# An Erdős-Ko-Rado theorem for multisets

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**Abstract**

Let $k$ and $m$ be positive integers. A collection of $k$-multisets from $\{1, \ldots, m\}$ is intersecting if every pair of multisets from the collection is intersecting. We prove that for $m \geq k+1$, the size of the largest such collection is $\binom{m+k-2}{k-1}$ and that when $m > k+1$, only a collection of all the $k$-multisets containing a fixed element will attain this bound. The size and structure of the largest intersecting collection of $k$-multisets for $m \leq k$ is also given.

## 1 Introduction

The Erdős-Ko-Rado Theorem [6] is an important result in extremal set theory that gives the size and structure of the largest pairwise intersecting $k$-subset system from $[n] = \{1, \ldots, n\}$. This theorem is commonly stated as follows:

**Theorem 1.1.** *Let $k$ and $n$ be positive integers with $n \geq 2k$. If $\mathcal{F}$ is a collection of intersecting $k$-subsets of $[n]$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

*Moreover, if $n > 2k$, equality holds if and only if $\mathcal{F}$ is a collection of all the $k$-subsets from $[n]$ that contain a fixed element from $[n]$.*

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Note that if $n < 2k$, any pair of $k$-subsets will be intersecting and so the largest intersecting collection will have size $\binom{n}{k}$.

A multiset is a generalization of a set in which an element may appear more than once. As with sets, the order of the elements is irrelevant. The cardinality of a multiset is the total number of elements including repetitions. A $k$-multiset system on an $m$-set is a collection of multisets of cardinality $k$ containing elements from $[m]$. We say that two multisets are intersecting if they have at least one element in common and that a collection of multisets is intersecting if every pair of multisets in the collection is intersecting.

In this paper, we give a generalization of the Erdős-Ko-Rado Theorem to intersecting multiset systems. Specifically, we prove the following two theorems for the cases when $m \geq k+1$ and $m \leq k$ respectively.

**Theorem 1.2.** *Let $k$ and $m$ be positive integers with $m \geq k+1$. If $\mathcal{A}$ is a collection of intersecting $k$-multisets of $[m]$, then*

$$|\mathcal{A}| \leq \binom{m+k-2}{k-1}.$$

*Moreover, if $m > k+1$, equality holds if and only if $\mathcal{A}$ is a collection of all the $k$-multisets from $[m]$ that contain a fixed element from $[m]$.*

If $m < k+1$, larger collections are possible. For example, if $m = k = 3$, the seven $k$-multisets containing either two or three distinct elements from $[m]$ will form an intersecting collection since each multiset contains more than half the elements from $[m]$. We will use $\mathcal{M}_{(>\frac{m}{2})}$ to denote the collection of all $k$-multisets that contain more than $\frac{m}{2}$ distinct elements from $[m]$ and $\mathcal{M}_{(\frac{m}{2})}$ to denote the collection of all $k$-multisets from $[m]$ containing exactly $\frac{m}{2}$ distinct elements. Then

$$\left|\mathcal{M}_{(\frac{m}{2})}\right| = \binom{m}{\frac{m}{2}}\binom{k-1}{k-\frac{m}{2}}$$

and

$$\left|\mathcal{M}_{(>\frac{m}{2})}\right| = \sum_{j=\lceil\frac{m+1}{2}\rceil}^{m}\binom{m}{j}\binom{k-1}{k-j}.$$

**Theorem 1.3.** *Let $k$ and $m$ be positive integers with $m \leq k$. If $\mathcal{A}$ is a collection of intersecting $k$-multisets of $[m]$, then:*

1. *If $m$ is odd, $|\mathcal{A}| \leq \left|\mathcal{M}_{(>\frac{m}{2})}\right|$ and equality holds if and only if $\mathcal{A} = \mathcal{M}_{(>\frac{m}{2})}$.*
2. *If $m$ is even, $|\mathcal{A}| \leq \left|\mathcal{M}_{(>\frac{m}{2})}\right| + \frac{1}{2}\left|\mathcal{M}_{(\frac{m}{2})}\right|$ and equality holds if and only if $\mathcal{A}$ consists of $\mathcal{M}_{(>\frac{m}{2})}$ and a maximal intersecting collection of $k$-multisets from $\mathcal{M}_{(\frac{m}{2})}$.*

A $k$-multiset on an $m$-set can be represented as an integer sequence of length $m$ with the integer in each position representing the number of repetitions of the corresponding element from $[m]$. For example, if $m = 6$, the multiset $\{1, 2, 2, 4\}$ can be represented by the integer sequence $(1, 2, 0, 1, 0, 0)$. For a $k$-multiset, the sum of the integers in the corresponding integer sequence will equal $k$.

Erdős-Ko-Rado type results for intersecting families of integer sequences are known (e.g. [9], [10], [11]). In these, the sum of the entries in the integer sequence is not restricted to $k$ and the definition of intersection is different from our definition for multisets. In [2], Anderson proves an Erdős-Ko-Rado type result for multisets but uses yet another definition of intersection. A definition of intersection equivalent to ours is used in several theorems for intersecting collections of vectors presented by Anderson in [3]. These theorems were originally written in terms of sets of noncoprime divisors of a number by Erdős et al. in [5] and [7], and again the sum of the entries is not restricted to $k$.

More recently, Brockman and Kay [4] proved the result in Theorem 1.2 provided that $m \geq 2k$. Mahdian [13] proved the bound on the size of a collection of intersecting $k$-multisets when $m > k$ using a method similar to Katona's cycle proof for sets [12]. Our results improve the bound on $m$ given in [4] and give the size and structure of the largest possible intersecting collection for all values of $m$ and $k$.

# 2 Proof of Theorem 1.2

Our proof of this theorem uses a homomorphism from a Kneser graph to a graph whose vertices are the $k$-multisets of $[m]$.

A Kneser graph, $K(n, k)$, is a graph whose vertices are all of the $k$-sets from an $n$-set, denoted by $\binom{[n]}{k}$, and where two vertices are adjacent if and only if the corresponding $k$-sets are disjoint. Thus an independent set of vertices in the Kneser graph is an intersecting $k$-set system. We will use $\alpha(K(n, k))$ to denote the size of the largest independent set in $K(n, k)$.

We now define a multiset analogue of the Kneser graph. For positive integers $k$ and $m$, let $M(m, k)$ be the graph whose vertices are the $k$-multisets from the set $[m]$, denoted by $\left(\binom{[m]}{k}\right)$, and where two vertices are adjacent if and only if the corresponding multisets are disjoint. For this graph, the number of vertices is equal to $\left(\binom{m}{k}\right) = \binom{m+k-1}{k}$ and an independent set is an intersecting $k$-multiset system.

Let $n = m + k - 1$. Then $K(n, k)$ has the same number of vertices as $M(m, k)$ and $B \cap [m] \neq \emptyset$ for any $B \in \binom{[n]}{k}$.

For a set $A \subseteq [m]$ of cardinality $a$ where $1 \leq a \leq k$, the number of $k$-sets, $B$, from $[n]$ such that $B \cap [m] = A$ will be equal to

$$\binom{n-m}{k-a} = \binom{k-1}{k-a}.$$

Similarly, the number of $k$-multisets from $[m]$ that contain all of the elements of $A$ and

no others will be equal to

$$\left(\binom{a}{k-a}\right) = \binom{a+(k-a)-1}{k-a} = \binom{k-1}{k-a}.$$

Hence there exists a bijection, $f : \binom{[n]}{k} \to \left(\binom{[m]}{k}\right)$, such that for any $B \in \binom{[n]}{k}$, the set of distinct elements in $f(B)$ will be equal to $B \cap [m]$.

If $A, B \in \binom{[n]}{k}$ are two adjacent vertices in the Kneser graph, then $(A \cap [m]) \cap (B \cap [m]) = \emptyset$ and hence $f(A) \cap f(B) = \emptyset$. Therefore $f(A)$ is adjacent to $f(B)$ if $A$ is adjacent to $B$ and so the bijection $f : \binom{[n]}{k} \to \left(\binom{[m]}{k}\right)$ is a graph homomorphism. In fact, $K(n,k)$ is isomorphic to a spanning subgraph of $M(m,k)$. Thus

$$\alpha(M(m,k)) \leq \alpha(K(n,k)).$$

From the Erdős-Ko-Rado Theorem, we have that if $n \geq 2k$,

$$\alpha(K(n,k)) = \binom{n-1}{k-1}.$$

Thus, for $m \geq k+1$,

$$\alpha(M(m,k)) \leq \binom{n-1}{k-1} = \binom{m+k-2}{k-1}.$$

An intersecting collection of $k$-multisets from $[m]$ consisting of all $k$-multisets containing a fixed element from $[m]$ will have size $\binom{m+(k-1)-1}{k-1} = \binom{m-k-2}{k-1}$. Therefore

$$\alpha(M(m,k)) = \binom{m+k-2}{k-1}$$

which gives the upper bound on $\mathcal{A}$ in Theorem 1.2.

To prove the uniqueness statement in the theorem, let $m > k+1$ and let $\mathcal{A}$ be an intersecting multiset system of size $\binom{m+k-2}{k-1}$. With the homomorphism defined above, the pre-image of $\mathcal{A}$ will be an independent set in $K(n,k)$ of size $\binom{n-1}{k-1}$. Since $m > k+1$ and $n = m+k-1$, it follows that $n > 2k$ so, by the Erdős-Ko-Rado theorem, $f^{-1}(\mathcal{A})$ will be a collection of all the $k$-subsets of $[n]$ that contain a fixed element from $[n]$. If the fixed element, $x$, is an element of $[m]$, then it follows from the definition of $f$ that every multiset in $\mathcal{A}$ will contain $x$. Thus $\mathcal{A}$ will be a collection of all the $k$-multisets from $[m]$ that contain a fixed element from $[m]$ as required. If $x \notin [m]$, then $f^{-1}(\mathcal{A})$ will include the sets $A = \{1, m+1, \ldots, n\}$ and $B = \{2, m+1, \ldots, n\}$ since $m > k+1$ implies that $m > 2$. But $f(A) \cap f(B) = \emptyset$ which contradicts our assumption that $\mathcal{A}$ is an intersecting collection of multisets. Therefore, when $m > k+1$, if $\mathcal{A}$ is an intersecting collection of multisets of the maximum possible size, then $\mathcal{A}$ is the collections of all $k$-multisets containing a fixed element from $[m]$. □

The case when $m = k+1$ is analogous to the case when $n = 2k$ in the Erdős-Ko-Rado theorem. The size of the largest possible intersecting collection is equal to $\binom{m+k-2}{k-1}$ but collections attaining this bound are not limited to those having a common element in all $k$-multisets.

# 3 Proof of Theorem 1.3

Although Theorem 1.2 is restricted to $m \geq k+1$, the inequality $\alpha(M(m,k)) \leq \alpha(K(n,k))$ still holds when $m \leq k$. However, the resulting inequality

$$\alpha(M(m,k)) \leq \binom{n}{k} = \binom{m+k-1}{k}$$

is not particularly useful since for $m > 1$ this bound is not attainable. Clearly, two multisets consisting of $k$ copies of different elements from $[m]$ will not intersect.

Before proceeding with our proof of Theorem 1.3, we define the support of a multiset. If $A$ is a $k$-multiset from $[m]$, the support of $A$, denoted by $S_A$, is the set of distinct integers from $[m]$ in $A$. Note that two $k$-multisets, $A, B \in \left(\binom{[m]}{k}\right)$, will be intersecting if and only if $S_A \cap S_B \neq \emptyset$ and that each $S_A$ will have a unique complement, $\overline{S_A} = [m] \backslash S_A$, in $[m]$.

Let $\mathcal{A}$ be an intersecting family of $k$-multisets of $[m]$ of maximum size and let $M \in \mathcal{A}$ be a $k$-multiset such that $|S_M| = \min\{\, |S_A| : A \in \mathcal{A}\}$. If $m = 2$, it is easy to see that the theorem holds, so we will assume that $m > 2$.

Suppose that $|S_M| < \frac{m}{2}$. Let $\mathcal{B}_1 = \{A \in \mathcal{A} : S_A = S_M\}$ and let $\mathcal{B}_2 = \{B \in \left(\binom{[m]}{k}\right) : S_B = \overline{S_M}\}$. Then $\mathcal{B}_1 \subseteq \mathcal{A}$ and $\mathcal{B}_2 \cap \mathcal{A} = \emptyset$.

We will now show that $\mathcal{A}' := (\mathcal{A} \backslash \mathcal{B}_1) \cup \mathcal{B}_2$ is an intersecting family of $k$-multisets from $[m]$ that is larger than $\mathcal{A}$. By construction, every multiset in $\mathcal{A} \backslash \mathcal{B}_1$ contains at least one element from $[m] \backslash S_M$, and $[m] \backslash S_M = S_B$ for all $B \in \mathcal{B}_2$. Thus $\mathcal{A}'$ is an intersecting collection of $k$-multisets.

Let $|S_M| = i$. Then

$$|\mathcal{B}_1| = \left(\binom{i}{k-i}\right) = \binom{k-1}{k-i}.$$

Since $\left|\overline{S_M}\right| = m - i$, it follows that

$$|\mathcal{B}_2| = \left(\binom{m-i}{k-(m-i)}\right) = \binom{k-1}{k-m+i}.$$

To show that $|\mathcal{A}'| > |\mathcal{A}|$, it is sufficient to show that

$$\binom{k-1}{k-m+i} > \binom{k-1}{k-i},$$

or equivalently, that

$$(k-i)!(i-1)! > (k-m+i)!(m-i-1)!\,.$$

Since $i < \frac{m}{2}$ and $m \leq k$, we have that $k - i > k - \frac{m}{2} > k - m + i \geq 1$. Therefore,

$$\begin{aligned}(k-i)!(i-1)! &= (k-i)(k-i-1)\dots(k-m+i+1)(k-m+i)!(i-1)! \\ &\geq (m-i)(m-i-1)\dots(i+1)(k-m+i)!(i-1)! \\ &= \frac{m-i}{i}(m-i-1)!(k-m+i)! \\ &> (m-i-1)!(k-m+i)!\end{aligned}$$

as required. Thus if $\mathcal{A}$ is of maximum size, it cannot contain a multiset with less than $\frac{m}{2}$ distinct elements from $[m]$.

It is easy to see that any $k$-multiset containing more than $\frac{m}{2}$ distinct elements from $[m]$ will intersect with any other such $k$-multiset. This completes the proof of the theorem for the case when $m$ is odd. When $m$ is even, it is necessary to consider the $k$-multisets which contain exactly $\frac{m}{2}$ distinct elements, that is, the $k$-multisets in $\mathcal{M}_{(\frac{m}{2})}$. These multisets will intersect with any multiset containing more than $\frac{m}{2}$ distinct elements. However, $\mathcal{M}_{(\frac{m}{2})}$ is not an intersecting collection. For any $A \in \mathcal{M}_{(\frac{m}{2})}$, all of the $k$-multisets, $B$, where $S_B = \overline{S_A}$ will be in $\mathcal{M}_{(\frac{m}{2})}$ and will not intersect with $A$. Since the size of a maximal intersecting collection of $\frac{m}{2}$-subsets of $[m]$ is $\frac{1}{2}\binom{m}{\frac{m}{2}}$ and each $\frac{m}{2}$-subset is the support for the same number of multisets in $\mathcal{M}_{(\frac{m}{2})}$, an intersecting collection of $k$-multisets will contain at most half of the $k$-multisets in $\mathcal{M}_{(\frac{m}{2})}$. □

# 4 Further work

An obvious open problem is determining the size and structure of the largest collection of $t$-intersecting $k$-multisets, i.e. collections of multisets where the size of the intersection for every pair of multisets is at least $t$. (We define the intersection of two multisets to be the multiset containing all elements common to both multisets with repetitions.) The following conjecture is a version of Conjecture 5.1 from [4].

**Conjecture 4.1.** *Let $k$, $m$ and $t$ be positive integers with $t \leq k$ and $m \geq t(k-t)+2$. If $\mathcal{A}$ is a collection of intersecting $k$-multisets of $[m]$, then*

$$|\mathcal{A}| \leq \binom{m+k-t-1}{k-t}.$$

*Moreover, if $m > t(k-t)+2$, equality holds if and only if $\mathcal{A}$ is a collection of all the $k$-multisets from $[m]$ that contain a fixed $t$-multiset from $[m]$.*

The lower bound on $m$ in this conjecture was obtained by substituting $m + k - 1$ for $n$ in the corresponding bound for sets given by Frankl [8] and Wilson [14]. The conjecture is supported by the fact that when $m > t(k-t)+2$, a collection consisting of all $k$-multisets containing a fixed $t$-multiset is larger than a collection consisting of all $k$-multisets containing $t+1$ elements from a set of $t+2$ distinct elements of $[m]$ and that these two collections are equal in size when $m = t(k-t)+2$. Furthermore, when

$m = t(k-t)+1$, collections larger than $\binom{m+k-t-1}{k-t}$ are possible. For example, if $t = 2$, $k = 5$ and $m = 7$, the cardinality of the collection of all $k$-multisets containing three or more elements from $\{1, 2, 3, 4\}$ is 91 while $\binom{m+k-t-1}{k-t} = 84$.

The existence of a graph homomorphism from the Kneser graph $K(n, k)$ to its multiset analogue $M(m, k)$ in the proof of Theorem 1.2 gave a simple and straight-forward way to show that the size of the largest independent set in $M(m, k)$ is no larger than the size of the largest independent set in $K(n, k)$. These graphs can be generalized as follows: let $K(n, k, t)$ be the graph whose vertices are the $k$-subsets of $[n]$ and where two vertices, $A, B$, are adjacent if $|A \cap B| < t$ and let $M(m, k, t)$ be the graph whose vertices are the $k$-multisets of $[m]$ and where two vertices, $C, D$ are adjacent if $|C \cap D| < t$.

If a bijective homomorphism from $K(n, k, t)$ to $M(m, k, t)$ exists, it could be used to prove a bound not only on the maximum size of a $t$-intersecting collection as given in Conjecture 4.1 but also on the maximum size when $k - t \leq m \leq t(k-t)+2$ using the Complete Erdős-Ko-Rado theorem of Ahlswede and Khachatrian [1]. However, it is not clear that such a homomorphism exists. The conditions placed on the bijection in the proof of Theorem 1.2 are not sufficient to ensure that the bijection is a homomorphism since for two $k$-multisets, $A$ and $B$, having $|S_A \cap S_B| < t$ does not imply that $|A \cap B| < t$.

The simple fact that if a graph $G$ is isomorphic to a spanning subgraph of a graph $H$, then $\alpha(H) \leq \alpha(G)$ may be useful in proving Erdős-Ko-Rado theorems for different objects. It would be interesting to determine if there are combinatorial objects other than multisets which have this relationship to an object for which an Erdős-Ko-Rado type result is known.

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# References

[1] R. Ahlswede and L.H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18(2):125–136, 1997.

[2] I. Anderson. An Erdős-Ko-Rado theorem for multisets. *Discrete Math.*, 69(1):1–9, 1988.

[3] I. Anderson. *Combinatorics of finite sets*. Dover Publications Inc., Mineola, NY, 2002.

[4] G. Brockman and B. Kay. Elementary techniques for Erdős-Ko-Rado-like theorems. *arXiv:* 0808.0774, August 2008.

[5] P. Erdős, M. Herzog, and J. Schönheim. An extremal problem on the set of noncoprime divisors of a number. *Israel J. Math.*, 8:408–412, 1970.

[6] P. Erdős, Chao Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[7] P. Erdős and J. Schönheim. On the set of non pairwise coprime divisors of a number. In *Combinatorial theory and its applications, I (Proc. Colloq., Balatonfüred, 1969)*, pages 369–376. North-Holland, Amsterdam, 1970.

[8] P. Frankl. The Erdős-Ko-Rado theorem is true for $n = ckt$. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 365–375. North-Holland, Amsterdam, 1978.

[9] P. Frankl and Z. Füredi. The Erdős-Ko-Rado theorem for integer sequences. *SIAM J. Algebraic Discrete Methods*, 1(4):376–381, 1980.

[10] P. Frankl and N. Tokushige. The Erdős-Ko-Rado theorem for integer sequences. *Combinatorica*, 19(1):55–63, 1999.

[11] H.-D. O. F. Gronau. More on the Erdős-Ko-Rado theorem for integer sequences. *J. Combin. Theory Ser. A*, 35(3):279–288, 1983.

[12] G. O. H. Katona. A simple proof of the Erdős-Chao Ko-Rado theorem. *J. Combinatorial Theory Ser. B*, 13:183–184, 1972.

[13] M. Mahdian. private communication, 2010.

[14] R.M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984.