogues of classical discrete geometry problems, see [3, 4, 6, 9, 11, 12, 14, 15, 16, 13, 19, 20, 21, 22, 23, 24] and references therein. The main purpose of this note is to study the finite field analogue of $g(n)$. Since there are only q possible areas for triangles in finite plane $\mathbb{F}_q^2$, one may expect that if $\mathcal{P}$ is large, then $\mathcal{A}(\mathcal{P})$ covers the whole or a positive proportion of $\mathbb{F}_q$.

Let $\mathcal{P}$ be a set of n points in the finite plane $\mathbb{F}_q^2$ over the finite field $\mathbb{F}_q$ of q elements, where q is an odd prime power. The area of triangles determined by three points $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$, $\boldsymbol{z} = (z_1, z_2) \in \mathbb{F}_q^2$ is defined as usual

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix} = \frac{1}{2}(\boldsymbol{x} * \boldsymbol{y} + \boldsymbol{y} * \boldsymbol{z} + \boldsymbol{z} * x), \tag{1.2}$$

where $(a_1, a_2) * (b_1, b_2) = a_1 b_2 - a_2 b_1 \in \mathbb{F}_q$. For any $s \in \mathbb{F}_q$, denote by $A(\mathcal{P}; s)$ the number of ordered triangles (i.e. triangles are determined by an ordered triple of vertices) whose vertices in $\mathcal{P}$ having area s. Our main result is the following theorem.

Theorem 1.2 *Let $\mathcal{P}$ be a set of n points in the finite plane $\mathbb{F}_q^2$. Suppose that $s \neq 0$ and $|\mathcal{P}| \gg q^{3/2}$ then $A(\mathcal{P}; s) = (1 + o(1))|\mathcal{P}|^3/q$. Moreover, if $|\mathcal{P}| \gg q^{5/3}$ then $A(\mathcal{P}; 0) = (1 + o(1))|\mathcal{P}|^3/q$.*

We conjecture that the exponent $3/2$ can be further improved to $1 + \epsilon$, or at least, we hope to show that if $|\mathcal{P}| \gg q^{1+\epsilon}$ then we can find a triangle of an arbitrary area from the set $\mathcal{A}$. Note that one can not go lower than 1 as we can take q points on a line.

In the nondegenerate case $s \neq 0$, the above result follows immediately from Hart and Iosevich's result ([9]) on the problem where one of the vertices of the triangle is assumed to be at the origin. Our result in the degenerate case $s = 0$, however, is stronger than the related result in Hart and Iosevich ([9]).

2 Proof of Theorem 1.2

To prove the first part of Theorem 1.2 we will need the following lemma.

Lemma 2.1 *Let $\mathcal{P}$ be a set of n points in the finite plane. For any $s \in \mathbb{F}_q$, denote by $\mu(\mathcal{P}, s)$ the number of ordered pair $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{P} \times \mathcal{P}$ such that $\boldsymbol{x} * \boldsymbol{y} = 2s$. For any $s \neq 0$, then*

$$\left| \mu(\mathcal{P}, s) - \frac{|\mathcal{P}|^2}{q} \right| \leq 2\sqrt{q}|\mathcal{P}|.$$

Proof See the proof of Theorem 1.4 in [10]. Note that [10, Theorem 1.4] only states for dot product but the proof works transparently for any non-degenerate bilinear form, in particular, the $*$-product defined here. The reader also can find a graph-theoretic proof of this lemma in [24, Section 8]. $\square$

For any $\boldsymbol{x} \in \mathcal{P}$, denote $\mathcal{P} - \boldsymbol{x} = \{\boldsymbol{y} - \boldsymbol{x} : \boldsymbol{y} \in \mathcal{P}\}$. Then

$$A(\mathcal{P}; s) = \sum_{\boldsymbol{x} \in \mathcal{P}} \mu(\mathcal{P} - \boldsymbol{x}, s). \tag{2.1}$$

It follows from Lemma 2.1 that $\mu(\mathcal{P} - \boldsymbol{x}, s) = (1 + o(1))\frac{|\mathcal{P}|^2}{q}$ if $|\mathcal{P}| \gg q^{3/2}$. Together with (2.1), we have $A(\mathcal{P}; s) = (1 + o(1))|\mathcal{P}|^3/q$ if $|\mathcal{P}| \gg q^{3/2}$.

Next we will follow the methods in [19] to prove the second part of the theorem. Note that Lemma 2.1 does not work when $s = 0$, so we cannot use the above methods to study the case of triangles with zero area. Let Ψ be the set of all additive characters of $\mathbb{F}_q$, and let $\Psi^* \subset \Psi$ be the set of all nonprincipal characters. We recall the following identity

$$\sum_{\psi \in \Psi} \psi(z) = \begin{cases} q & z = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that if the field is $\mathbb{Z}_p$, then the characters are just $e^{\frac{2\pi i a}{p}}$ and the identity follows by summing up the geometric series. For more information about the additive characters, we refer the reader to [17, Section 11.1].

For any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{F}_q^2$ and $\lambda \in \mathbb{F}_q$, the product $*$ satisfies the following properties:

$$\begin{aligned} \boldsymbol{a} * \boldsymbol{b} &= -\boldsymbol{b} * \boldsymbol{a} \\ \boldsymbol{a} * \boldsymbol{b} + \boldsymbol{a} * \boldsymbol{c} &= \boldsymbol{a} * (\boldsymbol{b} + \boldsymbol{c}) \\ \boldsymbol{a} * (\lambda \boldsymbol{b}) &= \lambda(\boldsymbol{a} * \boldsymbol{b}). \end{aligned}$$

It follows from (1.2) and (2.2) that

$$A(\mathcal{P}; s) = \frac{1}{q} \sum_{\psi \in \Psi} \sum_{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}} \psi(\boldsymbol{x} * \boldsymbol{y} + \boldsymbol{y} * \boldsymbol{z} + \boldsymbol{z} * \boldsymbol{x} - 2s).$$

Separating the principle character gives

$$\begin{aligned} \left| A(\mathcal{P}; 0) - \frac{|\mathcal{P}|^3}{q} \right| &= \frac{1}{q} \left| \sum_{\psi \in \Psi^*} \sum_{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}} \psi(\boldsymbol{x} * \boldsymbol{y} + \boldsymbol{y} * \boldsymbol{z} + \boldsymbol{z} * \boldsymbol{x}) \right| \\ &\leq \frac{1}{q} \sum_{\psi \in \Psi^*} \left| \sum_{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}} \psi(\boldsymbol{x} * (\boldsymbol{y} - \boldsymbol{z}) + \boldsymbol{y} * \boldsymbol{z}) \right|. \end{aligned}$$

Applying the Cauchy-Schwartz inequality twice, we have

$$
\begin{aligned}
& \leq \left|A(\mathcal{P};0)-\frac{|\mathcal{P}|^3}{q}\right|^2 \\
& \leq \frac{q-1}{q^2}\sum_{\psi\in\Psi^*}\left|\sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathcal{P}}\psi(\boldsymbol{x}*(\boldsymbol{y}-\boldsymbol{z})+\boldsymbol{y}*\boldsymbol{z})\right|^2 \\
& < \frac{(q-1)|\mathcal{P}|}{q^2}\sum_{\psi\in\Psi^*}\sum_{\boldsymbol{x}\in\mathcal{P}}\left|\sum_{\boldsymbol{y},\boldsymbol{z}\in\mathcal{P}}\psi(\boldsymbol{x}*(\boldsymbol{y}-\boldsymbol{z})+\boldsymbol{y}*\boldsymbol{z})\right|^2 \\
& \leq \frac{(q-1)|\mathcal{P}|}{q^2}\sum_{\psi\in\Psi^*}\sum_{\boldsymbol{x}\in\mathbb{F}_q^2}\left|\sum_{\boldsymbol{y},\boldsymbol{z}\in\mathcal{P}}\psi(\boldsymbol{x}*(\boldsymbol{y}-\boldsymbol{z})+\boldsymbol{y}*\boldsymbol{z})\right|^2 \\
& = \frac{(q-1)|\mathcal{P}|}{q^2}\sum_{\psi\in\Psi^*}\sum_{\boldsymbol{x}\in\mathbb{F}_q^2}\sum_{\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}',\boldsymbol{z}'\in\mathcal{P}}\psi(\boldsymbol{x}*(\boldsymbol{y}-\boldsymbol{z}-\boldsymbol{y}'+\boldsymbol{z}')+\boldsymbol{y}*\boldsymbol{z}-\boldsymbol{y}'*\boldsymbol{z}') \\
& = \frac{(q-1)|\mathcal{P}|}{q^2}\sum_{\psi\in\Psi^*}\sum_{\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}',\boldsymbol{z}'\in\mathcal{P}}\psi(\boldsymbol{y}*\boldsymbol{z}-\boldsymbol{y}'*\boldsymbol{z}')\sum_{\boldsymbol{x}\in\mathbb{F}_q^2}\psi(\boldsymbol{x}*(\boldsymbol{y}-\boldsymbol{z}-\boldsymbol{y}'+\boldsymbol{z}')).
\end{aligned}
$$

By the orthogonality property of additive characters (2.2), we see that the inner sum vanishes if and only if $\boldsymbol{y}-\boldsymbol{z}=\boldsymbol{y}'-\boldsymbol{z}'$, in which case it equals q^2. This implies that

$$
\begin{aligned}
\left|A(\mathcal{P};0)-\frac{|\mathcal{P}|^3}{q}\right|^2 & \leq (q-1)|\mathcal{P}|\sum_{\psi\in\Psi^*}\sum_{\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}',\boldsymbol{z}'\in\mathcal{P},\boldsymbol{y}-\boldsymbol{z}=\boldsymbol{y}'-\boldsymbol{z}'}\psi(\boldsymbol{y}*\boldsymbol{z}-\boldsymbol{y}'*\boldsymbol{z}') \\
& \leq (q-1)|\mathcal{P}|\sum_{\psi\in\Psi}\sum_{\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}',\boldsymbol{z}'\in\mathcal{P},\boldsymbol{y}-\boldsymbol{z}=\boldsymbol{y}'-\boldsymbol{z}'}\psi(\boldsymbol{y}*\boldsymbol{z}-\boldsymbol{y}'*\boldsymbol{z}') \\
& = q(q-1)|\mathcal{P}|V, \qquad (2.3)
\end{aligned}
$$

where V is the number of solutions to the system

$$\boldsymbol{y}-\boldsymbol{z}=\boldsymbol{y}'-\boldsymbol{z}' \text{ and } \boldsymbol{y}*\boldsymbol{z}=\boldsymbol{y}'*\boldsymbol{z}' \qquad (2.4)$$

in $\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}',\boldsymbol{z}'\in\mathcal{P}$. We consider two cases.

Case I. Suppose that $\boldsymbol{y}=\boldsymbol{z}$ or $\boldsymbol{y}'=\boldsymbol{z}'$. Since $\boldsymbol{y}=\boldsymbol{z}$ if and only if $\boldsymbol{y}'=\boldsymbol{z}'$, this case contributes $|\mathcal{P}|^2$ solutions to the system (2.4).

Case II. Suppose that $\boldsymbol{y}\neq\boldsymbol{z}$ and $\boldsymbol{y}'\neq\boldsymbol{z}'$. The number of solutions to the system (2.4) in this case is bounded by the number of solutions to the equation

$$\boldsymbol{y}*\boldsymbol{z}=\boldsymbol{y}'*(\boldsymbol{y}'+\boldsymbol{z}-\boldsymbol{y})=\boldsymbol{y}'*(\boldsymbol{z}-\boldsymbol{y}) \qquad (2.5)$$

in $\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}'\in\mathcal{P}$, $\boldsymbol{y}\neq\boldsymbol{z}$. When $\boldsymbol{y},\boldsymbol{z}\in\mathcal{P}$, $\boldsymbol{y}\neq\boldsymbol{z}$ are fixed, the equation (2.5) has at most q solutions. Hence, this case contributes at most $q|\mathcal{P}|^2$ solutions to the system (2.4).

Putting these cases together, we have

$$W\leq(q+1)|\mathcal{P}|^3. \qquad (2.6)$$

It follows from (2.3) and (2.6) that

$$\left| A(\mathcal{P}; 0) - \frac{|\mathcal{P}|^3}{q} \right|^2 \leq q(q-1)(q+1)|\mathcal{P}|^3 < q^3|\mathcal{P}|^3, \tag{2.7}$$

which also implies that $A(\mathcal{P}; 0) = (1 + o(1))\frac{|\mathcal{P}|^3}{q}$ if $|\mathcal{P}| \gg q^{5/3}$. This completes the proof of Theorem 1.2.

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