# A Ramsey theorem for indecomposable matchings

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**Abstract**

A matching is indecomposable if it does not contain a nontrivial contiguous segment of vertices whose neighbors are entirely contained in the segment. We prove a Ramsey-like result for indecomposable matchings, showing that every sufficiently long indecomposable matching contains a long indecomposable matching of one of three types: interleavings, broken nestings, and proper pin sequences.

## 1 Introduction

A (labeled, complete) *matching* is a graph on the vertex set $[2n] = \{1, 2, \ldots, 2n\}$ in which every vertex is incident to exactly one edge. An *interval* in a matching is a contiguous segment of vertices $[i, j] = \{i, i+1, \ldots, j\}$ such that no vertex in $[i, j]$ is adjacent to a vertex outside $[i, j]$. Every matching has two *trivial* intervals: the empty set and the set of all its vertices; it is worth noting that unlike other objects, there are no intervals containing only a single vertex. A matching is said to be *indecomposable* if it has no other intervals (and *decomposable* if it does have nontrivial intervals, see Figure 1).

Indecomposable matchings have been studied by Nijenhuis and Wilf [2], who provided a recursive algorithm for constructing all indecomposable matchings. From their recursion, it follows that the number, $s_n$, of indecomposable matchings of $[2n]$ satisfies the recurrence

$$s_n = (n-1)\sum_{i=1}^{n-1} s_i s_{n-i}.$$

The contribution of this paper is to show that every large indecomposable matching contains a large submatching in one of three explicit families.

**Theorem 1.1.** *Every indecomposable matching with at least $(2k)^{2k}$ edges contains a broken nesting, interleaving, or proper pin sequence with $k$ edges.*

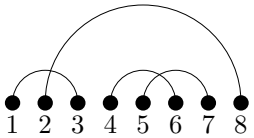


Figure 1: A decomposable matching with the nontrivial interval $[4, 7]$

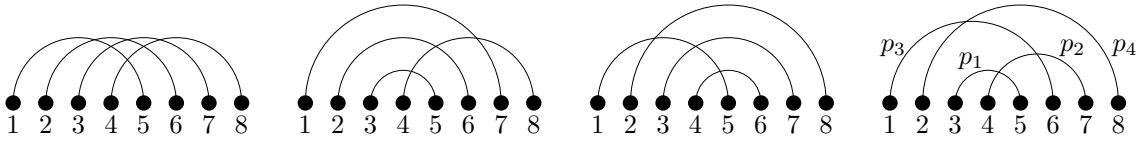


Figure 2: From left to right, the interleaving on [8], the right-broken nesting on [8], the left-broken nesting on [8], and a proper pin sequence on [8].

This result is the matching analogue of the results of Brignall, Huczynska, and Vatter [1], who proved a similar result for permutations.

In Theorem 1.1, we say that the matching $M$ *contains* the matching $N$ if $N$ can be obtained from $M$ by deleting a collection of edges and the vertices incident with those edges, and then relabeling the remaining vertices. For the remainder of this section we discuss the three types of indecomposable matchings mentioned in Theorem 1.1. The proof of the theorem follows in the next section.

The *interleaving* on $[2n]$ is the indecomposable matching defined by $i \sim i + n$ for all $i \in [n]$. The interleaving on [8] is depicted in the first matching of Figure 2.

The *nesting* on $[2n-2]$ is the matching defined by $i \sim 2n-2-i+1$ for $i \in [n]$. This matching is not indecomposable, but can be made indecomposable by adding a new edge which *breaks* the nesting. This new edge can break the nesting either to the left or the right. The *right-broken nesting* on $[2n]$ has edges $n \sim 2n$ and $i \sim 2n-i$ for $i \in [n-1]$, while the *left-broken nesting* on $[2n]$ has edges $1 \sim n+1$ and $i+1 \sim 2n-i+1$ for $i \in [n-1]$. The right- and left-broken nestings on [8] are depicted in the second and third matchings of Figure 2.

The most diverse family of indecomposable matchings is the family of *pin sequences*. In order to define pin sequences, we need a few preliminaries. Given a set of edges in a matching, its *shadow* is the smallest contiguous segment of vertices containing their endpoints. In an indecomposable matching, every nonempty shadow must either consist of all the vertices, or be *split*, meaning that there is a vertex in the shadow which is adjacent to a vertex outside of the shadow. We refer to such edges as *pins*.

A pin sequence is then a sequence of edges $p_1, p_2, \ldots$ such that each $p_i$ breaks the shadow of $\{p_1, \ldots, p_{i-1}\}$. Thus one pin sequences on [8] is $3 \sim 5, 4 \sim 7, 6 \sim 1, 2 \sim 8$, shown in the fourth matching of Figure 2. First we verify that all pin sequences are indecomposable.

**Proposition 1.2.** *Every pin sequence is an indecomposable matching.*

*Proof.* Let us say that an interval contains an edge if it contains one (and thus both) endpoints of that edge. We begin by noting that in any matching, if two edges cross, any interval containing one must contain both. It follows that if $I$ is an interval in a pin sequence containing pins $p_i$ and $p_j$, then $I$ contains all pins with indices between $i$ and $j$.

Suppose that $I$ is a maximal interval in a pin sequence $P = p_1, p_2, \ldots, p_n$. By the previous observation, $I$ consists of a contiguous sequence $p_i, p_{i+1}, \ldots, p_j$ for some $i \leq j$. If $i > 1$ there is a pin $p_{i-1}$ that splits the interval $I$ and if $j < n$ there is a pin $p_{j+1}$ thats splits $I$. Therefore, by the maximality of $I$, we see that $I$ contains the entire pin sequence, proving the proposition. □

We note that the permutation analogue of Proposition 1.2 does not hold.

# 2 Proof of Theorem 1.1

Our proof of Theorem 1.1 consists of analyzing two possibilities. First, we show that if a single edge is crossed by many different edges, then the matching contains an interleaving or broken nesting. The alternative is that no edge is crossed by many different edges, in which case we show that the matching contains a long proper pin sequence.

**Lemma 2.1.** *If a single edge $e$ is crossed by $2(k-1)^2+1$ edges of a matching, then the matching contains either a broken nesting or an interleaving with $k$ edges.*

*Proof.* Every edge that crosses $e$ crosses either to the left or to the right, thus at least $(k-1)^2+1$ of the edges must cross to the same side of $e$. By symmetry call that side left. Now order these $(k-1)^2+1$ edges by their left endpoints, preserving the natural order on the integers. Let $S$ be the unique sequence formed by the right endpoints of the these edges under this order.

By the Erdős-Szekeres Theorem, $S$ has a monotone subsequence of length $k$. If this subsequence is increasing, the matching contains an interleaving. Otherwise this subsequence is decreasing and the matching contains a nesting that is broken by $e$. □

In order to prove the main theorem, we will need two special types of pin sequences. *Proper* pin sequences satisfy, for each $1 < i < 2n$, $p_{i+1}$ splits the shadow cast by $\{p_1, \ldots, p_i\}$ but not the shadow cast by $\{p_1, \ldots, p_{i-1}\}$, and *right-reaching* pin sequences, which have their final pin incident with the vertex $2n$.

It is helpful to know that proper right-reaching pin sequences are always available in indecomposable matchings.

**Lemma 2.2.** *Every indecomposable matching has a proper right-reaching pin sequence beginning with any edge.*

*Proof.* Let $p_1$ be an arbitrary edge of the indecomposable matching $M$. If the vertex $2n$ is incident with $p_1$, then we are done. Otherwise, by the indecomposability of $M$, there is an edge which crosses $p_1$; label this edge $p_2$. If $2n$ is incident with $p_2$, then we are done. Otherwise, the edges $p_1$ and $p_2$ define a new shadow, which is still not an interval, so there

is an edge, $p_3$, which splits this shadow. Since the only interval is $[2n]$, by repeating this process, we can create a pin sequence $p_1, \ldots, p_m$ such that $2n$ is incident with $p_m$.

We now construct from this right-reaching pin sequence a proper right-reaching pin sequence $q_1, q_2, \ldots, q_s$. First, set $q_1 = p_1$. Then we successively extend this sequence by choosing $q_i$ to be the pin $p_j$ of the greatest index which crosses $q_{i-1}$. We stop when $q_i$ is incident with $2n$. Note that by this selection procedure, $q_i$ crosses $q_{i-1}$ but does not cross $q_1, \ldots, q_{i-2}$. Therefore the resulting sequence $q_1, \ldots$ is a proper right-reaching pin sequence, as desired. □

In the proof of Theorem 1.1, we use Lemma 2.2 to show that every indecomposable matching with $n$ edges contains at least $n$ distinct right-reaching proper pin sequences.

We can now derive the main result.

*Proof of Theorem 1.1.* Let $M$ be a matching which does not contain a broken nesting, interleaving, or proper pin sequence with at least $k$ edges. We construct a tree of all the proper right-reaching pin sequences of $M$ in the following manner. The parent of the pin sequence $p_1, \ldots, p_m$ $(m \geq 2)$ is the sequence $p_2, \ldots, p_m$, so the root of this tree is the edge (thought of as a pin sequence) incident with the vertex of the greatest label.

Since $M$ does not have a pin sequence with $k$ edges, this tree has height at most $k-1$. Because $M$ does not contain an interleaving or broken nesting with $k$ edges, Lemma 2.1 implies that no node may have $2(k-1)^2 + 2$ children. This bounds the size of the tree with the sum

$$\sum_{i=0}^{k-1} (2(k-1)^2 + 1)^i < \sum_{i=0}^{k-1} (2k^2)^i \leq (2k)^{2k}$$

By Lemma 2.2, every edge of $M$ begins a proper right-reaching pin sequence. Therefore $M$ can have at most $(2k)^{2k}$ edges, proving the theorem. □

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# References

[1] Brignall, R., Huczynska, S., and Vatter, V. Decomposing simple permutations, with enumerative consequences. *Combinatorica 28* (2008), 385–400.

[2] Nijenhuis, A., and Wilf, H. S. The enumeration of connected graphs and linked diagrams. *J. Combin. Theory Ser. A 27*, 3 (1979), 356–359.