# Equitable partitions into spanning trees in a graph*

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**Abstract**

In this paper we first prove that if the edge set of an undirected graph is the disjoint union of two of its spanning trees, then for every subset $P$ of edges there exists a spanning tree decomposition that cuts $P$ into two (almost) equal parts. The main result of the paper is a further extension of this claim: If the edge set of a graph is the disjoint union of two of its spanning trees, then for every stable set of vertices of size 3, there exists such a spanning tree decomposition that cuts the stars of these vertices into (almost) equal parts. This result fails for 4 instead of 3. The proofs are elementary.

**Keywords:** disjoint spanning trees, base partitions of matroids

## 1 Introduction

An undirected graph $G = (V, E)$ is a **2-tree-union** if $E$ is the disjoint union of the edge sets of two spanning trees of $G$. A coloring of the edges of a 2-tree-union to red and blue is a **2-tree-coloring** if both the red and the blue edges form a spanning tree. If $G = (V, E)$ is a 2-tree-union and $\mathcal{P}$ is a collection of disjoint subsets of $E$, then a 2-tree-coloring of $E$ is **equitable to** $\mathcal{P}$, if in every element of $\mathcal{P}$ the number of red and blue colors differ in at most 1. We say that a 2-tree-union $G = (V, E)$ is $k$-**equitable** if for any sub-partition $\mathcal{P}$ of $E$ consisting of $k$ disjoint subsets of edges, $G$ has a 2-tree-coloring equitable to $\mathcal{P}$. An

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edge set $F \subseteq E$ is a **star** if all the edges in $F$ have an end vertex, the **center**, in common. We say that a 2-tree-union $G = (V, E)$ is **$k$-star-equitable** if for any sub-partition $\mathcal{P}$ of $E$ consisting of $k$ disjoint *stars*, $G$ has a 2-tree-coloring equitable to $\mathcal{P}$.

In this paper we consider the question of equitability and star-equitability. In Section 2 we prove that 2-tree-unions are 1-equitable (Theorem 2.3), but not necessarily 2-equitable. In Section 3 we prove that 2-tree-unions are $k$-star-equitable for $k = 1, 2, 3$ (Theorem 3.1), but not necessarily 4-star-equitable. The proof of Theorem 3.1 is elementary but quite involved.

The **star of a vertex** $v \in V$, denoted by $\Delta(v) \subseteq E$, consists of the edges of $G$ incident to $v$. Theorem 3.1 implies that if $s_1$, $s_2$, $s_3$ are independent vertices in a 2-tree-union $G$, then $G$ has a 2-tree-coloring equitable to $\{\Delta(s_1), \Delta(s_2), \Delta(s_3)\}$. However, the same statement with four vertices would be false, by the counterexample in the beginning of Section 3.

Observe that our results imply also that if the edge set $E$ of an undirected graph can be partitioned into $l$ spanning trees, then one can choose such a partition to be equitable to a given set $P \subseteq E$, or to be equitable to a given sub-partition $\mathcal{P}$ of $E$ consisting of at most three stars.

The following conjecture gave some motivation to the above questions.

**Conjecture 1.1** ([3], Exercise 4.69)**.** Let $G = (V, E)$ be an undirected graph. For $X \subseteq V$ let $i_G(X)$ denote the number of edges of $G$ induced by $X$. If $|E| = 2|V| - 2$, $i_G(X) \leq 2|X| - 3$ for every $X \subsetneq V$, $|X| \geq 2$, and every vertex of $G$ has degree at most 4, then $E$ can be partitioned into two Hamiltonian paths.

Observe that a partition of $E$ into two Hamiltonian paths is just a partition into two spanning trees equitable to the set of all stars. Thus it would be interesting to investigate equitable partitions in 2-tree-unions which satisfy properties like $i_G(X) \leq 2|X| - 3$ or connectivity requirements.

The question of the paper can also be put in a matroidal setting. Call a matroid a **2-base** if its ground set is the disjoint union of two of its bases. If $M = (E, r)$ is a 2-base and $\mathcal{P}$ is a sub-partition of $E$, then call a partition of $E$ into two bases $B_1$, $B_2$ **equitable to $\mathcal{P}$** if $||B_1 \cap P| - |B_2 \cap P|| \leq 1$ for all $P \in \mathcal{P}$; and call a 2-base $M = (E, r)$ **$k$-equitable** if for any $k$-element sub-partition $\mathcal{P}$ of $E$ there is a partition of $E$ into two bases equitable to $\mathcal{P}$. Observe that the cycle matroid of a 2-tree-union is a 2-base.

It is an intriguing open problem whether every 2-base matroid is 1-equitable. This definitely holds for graphic matroids by Theorem 2.3. It is also true for weakly base orderable matroids, as one can greedily modify the current base decomposition $E = B_1 \dot\cup B_2$ to decrease $||B_1 \cap P| - |B_2 \cap P||$, until the bases cut $P$ into two (almost) equal parts. Finally, we mention a result of Davies and McDiarmid [1], who proved that if $M_1$ and $M_2$ are two strongly base orderable matroids on the same ground set $E$, and both of them can be partitioned into $l$ bases, then $E$ can be partitioned into $l$ common bases. It follows that 2-base strongly base orderable matroids are $k$-equitable for any $k$.

The paper was also motivated by coverings of common independent sets of two matroids, a problem to which no characterization is known yet. One solved example is

Kőnig's edge-coloring theorem ([4], see [6, p. 321]). If $G(V_1, V_2; E)$ is a bipartite graph, and we define matroids $M_1$ and $M_2$ on $E$ with $I \subseteq E$ independent in $M_i$ if $\deg_I(v) \leq 1$ for all $v \in V_i$, then Kőnig's edge-coloring theorem states that $E$ can be covered by $\Delta$ common independent sets if and only if both $M_1$ and $M_2$ can be covered by $\Delta$ independent sets. Another example is Edmonds' arborescence theorem ([2], see [6, p. 904]), stating that if a directed graph $D$ can be partitioned into $k$ undirected trees and every in-degree is $k$, except at a specified vertex $r$ where it is 0, then $D$ can be partitioned into $k$ arborescences rooted at $r$. In other words, the cycle matroid of $D$ has a partition into $k$ bases equitable to the sub-partition with classes the in-stars of the vertices.

In the rest of this paper all graphs $G$ are undirected. If $G = (V, E)$ is a graph and $X, Y \subseteq V$ are disjoint vertex sets, then $i_G(X)$ denotes the number of edges of $G$ induced by $X$; edge $e$ **enters** $X$ if exactly one end-vertex of $e$ is contained in $X$; $d_G(X, Y)$ denotes the number of edges between $X$ and $Y$; and $\delta_G(X) = d_G(X, V - X)$.

## 2 2-tree-unions are 1-equitable

We need some preliminaries on 2-tree-unions. First observe that a 2-tree-union may have double parallel edge pairs but no loops. A characterization of 2-tree-unions was given by Nash-Williams [5].

**Theorem 2.1** (Nash-Williams [5]). *The graph $G = (V, E)$ is a 2-tree-union if and only if $|E| = 2|V| - 2$ and $i_G(X) \leq 2|X| - 2$ for all $\emptyset \neq X \subseteq V$.*

We call a set $X \subseteq V$ **tight** if $i_G(X) = 2|X| - 2$. By the supermodularity of $i_G$ the next claim follows easily.

**Claim 2.2.** *If $G$ is a 2-tree-union then the union of two intersecting tight sets is tight. Moreover, if $X$ is tight and $u \notin X$ then $d_G(X, u) \leq 2$.*

**Pinching** edges $e$ and $f$ in a graph means subdividing these edges with two new vertices $v_e$ and $v_f$, and then identifying these nodes with one new node $v_{ef} = v_e = v_f$. Note that $\deg(v_{ef}) = 4$.

Let $G = (V, E)$ be a 2-tree-union. An operation used throughout is the **split** at vertex $v \in V$, defined below. We call a split **admissible** if it results in a 2-tree-union. The inverse operation of a split is called **unsplit**.

- If $\deg_G(v) = 2$ then **splitting** $v$ means simply deleting $v$ from $G$. Clearly, $G - v$ is also a 2-tree-union and any 2-tree-coloring of $G - v$ can be extended to a 2-tree-coloring of $G$ in two ways, by arbitrary coloring one of the edges of $v$ to blue and the other one to red.
- If $\deg_G(v) = 3$ then let the edges incident to $v$ be $e_i$ joining $v$ to $u_i$ for $i = 1, 2, 3$. **Splitting the edge-pair** $e_i, e_j$ $(i \neq j)$ means deleting $v$ from $G$ and adding the $u_i u_j$-edge $e$, resulting in the graph $H$. We also say that we **split** $v$ **to a** $u_i u_j$**-edge**. Note that splitting the pair $e_i, e_j$ is admissible unless $G$ has a tight set $X$ such that

$u_i, u_j \in X$ and $v \notin X$. So Claim 2.2 clearly implies that for at least two choices of the edge-pair $e_i$, $e_j$ the graph $H$ is a 2-tree-union. In this case any 2-tree-coloring of $H$ can be extended to a 2-tree-coloring of $G$ in the following way (the **unsplitting at** $v$). If the split edge $e$ is, say, blue then delete $e$ from $H$, add $v$, add the edges $e_i$, $e_j$ colored blue and let the third edge incident to $v$ be red.

- If $\deg_G(v) = 4$ then let the edges incident to $v$ be $e_i$ joining $v$ to $u_i$ for $1 \leq i \leq 4$. **Splitting the edge-pair** $e_1$, $e_2$ means deleting $v$ from $G$ and adding the $u_1u_2$-edge $e$ and the $u_3u_4$-edge $f$ resulting in the graph $H$. We also say that we **split** $v$ **to a** $u_1u_2$**-edge and to a** $u_3u_4$**-edge**. It is easy to see that this split is admissible unless $G$ has a tight set $v \notin X$ such that either $u_1, u_2 \in X$ or $u_3, u_4 \in X$. By Claim 2.2 at least two splits give a 2-tree-union. In this case any 2-tree-coloring of $H$ can be extended to a 2-tree-coloring of $G$ in the following way (the **unsplitting at** $v$). First pinch $e$ and $f$ by vertex $v$. If $e$ and $f$ had different colors then we are done. Otherwise, say, both $e$ and $f$ were blue so we produced a circuit $C$ in the blue tree. Now re-color an edge of $C$ incident to $v$ to red.

Theorem 2.1 implies that a 2-tree-union with at least two edges has either a vertex of degree 2 or two vertices of degree 3. Thus it is always possible to perform an admissible split.

Now we prove that 2-tree-unions are 1-equitable. That they are not necessarily 2-equitable is shown by $K_4$ and the sub-partition of $E(K_4)$ consisting of two disjoint perfect matchings.

**Theorem 2.3.** *2-tree-unions are 1-equitable.*

*Proof.* Let $G = (V, E)$ be a 2-tree-union and $P \subseteq E$. We prove by induction on $E$ that $G$ has a 2-tree-coloring equitable to $P$. If $E = \emptyset$ then the statement is trivially true. Recall that by Theorem 2.1, $G$ has a vertex of degree at most 3.

Assume that $G$ has a vertex $v$ of degree 2. Let $\Delta(v) = \{e, f\}$. If $|\{e, f\} \cap P| \in \{0, 2\}$ then apply the induction hypothesis to $G - v$ and $P - \{e, f\}$. If, say, $e \in P$ and $f \notin P$ then by induction, $G - v$ has two disjoint spanning trees $F_1$ and $F_2$ equitable to $P - e$. Assume that, say, $|F_1 \cap (P - e)| \leq |F_2 \cap (P - e)|$. Now $F_1 + e$ and $F_2 + f$ are two disjoint spanning trees of $G$ equitable to $P$.

Assume now that $G$ has a vertex $v$ of degree 3. Let $\Delta(v) = \{e_1, e_2, e_3\}$ such that $e_i$ joins $v$ to $u_i \in V$ for $i = 1, 2, 3$. Recall that $G - v + u_iu_j$ is a 2-tree-union for at least two choices of $1 \leq i < j \leq 3$. We distinguish four cases.

- $|\Delta(v) \cap P| = 0$. Here we apply induction to any admissible split at $v$. Now the unsplitting at $v$ gives a 2-tree-coloring of $G$ equitable to $P$.

- $|\Delta(v) \cap P| = 1$. We can assume that, say, $G - v + u_1u_2$ is a 2-tree-union and $e_1 \in P$. We apply induction to $G - v + u_1u_2$ and $P - e_1 + u_1u_2$. Then the unsplitting at $v$ gives a 2-tree-coloring of $G$ equitable to $P$.

- $|\Delta(v) \cap P| = 2$. We can assume that, say, $G - v + u_1u_2$ is a 2-tree-union and $e_1, e_3 \in P$. We apply induction to $G - v + u_1u_2$ and $P - \{e_1, e_3\}$. As before, the unsplitting at $v$ gives a 2-tree-coloring of $G$ equitable to $P$.
- $|\Delta(v) \cap P| = 3$. We apply induction to an admissible split at $v$ to some edge $u_iu_j$ and $P - \Delta(v) + u_iu_j$.

□

## 3 2-tree-unions are 3-star-equitable

In this section we prove that 2-tree-unions are $k$-star-equitable for $k = 1, 2, 3$. On the other hand, the following 2-tree-union is not 4-star-equitable. Consider the 2-tree-union $H$ in Figure 1 and the sub-partition $\mathcal{P} = \{\{ae, ab\}, \{ce, cd\}, \{bf, bc\}, \{df, da\}\}$. One can check that $H$ has no 2-tree-coloring equitable to $\mathcal{P}$. Now pinch each edge pair $\{e_i^1, e_i^2\} \in \mathcal{P}$ by a new vertex $v_i$ for $1 \leq i \leq 4$ resulting in the 2-tree-union $G$. The set of new vertices is stable in $G$. Assume that $G$ has two disjoint spanning trees equitable to $\{\Delta(v_i) : 1 \leq i \leq 4\}$. Observe that each vertex $v_i$ is incident to exactly one parallel edge-pair. Contracting this edge-pair for each $v_i$ would give a 2-tree-coloring of $H$, which is impossible.

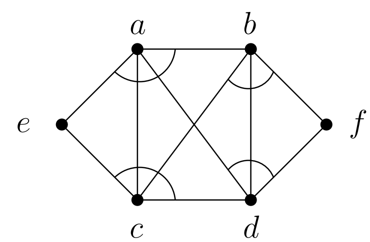


Figure 1: The graph $H$

**Theorem 3.1.** *Let $G$ be a 2-tree-union, $S \subseteq V$ with $|S| \leq 3$ and $\emptyset \neq P_s \subseteq \Delta(s)$ be a star for all $s \in S$. If the stars $P_s$ are disjoint then $G$ has a 2-tree-coloring equitable to $\mathcal{P} = \{P_s : s \in S\}$.*

Theorem 3.1 implies that 2-tree-unions are 3-star-equitable. Indeed, if the centers of the stars of $\mathcal{P}$ are different then we are done by Theorem 3.1. Otherwise, if the centers of $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ are the same vertex $v \in V$, then replace $v$ by two vertices $v_1$ and $v_2$ joined by a parallel edge-pair, and detach the incident edges of $v$ in such a way that $P_i \subseteq \Delta(v_i)$ holds for $i = 1, 2$.

### Proof of Theorem 3.1

The proof proceeds as follows. First we show some properties which a counterexample minimizing $|S| + |V|$ must have, and then we explore the possible connected components of an auxiliary graph $G_{\text{aux}}$ (definition below). Finally, using the description of the components of $G_{\text{aux}}$, we prove that no counterexample exists. For $S = \emptyset$ the statement clearly holds, so we assume otherwise.

**Definition 3.2.** Replace each vertex $s \in S$ by $\deg_G(s)$ vertices of degree 1, called **leaves**, see Figure 2. The resulting graph is $G_{\text{aux}}$. For a connected component $C$ of $G_{\text{aux}}$ let $V_2(C)$ denote the set of the non-leaf vertices of $C$, that is $V_2(C) = \{v \in V(C) : \deg_C(v) \geq 2\}$.

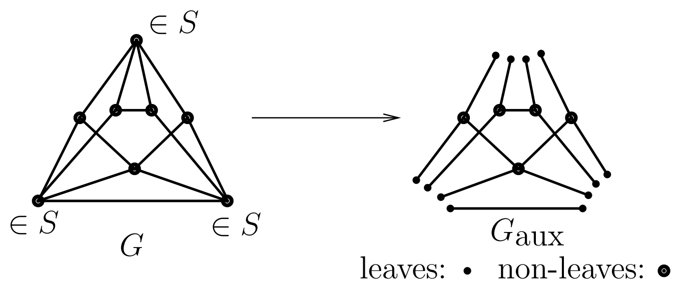


Figure 2: The construction of $G_{\text{aux}}$

**Definition 3.3.** The edges of $\bigcup\{P_s : s \in S\}$ are called **significant**. For a vertex $v \notin S$ we denote by $\text{s-deg}_G(v)$ the number of significant edges incident to $v$.

Let the pair $(G, \mathcal{P})$ be a counterexample to the theorem minimizing $|S| + |V|$. We may assume that $|P_s|$ is even for all $s \in S$. Otherwise, delete one edge from each $P_s$ of odd size, resulting in a new sub-partition $\mathcal{P}'$. Since each 2-tree-coloring of $G$ which is equitable to $\mathcal{P}'$ is also equitable to $\mathcal{P}$, we get that $(G, \mathcal{P}')$ is also a counterexample to the theorem. Thus we assume that $(G, \mathcal{P})$ is a counterexample to the theorem minimizing $|S| + |V|$, and $|P_s|$ is even for all $s \in S$.

**Proposition 3.4.** $\deg_G(s) \geq 4$ *for all* $s \in S$.

*Proof.* If $\deg_G(s) = 2$ then a 2-tree-coloring equitable to $\{P_t : t \in S - s\}$ guaranteed by the minimality of $(G, \mathcal{P})$ is equitable to $P_s$ as well. Similarly, if $\deg_G(s_1) = 3$ then a 2-tree-coloring equitable to $\{P_t : t \in S - s\}$ is equitable to $P_s$ as well, except possibly when $|P_s| = 2$. So assume that $P_s = \{e_1, e_2\}$ and $\Delta(s) = \{e_1, e_2, e_3\}$, where $e_i$ joins $s$ to $v_i$ for $i = 1, 2, 3$. Assume that, say, splitting the edge-pair $e_1$, $e_3$ to the $v_1v_3$-edge $f$ results in a 2-tree-union $H$. If $e_3 \in P_t$ for $t \in S - s$ then let $P_t^H = P_t - e_3 + f$, otherwise let $P_t^H = P_t$. By the minimality of $(G, \mathcal{P})$, the graph $H$ has a 2-tree-coloring equitable to $\{P_t^H : t \in S - s\}$. Now the unsplitting at $v$ results in a 2-tree-coloring of $G$ equitable to $\{P_t : t \in S - s\}$ such that also $e_1$, $e_2$ have different colors, a contradiction. $\square$

**Proposition 3.5.** *$G$ has at most one vertex of degree 2. If $v$ is such a vertex then* $\text{s-deg}_G(v) = 2$ *and the edges incident to $v$ are not parallel.*

*Proof.* Let $v \in V$ be a vertex with $\deg_G(v) = 2$. By Proposition 3.4, $v \notin S$. Let $P'_s = P_s - \Delta(v)$ for $s \in S$. Suppose first that $\text{s-deg}_G(v) \leq 1$. By the minimality of $G$, the graph $G - v$ has a 2-tree-coloring equitable to $\mathcal{P}' = \{P'_s : s \in S\}$, which can trivially be extended to a 2-tree-coloring of $G$ equitable to $\mathcal{P}$. We could also extend this 2-tree-coloring of $G - v$ equitable to $\mathcal{P}'$ if $\text{s-deg}_G(v) = 2$ and the edges of $v$ are parallel. So $\deg_G(v) = 2$ implies that $\text{s-deg}_G(v) = 2$ and that the edges incident to $v$ are not parallel. Suppose that $v_1$ and $v_2$ are two such vertices with neighbors $s_1$, $s_2$ and $s_1$, $s$, resp. If $s = s_2$ then a 2-tree-coloring of $G - \{v_1, v_2\}$ equitable to $\{P_s - \Delta(v_1) - \Delta(v_2) : s \in S\}$ can be easily extended to a 2-tree-coloring of $G$ equitable to $\mathcal{P}$. If $s = s_3$ then let $H$ be the following 2-tree-union: add to $G$ a vertex $v$ of degree 2 with neighbors $s_2$ and $s_3$ and delete $v_1$ and $v_2$. Let $P^H_{s_1} = P_{s_1} - v_1 s_1 - v_2 s_1$, $P^H_{s_2} = P_{s_2} - v_1 s_2 + v s_2$ and $P^H_{s_3} = P_{s_3} - v_2 s_3 + v s_3$. By the minimality of $G$, the graph $H$ has a 2-tree-coloring equitable to $\{P^H_{s_1}, P^H_{s_2}, P^H_{s_3}\}$, and this coloring can be easily extended to a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction. □

**Corollary 3.6.** *There exists at most one component $C$ of $G_{\text{aux}}$ such that $V_2(C)$ contains a vertex $v$ with $\deg_G(v) = 2$. Such a component is called the* **null-component***, and it has the property that $V_2(C) = \{v\}$, and that $v$ is adjacent to two distinct vertices in $S$.*

**Proposition 3.7.** $\deg_G(v) = 3$ *implies* $\text{s-deg}_G(v) \geq 2$.

*Proof.* Suppose that $\text{s-deg}_G(v) \leq 1$ and let the edges incident to $v$ be $e_1$, $e_2$, $e_3$ such that $e_2, e_3 \notin P_s$ for any $s \in S$. We may assume that splitting the edge-pair $e_1$, $e_2$ to edge $e$ results in a 2-tree-union $H$. For $s \in S$, if $e_1 \in P_s$ then let $P^H_s = P_s - e_1 + e$, otherwise let $P^H_s = P_s$. Now $H$ has a 2-tree-coloring equitable to $\{P^H_s : s \in S\}$ by the minimality of $G$. This coloring gives a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction. □

**Theorem 3.8.** *Only the following type of sets can be tight in $G$:*

1. *a singleton,*
2. *$V$,*
3. *$V - v$ where* $\deg_G(v) = 2$*,*
4. *$\{s, t\}$ such that $s, t \in S$ and $E$ contains a parallel st-edge-pair.*

*Proof.* The graph we get when contracting $X \subseteq V$ to one vertex and deleting the loops created is denoted by $G/X$. Suppose that $X \subseteq V$ is a tight set of $G$ not listed in the theorem. Observe that by Theorem 2.1 both $G/X$ and $G[X]$ are 2-tree-unions. We have four cases depending on the size of $X \cap S$.

- $X \cap S = \emptyset$. $X$ is not a singleton so by the minimality of $G$, the graph $G/X$ has a 2-tree-coloring equitable to $\mathcal{P}$. Extending this by an arbitrary 2-tree-coloring of $G[X]$ gives a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction.

- $|X \cap S| = 1$. Let $s \in X \cap S$. $X$ is not a singleton so by the minimality of $G$, the graph $G/X$ has a 2-tree-coloring equitable to $\{P_s \cap E(G/X)\} \cup \{P_t : t \in S - s\}$. Moreover, $G[X]$ has a 2-tree-coloring equitable to $P_1 \cap E(G[X])$. By possibly oppositely coloring the edges of $G[X]$, these give a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction.

- $|X \cap S| = 2$, see Figure 3. Observe that $|X| \geq 3$. Let $s_1, s_2 \in X \cap S$ and let $P_i^1 = E(G/X) \cap P_{s_i}$ and $P_i^2 = E(G[X]) \cap P_{s_i}$ for $i = 1, 2$. If $|S| = 3$ then denote the third vertex by $s_3 \notin X$. Denote the vertex of $G/X$ to which $X$ was contracted by $w_1$. Let $G_1$ be the following graph: add to $G/X$ a new vertex $w_2$, join it by two parallel edges $e_1$, $e_2$ to $w_1$ and re-join the edges of $P_2^1$ to $w_2$ instead of $w_1$. Moreover, let $G_2 = G[X]$. By the minimality of $G$, the graph $G_1$ has a 2-tree-coloring equitable to $\{P_1^1, P_2^1, P_{s_3}\}$ (or to $\{P_1^1, P_2^1\}$ if $|S| = 2$) and $G_2$ has a 2-tree-coloring equitable to $\{P_1^2, P_2^2\}$. If not all $|P_i^1|$, $|P_i^2|$ are odd for $i = 1, 2$, then by possibly oppositely coloring the edges of $G_2$, these 2-tree-colorings of $G_1$ and $G_2$ give a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction. If $|P_i^1|$, $|P_i^2|$ are odd for $i = 1, 2$, then let $G_2'$ be the graph we get when adding a new vertex $v$ to $G_2$ and joining it to $s_1$ and $s_2$. Note that $|V(G_2')| < |V|$ since $\deg_G(s_3) \geq 4$ by Proposition 3.4. Now $G_1$ has a 2-tree-coloring equitable to $\{P_1^1 + e_1, P_2^1 + e_2, P_{s_3}\}$ (or to $\{P_1^1, P_2^1\}$ if $|S| = 2$), and $G_2'$ has a 2-tree-coloring equitable to $\{P_1^2 + vs_1, P_2^2 + vs_2\}$. By a possible opposite coloring these 2-tree-colorings give a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction.

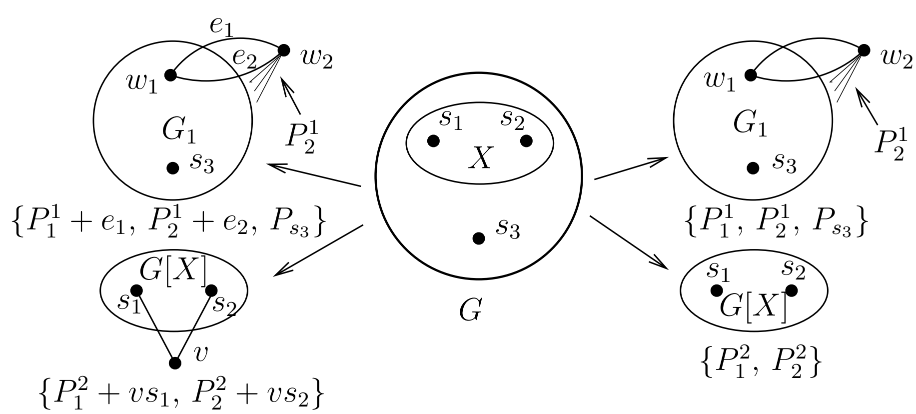


Figure 3: The case $|X \cap S| = 2$

- $|X \cap S| = 3$. Assume that $X$ is a maximal tight set which is not of the form $V$ or $V - v$ for $\deg_G(v) = 2$. Now there exists a component $C$ of $G_{\text{aux}}$ different from the null-component such that $Y = V_2(C) - X \neq \emptyset$. By Corollary 3.6, $\deg_G(y) \geq 3$ holds for all $y \in Y$. Suppose that $y \in Y$ is a vertex with $\deg_G(y) = 3$. Now s-$\deg_G(y) \geq 2$ holds by Proposition 3.7, hence $d_G(y, X) = 2$ by Claim 2.2. But then $Y - y \neq \emptyset$ so the tight set $X + y$ would contradict to the maximality of $X$. So even $\deg_G(y) \geq 4$

holds for all $y \in Y$, implying $2i_G(Y) + \delta_G(Y) \geq 4|Y|$. But now

$$\begin{aligned} i_G(X \cup Y) &= i_G(X) + (2i_G(Y) + \delta_G(Y)) - i_G(Y) \\ &\geq (2|X| - 2) + 4|Y| - (2|Y| - 2) \\ &= 2|X \cup Y|, \end{aligned}$$

contradicting Theorem 2.1.

□

**Corollary 3.9.** *$G$ contains no parallel edges except possibly induced by $S$.*

Corollary 3.9 implies that if $e$ is an $xy$-edge such that $\{x, y\} \not\subseteq S$, then it has multiplicity 1, hence we may use the notation '$xy$' for $e$.

**Corollary 3.10.** *If $v \in V - S$ and $\deg_G(v) = 3$ or $4$ then all three splits at $v$ give 2-tree-unions, except a split to a parallel st-edge-pair with $s, t \in S$.*

**Proposition 3.11.** *If $v \in V - S$ with $\deg_G(v) = 4$ then $\text{s-deg}_G(v) = 2$ or $3$.*

*Proof.* Corollary 3.9 implies that the edges incident to $v$ go to 4 distinct vertices $u_1$, $u_2$, $u_3$, $u_4$. This already excludes $\text{s-deg}_G(v) = 4$. Suppose that $\text{s-deg}_G(v) \leq 1$. We know that at least one split at $v$ gives a 2-tree-union, say, splitting $v$ to the $u_1u_2$-edge $e$ and to the $u_3u_4$-edge $f$ results in a 2-tree-union $H$. If $\text{s-deg}_G(v) = 1$ and, say, $u_1 = s \in S$ and $vs \in P_s$, then let $P'_s = P_s - vs + e$ and $P'_t = P_t$ for $t \in S - s$. If $\text{s-deg}_G(v) = 0$ then let $P'_t = P_t$ for $t \in S$. By the minimality of $G$, the graph $H$ has a 2-tree-coloring equitable to $\{P'_s : s \in S\}$. Now pinch the edges $e$ and $f$ by the vertex $v$ and if $e$ and $f$ had the same color, then re-color an edge $vu_i$ different from $vs$. This gives a 2-tree-coloring of $G$ equitable to $\mathcal{P}$. □

## The components of $G_{\text{aux}}$

Our next step in proving Theorem 3.1 is to describe the possible connected components of $G_{\text{aux}}$. There are altogether 24 of them.

**Definition 3.12.** For a component $C$ of $G_{\text{aux}}$ let $b = b(C) = |E(C)| - 2|V_2(C)|$. We denote the number of vertices $v \in V(C)$ with $\deg_C(v) = k$ by $d_k = d_k(C)$, especially, the number of leaves of $C$ by $\delta = \delta(C) = d_1$. Moreover, $V_4 = V_4(C) = \{v \in V(C) : \deg_G(v) \geq 4\}$.

Observe that $b(C) \geq 0$ because $|E(C)| = |E| - i_G(V - V_2(C)) \geq 2|V| - 2 - (2|V - V_2(C)| - 2) = 2|V_2(C)|$. E.g. the null-component $C$ has $b(C) = 0$. Besides,

$$\sum\{b(C) : C \text{ is a component of } G_{\text{aux}}\} = |E| - 2|V - S| = 2|S| - 2.$$

**Proposition 3.13.** $\sum\{\delta(C) : C$ *is a component of* $G_{\text{aux}}\} \geq 4|S|$.

*Proof.* $\sum_C \delta(C) = \sum\{\deg_G(s) : s \in S\} \geq 4|S|$ by Proposition 3.4. □

Next we list some properties of these components.

**Proposition 3.14.** *For each component $C$ of $G_{\text{aux}}$ the following properties hold.*

$$b \geq 1 \;\Rightarrow\; d_2 = 0 \tag{1}$$

$$\sum_{k\geq 2}(4-k)d_k = \delta - 2b \tag{2}$$

$$d_2 = 0 \;\Rightarrow\; \delta - 2b \leq d_3 \leq \left\lfloor \frac{\delta}{2} \right\rfloor \tag{3}$$

$$d_2 = 0\,,\;\; \delta - 2b \geq 3 \;\Rightarrow\; \delta - 2b + 1 \leq d_3 \tag{4}$$

$$b \geq 2 \;\Rightarrow\; d_3 \leq \delta - b - 2 \textit{ and equality implies } |V_4| = 1 \tag{5}$$

$$b \geq 1 \;\Rightarrow\; 2 \leq \delta \leq 4b \tag{6}$$

*Proof.* (1) If $\deg_C(v) = 2$ for $v \in V(C)$ then trivially $\deg_G(v) = 2$ so Corollary 3.6 implies that $C$ is the null-component, which has $b(C) = 0$.

(2) $\sum_{k\geq 2}(4-k)d_k = \sum\{4 - \deg_C(v) : v \in V_2(C)\} = 4|V_2(C)| - 2|E(C)| + \delta = \delta - 2b$.

(3) The lower bound is implied by (2) while the upper by Proposition 3.7.

(4) Let $C' = C[V_2(C)]$. Now $\delta - 2b \leq d_3$ holds by (3) so suppose that $\delta - 2b = d_3$. This implies that $\deg_C(v) \leq 4$ holds for all vertices $v \in V_2$ by (2). Moreover, $\deg_C(v) \geq 3$ for $v \in V_2$ since $d_2 = 0$. Let $v \in V_2$. Now $\deg_{C'}(v) \leq 1$ if $\deg_C(v) = 3$ by Proposition 3.7 and $\deg_{C'}(v) \leq 2$ if $\deg_C(v) = 4$ by Proposition 3.11. Thus the highest degree of $C'$ is at most 2. So $C'$ is a path or a circuit because it is connected. So $C'$ has at most 2 vertices of degree one hence $\delta - 2b = d_3 \leq 2$, a contradiction.

(5) Note that $d_2 = 0$ by (1). Let $v \in V_2$ be a vertex with $\deg_G(v) = 3$. If s-$\deg_G(v) = 3$ then $v$ and its three leaves would form a component $C$ of $G_{\text{aux}}$ with $b(C) = |E(C)| - 2|V_2(C)| = 1$. Thus s-$\deg_G(v) = 2$ holds by Proposition 3.7. Denote by $e$ the non-significant edge incident to $v$. If $e$ joins $v$ to a vertex $w \in V(C)$ with $\deg_G(w) = 3$, then the vertices $v$, $w$ together with the 4 incident leaves would form a component of $G_{\text{aux}}$ with $b = 1$. Hence $e$ joins $v$ to $V_4$ implying that $V_4 \neq \emptyset$ and $\delta_G(V_4) = \delta - d_3$. Moreover,

$$\delta - 2b = \sum_{k\geq 2}(4-k)d_k = d_3 + \sum\{4 - \deg_C(v) : v \in V_4(C)\}$$
$$= d_3 + 4|V_4| - 2i_G(V_4) - \delta_G(V_4).$$

So

$$\delta - b = d_3 + 2|V_4| - i_G(V_4) \geq d_3 + 2,$$

where the last inequality is due to Theorem 2.1. Equality holds only if $V_4$ is tight, and thus a singleton by Theorem 3.8.

(6) The lower bound is due to the 2-edge connectivity of $G$. The upper is implied by the inequalities $\delta - 2b \leq d_3$ (by Properties (1) and (3)) and $2d_3 \leq \delta$ (by Proposition 3.7). □

Now we are ready to describe the connected components of $G_{\text{aux}}$. These components are depicted in Figures 4–7 and in the rest of the proof of Theorem 3.1 we refer to them using the notations ***(a)*** - ***(x)*** of these figures. The leaves of the components are not shown in the figures at all, only their incident edges. The notations (1) – (6) refer to the statements of Proposition 3.14. Without even mentioning, we frequently use Corollary 3.9, Propositions 3.7, 3.11 and statements (1) – (6) of Proposition 3.14.

**Components with** $b = 0$

Assume that $d_2 = 0$. (3) yields that $d_3 \geq \delta - 2b = \delta \geq 2$. For all vertices $v \in V(C)$ with $\deg_G(v) = 3$ we know that s-$\deg_G(v) \geq 2$ by Proposition 3.7. Thus $\delta \geq 2d_3$, a contradiction. So $d_2 \geq 1$. Now Proposition 3.6 implies that $C$ is the null-component (see Figure 4 ***(a)***) and that $G_{\text{aux}}$ contains no other components with $b = 0$.

**Components with** $b = 1$

$\delta = 2$: Now $d_3 \leq 1$ by (3). If $v \in V(C)$ with $\deg_C(v) = 3$, then two edges would join $v$ to $S$ in $G$ and the third edge incident to $v$ would be a cut edge of $G$. So $d_3 = 0$. (2) implies that $\deg_C(v) = 4$ for all $v \in V_2(C)$. $d_4 \leq 1$ by Proposition 3.11 but $d_4 = 1$ is impossible. So $d_4 = 0$ and we get the edge-graph shown in 4 ***(b)***. Such a component comes from an edge induced by $S$.

$\delta = 3$: Now $d_3 = 1$ by (3). (2) implies that $\deg_C(v) = 4$ for all $v \in V_4(C)$ but $d_4 = 0$ by Proposition 3.11. So $|V_2| = 1$ and we get 4 ***(c)***. Observe that by Corollary 3.9, such a component comes from a subgraph of $G$ where the three edges are incident to three distinct vertices of $S$.

$\delta = 4$: (3) implies that $d_3 = 2$. $\deg_C(v) \leq 4$ for $v \in V(C)$ by (2) and $d_4 = 0$ by Proposition 3.11. Hence this component is as shown in 4 ***(d)***. There is a strong restriction on the position of this component in $G$. First, there are no parallel edges in $G$ by Corollary 3.9. Second, assume that $C$ comes from a subgraph depicted in 4 ***(d')***. Proposition 3.7 implies that $e_1, e_2 \in P_1$ and $e_3, e_4 \in P_2$. Now replace this subgraph by an edge $e$ joining $s_1$ to $s_2$ (that is with component ***(b)***) resulting in the 2-tree-union $H$. Let $P_1^H = P_1 - \{e_1, e_2\}$, $P_2^H = P_2 - \{e_3, e_4\}$ and $P_3^H = P_3$. By the minimality of $G$, $H$ has a 2-tree-coloring equitable to $\{P_1^H, P_2^H, P_3^H\}$, which gives a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, a contradiction. So only the subgraph of 4 ***(d”)*** remained.

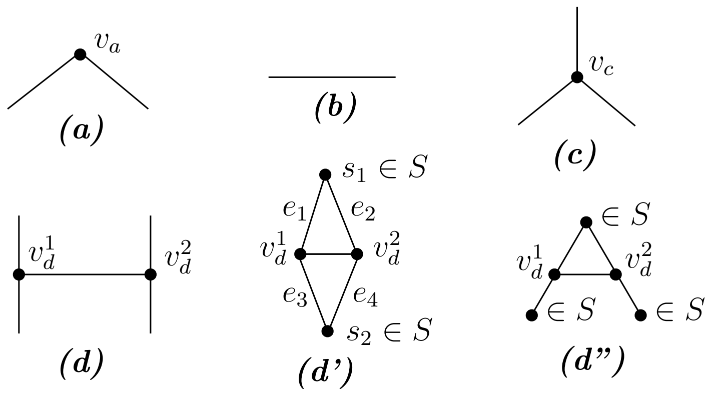


Figure 4: The components of $G_{\text{aux}}$, I. ***(a)*** has $b = 0$, the others $b = 1$

**Components with** $b = 2$

(3) and (5) give that $d_3 = \delta - 4$, moreover, $|V_4| = 1$ by (5). Let $V_4 = \{w\}$. Now $\deg_C(w) = 4$ by (2). Finally, $d_3 = \delta - 4$ gives that $\delta \geq 4$.

$\delta = 4$: $d_3 = 0$ so $\deg_C(w) = 4$ contradicts to Corollary 3.9.

$\delta = 5$: Now $d_3 = 1$, giving the component of Figure 5 ***(e)***.

$\delta = 6$: Now $d_3 = 2$, giving a component shown in Figure 5 ***(f)***.

$\delta = 7$: Contradicts to (4).

$\delta = 8$: Contradicts to (4).

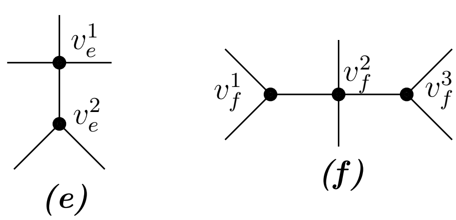


Figure 5: The components of $G_{\text{aux}}$, II. Components with $b = 2$

**Components with** $b = 3$

Such a component is accompanied in $G_{\text{aux}}$ with one component of $b = 1$ for which $\delta \leq 4$ by (6), and possibly with the null-component, where $\delta = 2$. So $\delta \geq 6$ holds by Proposition 3.13. (3) and (5) give that $\delta - 6 \leq d_3 \leq \delta - 5$. If $d_3 = \delta - 6$ then $\deg_C(v) = 4$ for all $v \in V_4$ by (2), and if $d_3 = \delta - 5$ then $V_4 = \{w\}$ by (5) and $\deg_C(w) = 5$ by (2).

$\delta = 6$: If $d_3 = 0$ then the vertices of $V_4$ are adjacent to altogether 6 leaves so $d_4 = 2$ or 3 by Proposition 3.11. The first case gives Figure 6 ***(g)*** and the second Figure 6 ***(h)***. Finally, if $d_3 = 1$ then $\deg_C(w) = 5$ would contradict to Corollary 3.9.

$\delta = 7$: If $d_3 = 1$ then the vertices of $V_4$ are adjacent to altogether 5 leaves so $|V_4| = 2$ by Proposition 3.11, and one vertex of $V_4$ is adjacent to 3 leaves and the other one to 2 leaves, see Figure 6 ***(i)***. If $d_3 = 2$ then we get Figure 6 ***(j)***.

$\delta = 8$: If $d_3 = 2$ then $|V_4| = 2$ by Proposition 3.11, and both vertices of $V_4$ are adjacent to 2 leaves, see Figure 6 ***(k)***. In the case $d_3 = 3$ we get Figure 6 ***(l)***.

$\delta = 9$: $d_3 = 3$ is excluded by (4) so $d_3 = 4$. Now we get Figure 6 ***(m)***.

$\delta = 10$: $d_3 = 4$ is excluded by (4) so $d_3 = 5$ yielding Figure 6 ***(n)***.

$\delta = 11$: $d_3 = 5$ by (3) but this is excluded by (4).

$\delta = 12$: $d_3 = 6$ by (3) but this is excluded by (4).

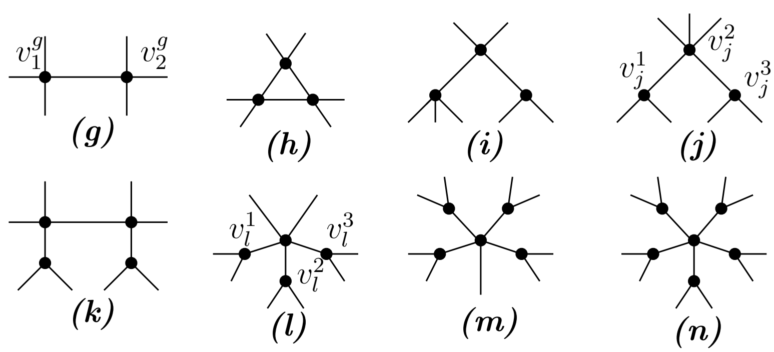


Figure 6: The components of $G_{\text{aux}}$, III. Components with $b = 3$

**Components with** $b = 4$

Such a component can be accompanied in $G_{\text{aux}}$ only with the null-component. The null-component has $\delta = 2$ so now $\delta \geq 10$ by Proposition 3.13. Moreover, $\delta \leq 16$ by (6). (3) and (5) give that $\delta - 8 \leq d_3 \leq \delta - 6$. If $d_3 = \delta - 8$ then $\deg_C(v) = 4$ for all $v \in V_4$ by (2). If $d_3 = \delta - 7$ then $\deg_C(w) = 5$ for a specified vertex $w \in V_4$ and $\deg_C(v) = 4$ for $v \in V_4 - w$ by (2). Finally, if $d_3 = \delta - 6$ then $V_4 = \{w\}$ by (5) and $\deg_C(w) = 6$ by (2).

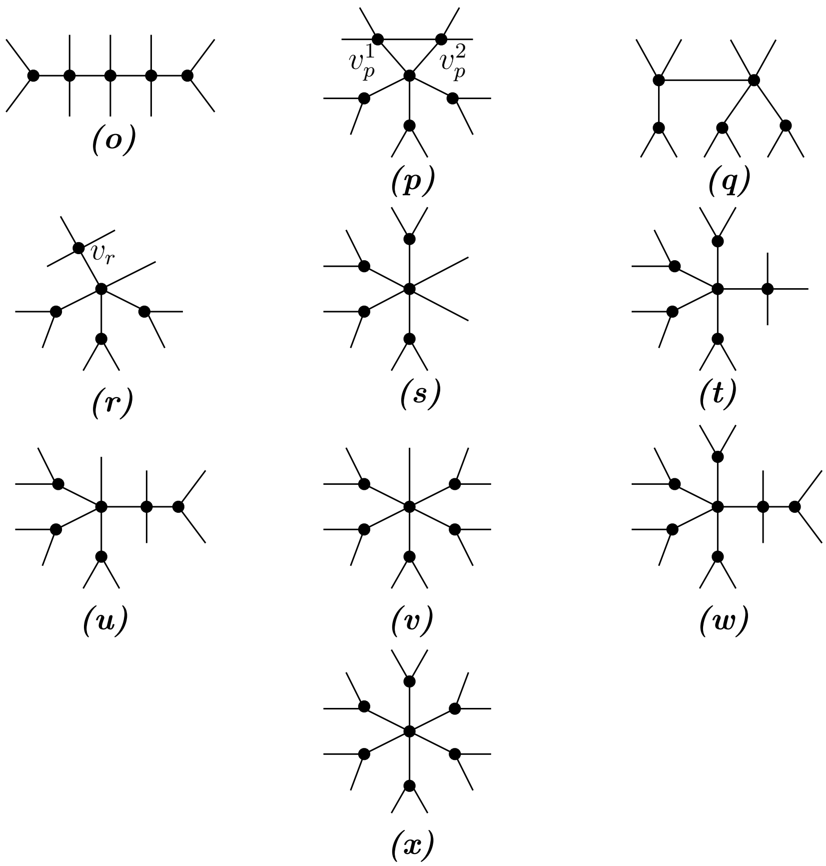


Figure 7: The components of $G_{\text{aux}}$, IV. Components with $b = 4$

$\delta = 10$: Let first $d_3 = 2$. The vertices of $V_4$ are adjacent to altogether 6 leaves so Proposition 3.11 yields that $d_4 = 2$ or 3. Now $d_4 = 2$ would give a disconnected graph and $d_4 = 3$ gives Figure 7 ***(o)***. If $d_3 = 3$ then $d_4 \leq 2$ by Proposition 3.11. Now $d_4 = 2$ gives Figure 7 ***(p)***, $d_4 = 1$ gives Figure 7 ***(q)*** and ***(r)***, while $d_4 = 0$ is impossible. Finally, $d_3 = 4$ gives Figure 7 ***(s)***.

$\delta = 11$: $d_3 = 3$ is excluded by (4). If $d_3 = 4$ then the vertices of $V_4$ are adjacent to altogether 3 leaves so $d_4 \leq 1$ by Proposition 3.11. Now $d_4 = 1$ gives Figure 7 ***(t)*** and Figure 7 ***(u)***, while $d_4 = 0$ is impossible. $d_3 = 5$ gives Figure 7 ***(v)***.

$\delta = 12$: $d_3 = 4$ is excluded by (4). If $d_3 = 5$ then $d_4 \leq 1$ by Proposition 3.11. Now $d_4 = 1$ gives Figure 7 ***(w)***, while $d_4 = 0$ is impossible. Finally, $d_3 = 6$ gives Figure 7 ***(x)***.

$\delta = 13$: $d_3 = 6$ by (4) and by the upper bound of (3). $d_4 = 0$ by Proposition 3.11, so this graph would be disconnected.

$\delta = 14$: $d_3 = 7$ by (3) and (4). $d_4 = 0$ by Proposition 3.11, so this graph would be disconnected.

$\delta = 15$: Impossible by (3) and (4).

$\delta = 16$: Impossible by (3) and (4).

## Reductions to smaller graphs

Using the above description of the components we enumerate all possibilities for $G_{\text{aux}}$. For two cases of $G_{\text{aux}}$ we cannot do else than directly giving a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, see Figures 12–13. However, in all the other cases we prove that we can apply admissible splits to $G$ to reduce the problem to a smaller 2-tree-union $H$ with sub-partition $\mathcal{P}^H = \{P_s^H : s \in S\}$. We use Reductions 1 – 3 below. These reductions all have the property that if $H$ is really a 2-tree-union, then a 2-tree-coloring of $H$ equitable to $\mathcal{P}^H$ can be extended to a 2-tree-coloring of $G$ equitable to $\mathcal{P}$. So our only task will be to prove that $H$ is indeed a 2-tree-union. Since proving that $H$ is a 2-tree-union will be always easy, we will not consider this issue, we only show a general scheme after Reduction 2 and an example in Figure 11.

We will apply the following reductions. We use Corollary 3.9 and Propositions 3.7, 3.11 without mentioning. In Figures 8–13 the vertices of $S$ are shown as big dots and *each edge $vs \in P_s$ is indicated by an arrow showing from $v$ to $s$.*

**Reduction 1.** (Figure 8.) Let $x_1, x_2 \in V - S$ be two vertices such that $\deg_G(x_i) \in \{3, 4\}$, s-$\deg_G(x_1) = 2$, s-$\deg_G(x_2) \leq 3$ and $x_i s \in P_s$, $x_i t \in P_t$ for $s, t \in S$. We pose the restriction that if $x_1$ and $x_2$ are adjacent in $G$, then $\deg_G(x_2) = 4$ must hold. Now first split $x_1$ to the $st$-edge $e_1$ resulting in the graph $G_2$. Then in $G_2$ split $x_2$ to the $st$-edge $e_2$ (note that if $x_1$ and $x_2$ are adjacent in $G$ and $\deg_G(x_1) = 3$, then $\deg_{G_2}(x_2) = 3$ holds.) The second splitting results in the graph $H$, see Figure 8 ***(1)***. Let $P_s^H = P_s - x_1 s - x_2 s$ and $P_t^H = P_t - x_1 t - x_2 t$. If s-$\deg_G(x_2) = 2$ then let $P_u^H = P_u$ for $u \in S - \{s, t\}$. If s-$\deg_G(x_2) = 3$ then let $P_u^H = P_u - x_2 u$ in case $\deg_{G_2}(x_2) = 3$, and let $P_u^H = P_u - x_2 u + e$ in case $\deg_{G_2}(x_2) = 4$ and $x_2$ was split to the edges $e_2$ and $e$. If $H$ is a 2-tree-union then it has a 2-tree-coloring equitable to $\mathcal{P}^H$ by the minimality of $G$. In this coloring $e_1$ and $e_2$ have different colors. By possibly exchanging the colors of $e_1$ and $e_2$ we can achieve that at the unsplitting at $x_2$

- we can keep equitability to $P_u^H$ in case s-$\deg_G(x_2) = \deg_{G_2}(x_2) = 3$, and
- we do not need to re-color any edges in case $\deg_{G_2}(x_2) = 4$.

Next unsplit at $x_1$ yielding a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, see Figure 8 ***(2)***. Note that if $\deg_G(x_1) = 4$ and both split edges of $x_1$ had the same color before the unsplitting at $x_1$, then it is possible to re-color an edge incident to $x_1$ keeping equitability.

**Reduction 2.** (Figure 9.) We assume that $S = \{s_1, s_2, s_3\}$. Let $x_1, x_2 \in V - S$ be two vertices such that $\deg_G(x_i) \in \{3, 4\}$, s-$\deg_G(x_i) = 2$ for $i = 1, 2$ and $x_2 s_1 \in P_1$, $x_1 s_2 \in P_2$ and $x_1 s_3, x_2 s_3 \in P_3$. We pose the restriction that if $x_1$ and $x_2$ are adjacent in $G$, then $\deg_G(x_2) = 4$ must hold. Now first split $x_1$ to the $s_2 s_3$-edge $e_1$ resulting in the graph

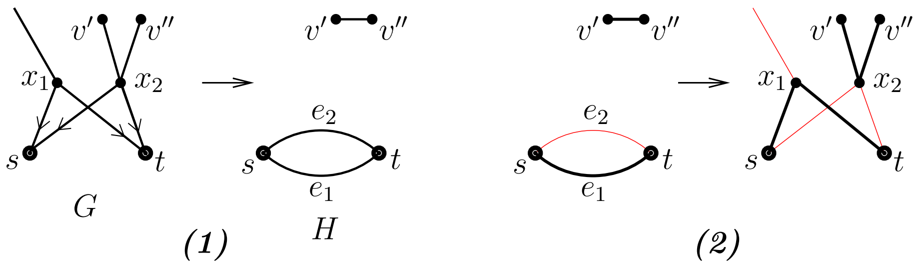


Figure 8: Reduction 1.

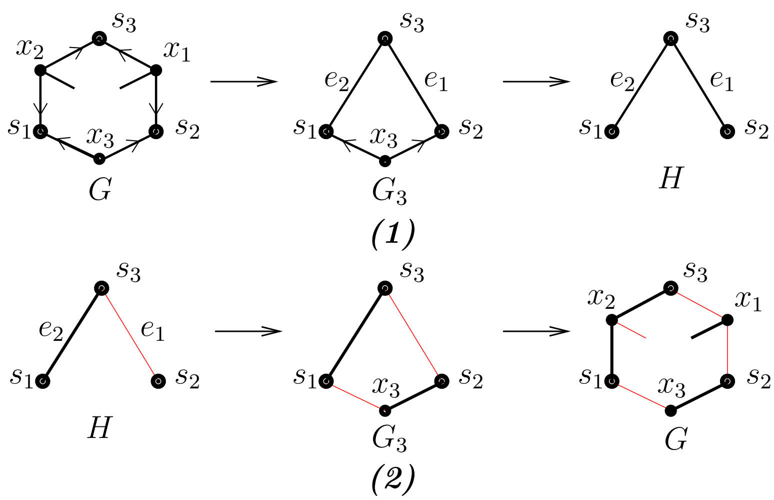


Figure 9: Reduction 2.

$G_2$. Then in $G_2$ split $x_2$ to the $s_1s_3$-edge $e_2$ resulting in the graph $G_3$. Finally, let $\deg_{G_3}(x_3) = 2$ for some $x_3 \in V - S$ such that the neighbors of $x_3$ in $G_2$ are $s_1$ and $s_2$ and $x_3s_1 \in P_1$, $x_3s_2 \in P_2$. Now delete $x_3$ from $G_3$ resulting in the graph $H$, see Figure 9 ***(1)***. Let $P_1^H = P_1 - x_2s_1 - x_3s_1$, $P_2^H = P_2 - x_1s_2 - x_3s_2$ and $P_3^H = P_3 - x_1s_3 - x_2s_3$. Assume that $H$ is a 2-tree-union and that it has a 2-tree-coloring equitable to $\mathcal{P}^H$ such that $e_1$ *and* $e_2$ *have different colors*. First unsplit at $x_3$ such that $x_3s_1$ has the color of $e_1$ and $x_3s_2$ has the color of $e_2$. Next unsplitting at $x_2$, and then at $x_1$ gives a 2-tree-coloring of $G$ equitable to $\mathcal{P}$, see Figure 9 ***(2)***.

These reductions are of no use unless $H$ is a 2-tree-union. To show that $H$ is really a 2-tree-union it is enough to show sequential splits described in page 3 which reduce $H$ to a 2-tree-union with vertex set $S$. Observe that a graph with vertex set $S$ and with 4 edges is always a 2-tree-union unless it has a loop or an edge with multiplicity at least 3. Every time we apply Reductions 1 and 2 it will be an easy task to show such sequential splits. For an example see one case below (Figure 11.). Recall that when using Reduction 2 one also has to check whether $H$ has a 2-tree-coloring equitable to $\mathcal{P}^H$ such that the split edges $e_1$ and $e_2$ have different colors. We will leave this to the reader when applying

Reduction 2.

Unlike in Reductions 1 and 2, in the next Reduction there is no need to check if $H$ is a 2-tree-union.

**Reduction 3.** We assume that $S = \{s_1, s_2, s_3\}$. Let $x_1,\, x_2 \in V - S$ be two non-adjacent vertices such that $\deg_G(x_i) = 3$ and $x_i s_1 \in P_1$, $x_i s_2 \in P_2$ hold for $s_1,\, s_2 \in S$. Assume also that the edge $s_1 s_2$ has multiplicity 1 in $G$. Let the neighbor of $x_i$ distinct from $s_1$, $s_2$ be $v_i$ for $i = 1, 2$. Splitting the vertices $x_i$ to the $s_i v_i$-edge $e_i$ for $i = 1, 2$ results in a graph $H$ (see Figure 10 ***(1)***). If $s_i v_i$ had multiplicity 2, then $G_{\text{aux}}$ would have 3 components of type ***(b)***, so $x_1$, $x_2$ would belong to a component ***(d)*** which is impossible. Thus if $H$ is not a 2-tree-union, then by Theorem 2.1, there exists a vertex set $W \subseteq V(H)$ such that $i_H(W) \geq 2|W| - 1$. Corollary 3.10 implies that $s_i,\, v_i \in W$ for $i = 1, 2$. But then $i_G(W \cup \{x_1,\, x_2\}) \geq 2|W \cup \{x_1,\, x_2\}| - 1$, a contradiction. So $H$ is *always a 2-tree-union*. Let $P_1^H = P_{s_1} - x_1 s_1 - x_2 s_1$, $P_2^H = P_{s_2} - x_1 s_2 - x_2 s_2$ and if $|S| = 3$, then $P_3^H = (P_3 \setminus \{x_1 s_3,\, x_2 s_3\}) \cup \{e_i :\ x_i s_3 \in P_3\}$. $H$ has a 2-tree-coloring equitable to $\mathcal{P}^H$ by the minimality of $G$. If $e_1$ and $e_2$ have the same colors in this coloring, then simply unsplit $x_1$ and $x_2$, see Figure 10 ***(2)***. If $e_1$ and $e_2$ have different colors, then use the extension of Figure 10 ***(3)***. In both cases we get a 2-tree-coloring of $G$ equitable to $\mathcal{P}$.

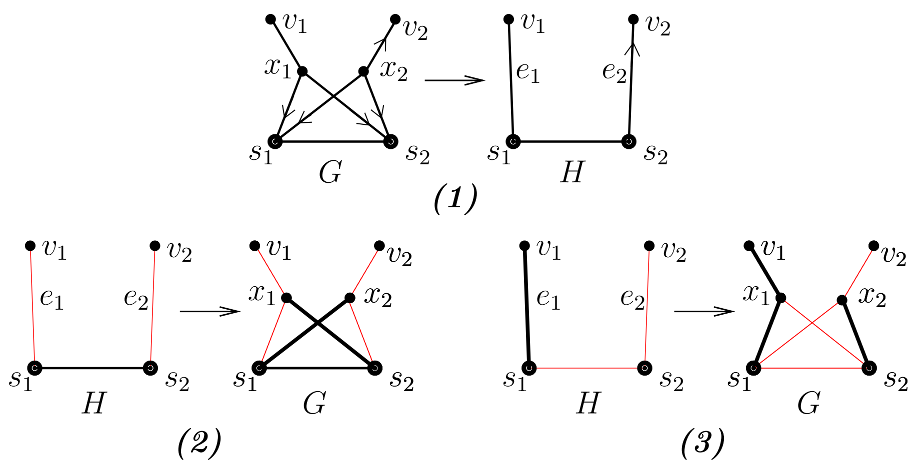


Figure 10: Reduction 3.

Now we prove that 2-tree-unions are 1-, 2-, and 3-star-equitable, by enumerating the possibilities for $G_{\text{aux}}$, according to how the $b$ values of the components can sum up to $2|S| - 2$. We use the notations of Figures 4–7, that is we refer to the components of $G_{\text{aux}}$ as ***(a)*** - ***(x)*** and to specified vertices of these components as $v_a$, $v_d^1$, $v_d^2$ etc. (see Figure 4).

## 2-tree-unions are 1-star-equitable

As $\sum b = 0$, $G_{\text{aux}}$ can consist only the null-component, but then $|S| \geq 2$ by Corollary 3.6. So no counterexample exists.

## 2-tree-unions are 2-star-equitable

$0 \leq b(C) \leq 2$ holds for each component $C$ of $G_{\text{aux}}$. As $G_{\text{aux}}$ contains at most one component with $b = 0$, the null-component ***(a)***, the $b$ values of the components can sum up to 2 in four ways: 2, 2+0, 1+1, 1+1+0. Observe that $G_{\text{aux}}$ has no component ***(c)*** by Theorem 3.8.

### 2(+0)

By Proposition 3.13, $G_{\text{aux}}$ must consist of ***(a)*** and ***(f)***. Apply Reduction 1.

### 1+1(+0)

Taking Proposition 3.13 into account, the possible components of $G_{\text{aux}}$ are as follows.

- ***(b)*** + ***(d)*** + ***(a)***. By our assumption every $P \in \mathcal{P}$ has even cardinality. However, by Corollary 3.6 and Proposition 3.7, at least one of them must be 3.
- ***(d)*** + ***(d)***, and perhaps ***(a)***. Apply Reduction 1.

## 2-tree-unions are 3-star-equitable

$0 \leq b(C) \leq 4$ holds for each component $C$ of $G_{\text{aux}}$, and the $b$ values of the components can sum up to 4 in five ways (not taking into consideration the components with $b = 0$): 4, 3+1, 2+2, 2+1+1, 1+1+1+1.

### 4(+0)

Denote the component of $G_{\text{aux}}$ with $b = 4$ by $C_4$. $C_4$ has at least 4 vertices $x$ with $\deg_G(x) \in \{3, 4\}$ and s-$\deg_G(x) = 2$ by Propositions 3.7 and 3.11 (except if $C_4 =$ ***(r)*** in the case s-$\deg_G(v_r) = 3$). In any case we can choose two vertices $x_1, x_2 \in V_2(C)$ such that, say, $x_i s_1 \in P_1$ and $x_i s_2 \in P_2$ for $i = 1, 2$. If $x_1$ and $x_2$ are adjacent in $G$ then make sure that $\deg_G(x_2) = 4$ holds. Now apply Reduction 1 to $x_1$, $x_2$ resulting in the graph $H$. We have to prove that $H$ is a 2-tree-union. $\deg_G(s_3) \geq 4$ by Proposition 3.4 which clearly implies that $\deg_H(s_3) \geq 3$, so it is straightforward to show a sequence of splits in $H$ which gives a 2-tree-union with vertex set $S$. We illustrate this in the case $C_4 =$ ***(o)***, see Figure 11. If $C_4 =$ ***(o)*** then $G_{\text{aux}}$ also contains the null-component ***(a)*** by Proposition 3.13. For instance, assume that $H$ is the graph shown in Figure 11 ***(1)***. Now split $v_1$ to $s_1 s_3$ resulting in the graph $H_1$. Then split $v_2$ to $s_1 s_3$ resulting in the graph $H_2$. Finally delete from $H_2$ the vertices $v_3$ and $v_a$ resulting in $H_3$, see Figure 11 ***(4)***. Since $H_3$ is trivially a 2-tree-union, we get that $H$ is a 2-tree-union as well so we are done. You only have to be careful if $C_4 = \boldsymbol{(p)}$ where it is forbidden to choose $x_1$, $x_2$ to be $v_p^1$ and $v_p^2$ because $H$ would contain a loop.

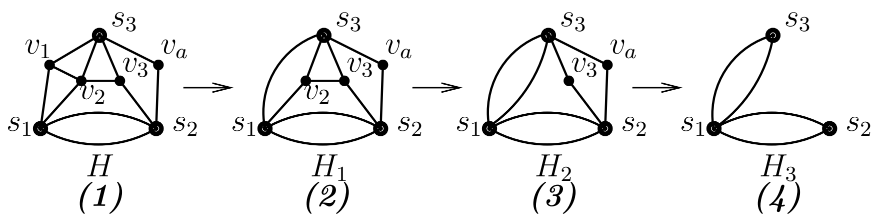


Figure 11: Proving that $H$ is a 2-tree-union

**3+1(+0)**

Let these components be denoted by $C_3$, $C_1$ (and $C_0$), resp.

- $C_3 = \boldsymbol{(g)}$. Proposition 3.13 gives that $C_1 = \boldsymbol{(d)}$ and also the null-component $C_0 = \boldsymbol{(a)}$ is present. Independently of the value of s-deg$_G(v_i^g)$ for $i = 1, 2$, we can apply Reduction 1.

- $C_3 = \boldsymbol{(h)}$. Proposition 3.13 gives that $C_1 = \boldsymbol{(d)}$ and also the null-component $C_0 = \boldsymbol{(a)}$ is present. Proposition 3.4 yields that $V_2(C_3)$ is adjacent to each $s_i \in S$. So we can apply Reduction 1 by appropriately choosing $x_1 \in V_2(C_3)$ and $x_2 \in V_2(C_1)$.

- $C_3 = \boldsymbol{(i)}$. If $C_1 = \boldsymbol{(c)}$ or $C_1 = \boldsymbol{(d)}$ then we can apply Reduction 1. $C_1 = \boldsymbol{(b)}$ is excluded by Proposition 3.4.

- $C_3 = \boldsymbol{(j)}$. $C_1 \neq \boldsymbol{(b)}$ by Proposition 3.4 and if $C_1 = \boldsymbol{(d)}$ then we are done by Reduction 1. If $C_1 = \boldsymbol{(c)}$ then also the null-component $C_0 = \boldsymbol{(a)}$ is present. The only possibility when we cannot apply Reduction 1 is when s-deg$_G(v_c) = 2$ and the two significant edges incident to $v_j^1$, $v_j^3$ and $v_c$ go to pairwise distinct pairs of vertices in $S$. So we can apply Reduction 2 by appropriately choosing $x_1$, $x_2 \in \{v_j^1, v_j^3, v_c\}$ and $x_3 = v_a$.

- $C_3 = \boldsymbol{(k)}$. If $C_1 = \boldsymbol{(c)}$ or $\boldsymbol{(d)}$ then we can apply Reduction 1. Assume $C_1 = \boldsymbol{(b)}$, that is an edge $s_1s_2$. Proposition 3.4 implies that $\deg_G(s_3) \geq 4$ so at least three vertices of $V_2(C_3)$ are adjacent to $s_3$. Thus Reduction 1 can be applied.

- $C_3 = \boldsymbol{(l)}$. If $C_1 = \boldsymbol{(c)}$ or $C_1 = \boldsymbol{(d)}$ then Reduction 1 can be applied. Assume that $C_1 = \boldsymbol{(b)}$, that is an edge $s_1s_2$. Proposition 3.13 implies that $G_{\text{aux}}$ contains the null-component $\boldsymbol{(a)}$ as well. Now the only case when we cannot apply Reduction 1 or 3 is when the two significant edges incident to $v_l^i$ go to pairwise distinct pairs of vertices in $S$ for $i = 1, 2, 3$. So we can apply Reduction 2 by appropriately choosing $x_1$, $x_2 \in \{v_l^1, v_l^2, v_l^3\}$ and $x_3 = v_a$.

- $C_3 = \boldsymbol{(m)}$ or $C_3 = \boldsymbol{(n)}$. If $C_1 = \boldsymbol{(c)}$ or $\boldsymbol{(d)}$ then we can apply Reduction 1. If $C_1 = \boldsymbol{(b)}$ then Reduction 1 or 3 can be applied.

**2+2(+0)**

In all cases Reduction 1 can be applied.

**2+1+1(+0)**

Denote the component with $b = 2$ by $C_2$. We list the cases according to the two components with $b = 1$.

- $\boldsymbol{(b)} + \boldsymbol{(b)}$. Proposition 3.4 implies that $G_{\text{aux}}$ contains the null-component $\boldsymbol{(a)}$, $C_2 = \boldsymbol{(f)}$ and $\deg_G(s_i) = 4$ for $i = 1, 2, 3$. Assume that the two edges of the components $\boldsymbol{(b)}$ are parallel, say, $s_1s_2$-edges. Then each vertex of $V_2(C_2)$ is adjacent to $s_3$ because $\deg_G(s_3) = 4$. Thus we can apply Reduction 1. So assume that the two edges of the components $\boldsymbol{(b)}$ are, say, $s_1s_2$ and $s_1s_3$. If the two significant edges incident to $v_f^i$ go to pairwise distinct pairs of vertices in $S$ for $i = 1, 2, 3$, then we can apply Reduction 2 by choosing $x_i = v_f^i$ for $i = 1, 2, 3$. The fact that $\deg_G(s) = 4$ for $s \in S$ implies that otherwise for at least two indices $i$ the two significant edges incident to $v_f^i$ go to $s_2$ and $s_3$. So we can apply Reduction 1.
- $\boldsymbol{(b)} + \boldsymbol{(c)}$. Assume that the edge of the component $\boldsymbol{(b)}$ joins $s_1$ to $s_2$. Proposition 3.4 implies that $G_{\text{aux}}$ contains the null-component $\boldsymbol{(a)}$, too.
  - $C_2 = \boldsymbol{(e)}$. Proposition 3.4 implies that $\deg_G(s_i) = 4$ for $i = 1, 2, 3$. So there is only one choice for $G$ up to isomorphism, namely, say, $v_a$ is adjacent to $s_2$ and $s_3$ and $v_e^2$ is adjacent to $s_1$ and $s_3$. If $\text{s-deg}_G(v_c) = 3$ or $\text{s-deg}_G(v_e^1) = 3$, then we can apply Reduction 1 with $x_1 = v_e^2$ and $x_2 = v_c$ or $v_e^1$ resp. So assume that $\text{s-deg}_G(v_c) = \text{s-deg}_G(v_e^1) = 2$. For $v = v_c$ or $v_e^1$, if the two significant edges incident to $v$ go to $s_1$ and $s_2$, then we can apply Reduction 2 with $x_1 = v_e^2$, $x_2 = v$, $x_3 = v_a$. Otherwise $v_cs_3$, $v_e^1s_3 \in P_3$ so we can apply Reduction 1.
  - $C_2 = \boldsymbol{(f)}$. If the two significant edges incident to $v_f^i$ go to pairwise distinct pairs of vertices in $S$ for $i = 1, 2, 3$, then we can apply Reduction 2 with $x_i = v_f^i$. Otherwise $v_f^i$ and $v_f^j$ are adjacent to the same pair of vertices in $S$ for some $1 \leq i < j \leq 3$. This pair cannot be $s_1$, $s_2$ since $\deg_G(s_3) \leq 3$ would hold. Thus we can apply Reduction 1.
- $\boldsymbol{(b)} + \boldsymbol{(d)}$. If $C_2 = \boldsymbol{(f)}$, then $G$ contains 4 vertices $v$ with $\deg_G(v) = 3$ and $\text{s-deg}_G(v) = 2$ so we can apply Reduction 1 or 3. So assume that $C_2 = \boldsymbol{(e)}$ and that component $\boldsymbol{(b)}$ joins $s_1$ to $s_2$. We can apply Reduction 1 or 3 unless the two significant edges incident to $v_d^1$, $v_d^2$ and $v_e^2$ go to pairwise distinct pairs of vertices in $S$. In this case Reduction 1 can be applied unless $\text{s-deg}_G(v_e^1) = 2$ and the two

significant edges incident to $v_e^1$ go to $s_1$ and $s_2$. $|P_i|$ is even for $i = 1, 2, 3$ so also $v_a$ is adjacent to $s_1$ and $s_2$. But then $\deg_G(s_3) \le 3$ would hold, which is impossible.

- ***(c)*** + ***(c)***, ***(c)*** + ***(d)*** and ***(d)*** + ***(d)***. Apply Reduction 1.

**1+1+1+1(+0)**

We list all possible cases.

- ***(b)*** + ***(b)*** + ***(b)*** + ***(b)*** and ***(b)*** + ***(b)*** + ***(b)*** + ***(c)*** are impossible by Proposition 3.13.

- ***(b)*** + ***(b)*** + ***(b)*** + ***(d)***. $G_{\mathrm{aux}}$ contains also the null-component by Proposition 3.13. Here we cannot apply any reductions. Assume that $v_d^1$ is adjacent to $s_1$, $s_3$ and $v_d^2$ is adjacent to $s_2$, $s_3$. There are two cases on the position of the null-component up to isomorphism.

  - First, let $v_a$ be adjacent to $s_1$ and $s_2$, see Figure 12 ***(1)***. Denote $P^1 = \{v_d^1 s_1,\ v_a s_1\}$, $P^2 = \{v_d^2 s_2,\ v_a s_2\}$ and $P^3 = \{v_d^1 s_3,\ v_d^2 s_3\}$. $|P_i|$ is even for $i = 1, 2, 3$ so there are 3 possibilities for $\mathcal{P}$ up to isomorphism: $\{P^1,\ P^2,\ P^3\}$, $\{P^1 + s_3 s_1 + s_2 s_1,\ P^2,\ P^3\}$ and $\{P^1,\ P^2,\ P^3 + s_1 s_3 + s_2 s_3\}$. The 2-tree-coloring of the graph in Figure 12 ***(1)*** is equitable to all these 3 cases of $\mathcal{P}$.
  - Let $v_a$ be adjacent to $s_1$ and $s_3$, see Figure 12 ***(2)***. Now the evenness of $|P_i|$ implies that with an $s_1 s_2$-edge $e$ it holds that $\mathcal{P} = \{\{v_a s_1,\ v_d^1 s_1\},\ \{e,\ v_d^2 s_2\},\ \{v_a s_3,\ v_d^1 s_3,\ v_d^2 s_3,\ s_2 s_3\}\}$. Figure 12 ***(2)*** shows a 2-tree-coloring equitable to $\mathcal{P}$.

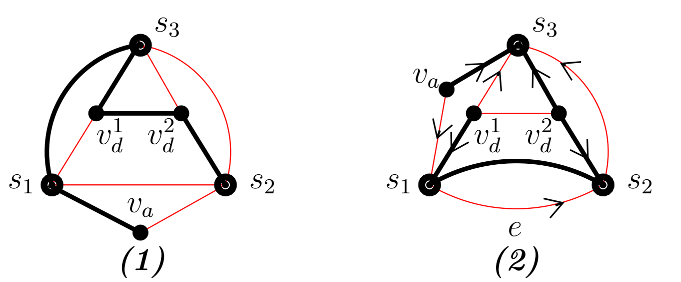


Figure 12: ***(b)*** + ***(b)*** + ***(b)*** + ***(d)***

- ***(b)*** + ***(b)*** + ***(c)*** + ***(c)***. $G_{\mathrm{aux}}$ contains also the null-component by Proposition 3.13. Denote the two vertices $v_c$ of the two components ***(c)*** by $v_c'$ and $v_c''$. Proposition 3.4 implies that, say, the two edges of the components ***(b)*** are $s_1 s_3$- and $s_2 s_3$-edges and $v_a$ is adjacent to $s_1$ and $s_2$. If s-$\deg_G(v_c') = 3$ or s-$\deg_G(v_c'') = 3$, then we can apply Reduction 1 or 3 so assume otherwise. If the two significant edges incident to $v_c'$ and $v_c''$ go to the same pair of vertices in $S$, then apply Reduction 1 or 3. Otherwise there are two cases up to isomorphism. First, if the significant edges incident to $v'_c$ go to $s_2$, $s_3$ and the significant edges incident to $v''_c$ go to $s_1$, $s_3$, then apply Reduction 2 with $x_1 = v'_c$, $x_2 = v''_c$, $x_3 = v_a$. Second, if the significant edges incident to $v'_c$ go to $s_1$, $s_2$ and the significant edges incident to $v''_c$ go to $s_2$, $s_3$, then the evenness of $|P_i|$ implies that there is only one choice for $\mathcal{P}$. A 2-tree-coloring of $G$ equitable to $\mathcal{P}$ is shown in Figure 13.

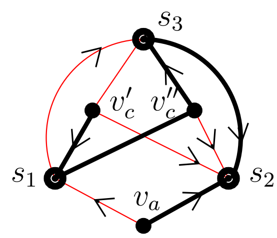


Figure 13: $(\boldsymbol{b}) + (\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{c})$

- $(\boldsymbol{b}) + (\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{d})$. If s-$\deg_G(v_c) = 3$ then apply Reduction 1 or 3. Assume that s-$\deg_G(v_c) = 2$. If the edges of the components $(\boldsymbol{b})$ are not parallel, then we can apply Reduction 1 or 3 unless the two significant edges incident to $v_c$, $v_d^1$ and $v_d^2$ go to pairwise distinct pairs of vertices in $S$. In this latter case Reduction 2 can be applied with $x_1 = v_c$, $x_2 = v_d^1$ and $x_3 = v_d^2$. Now assume that the edges of the components $(\boldsymbol{b})$ are parallel $s_1s_2$-edges. $\deg_G(s_3) \geq 4$ so $v_a$ is adjacent to $s_3$ and, say, $v_d^i$ is adjacent to $s_i$ for $i = 1, 2$. $|P_3|$ is even thus $v_cs_3 \in P_3$. So we can apply Reduction 1 with $x_1 = v_c$ and $x_2 = v_d^i$ for $i = 1$ or 2.

- $(\boldsymbol{b}) + (\boldsymbol{b}) + (\boldsymbol{d}) + (\boldsymbol{d})$. If the edges of the components $(\boldsymbol{b})$ are not parallel, then apply Reduction 1 or 3. Assume that the edges of the components $(\boldsymbol{b})$ are parallel $s_1s_2$-edges. $\deg_G(s_3) \geq 4$ so at least three vertices of type $v_d^i$ $(i = 1, 2)$ are adjacent to $s_3$, so it is possible to apply Reduction 1.

- $(\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{c}) + (\boldsymbol{c})$. Denote the vertices $v_c$ of the components $(\boldsymbol{c})$ by $v_c^i$ for $i = 1, 2, 3$. If s-$\deg_G(v_c^i) = 3$ for at least one index $i$, then apply Reduction 1 or 3. Otherwise we can apply Reduction 1 or 3 unless the two significant edges incident to $v_c^i$ go to pairwise distinct pairs of vertices in $S$ for $i = 1, 2, 3$. Suppose that the edge of the component $(\boldsymbol{b})$ is an $s_1s_2$-edge. $\deg_G(s_3) \geq 4$ so $G_{\text{aux}}$ contains also the null-component $(\boldsymbol{a})$ and $v_a$ is adjacent to $s_3$. But then $|P_3| = 3$ would hold, which is impossible.

- $(\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{c}) + (\boldsymbol{d})$ and $(\boldsymbol{b}) + (\boldsymbol{c}) + (\boldsymbol{d}) + (\boldsymbol{d})$ and $(\boldsymbol{b}) + (\boldsymbol{d}) + (\boldsymbol{d}) + (\boldsymbol{d})$. Apply Reduction 1 or 3.

- $(\boldsymbol{c}) + (\boldsymbol{c}) + (\boldsymbol{c}) + (\boldsymbol{c})$. Denote the vertices $v_c$ of the components $(\boldsymbol{c})$ by $v_c^i$ for $1 \leq i \leq 4$. If s-$\deg_G(v_c^i) = 2$ for some $1 \leq i \leq 4$, then apply Reduction 1. Assume that s-$\deg_G(v_c^i) = 3$ for all $i$. Now independently of the existence of the null-component, split $v_c^1$ and $v_c^2$ to $s_1 s_2$-edges and split $v_c^3$ and $v_c^4$ to $s_2 s_3$-edges. The resulting graph is $H$ with $V(H) = S$. Now take any 2-tree-coloring of $H$ and unsplit the vertices $v_c^i$ giving a 2-tree-coloring of $G$ equitable to $\mathcal{P} = \{\Delta(s_1), \Delta(s_2), \Delta(s_3)\}$.

- $\boldsymbol{(c)} + \boldsymbol{(c)} + \boldsymbol{(c)} + \boldsymbol{(d)}$ and $\boldsymbol{(c)} + \boldsymbol{(c)} + \boldsymbol{(d)} + \boldsymbol{(d)}$ and $\boldsymbol{(c)} + \boldsymbol{(d)} + \boldsymbol{(d)} + \boldsymbol{(d)}$ and $\boldsymbol{(d)} + \boldsymbol{(d)} + \boldsymbol{(d)} + \boldsymbol{(d)}$. Apply Reduction 1.

End of proof of Theorem 3.1. □

# References


[1] J. Davies, C. McDiarmid, Disjoint common transversals and exchange structures. *J. London Math. Soc.* (1964) **14** 55–62.

[2] J. Edmonds, Edge-disjoint branchings. *Combinatorial algorithms* (Courant Comput. Sci. Sympos. 9, New York Univ., New York, 1972), Algorithmics Press, New York, 1973, 91–96.

[3] J. Graver, B. Servatius, H. Servatius, Combinatorial rigidity. *Graduate Studies in Mathematics*, AMS, 1993.

[4] D. König, Graphok és alkalmazásuk a determinánsok és alkalmazásuk elméletére [in Hungarian]. *Mathematikai és Természettudományi Értesitő* (1916) **34** 104–119.

[5] C. St. J. A. Nash-Williams, Decomposition of finite graphs into forests. *J. London Math. Soc.* (1964) **39** 12.

[6] A. Schrijver, Combinatorial optimization. Polyhedra and efficiency. volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003.