On the Unitary Cayley Signed Graphs*†

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Abstract

A *signed graph* (or *sigraph* in short) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$ and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$. For a positive integer $n > 1$, the *unitary Cayley graph* X_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a - b \in U_n$. For a positive integer $n > 1$, the *unitary Cayley sigraph* $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$ is defined as the sigraph, where $\mathcal{S}_n^u$ is the unitary Cayley graph and for an edge ab of $\mathcal{S}_n$,

$$\sigma(ab) = \begin{cases} + & \text{if } a \in U_n \text{ or } b \in U_n, \\ - & \text{otherwise.} \end{cases}$$

In this paper, we have obtained a characterization of balanced unitary Cayley sigraphs. Further, we have established a characterization of canonically consistent unitary Cayley sigraphs $\mathcal{S}_n$, where n has at most two distinct odd prime factors.

1 Introduction

For standard terminology and notation in graph theory we refer Harary [21] and West [34] and Zaslavsky [35, 36] for sigraphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

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A *signed graph* (or *sigraph* in short) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the *signature* (or *sign* in short) of S. Alternatively, the sigraph can be written as $S = (V, E, \sigma)$, with V, E, σ in the above sense. Let $E^+(S) = \{e \in E : \sigma(e) = +\}$ and $E^-(S) = \{e \in E : \sigma(e) = -\}$. The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative* edges of S, respectively. A sigraph is *all-positive* (respectively, *all-negative*) if all its edges are positive (negative). Further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

The *negative degree* $d^-(v)$ of a vertex v in S is the number of negative edges incident at v in S. For a sigraph S, Behzad and Chartrand [9] defined its *line sigraph* $L(S)$ as the sigraph in which the edges of S are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in S have a vertex in common, any such edge ef is defined to be negative whenever both e and f are negative edges in S. The *negation* $\eta(S)$ of a sigraph S is a sigraph obtained from S by negating the sign of every edge of S, that means to find, $\eta(S)$ we change the sign of every edge to its opposite in S.

A cycle in a sigraph S is said to be *positive* if it contains an even number of negative edges. A given sigraph S is said to be *balanced* if every cycle in S is positive (*see* [20]). A spectral characterization of balanced sigraphs was given by Acharya [2]. Harary and Kabell [22, 23] developed a simple algorithm to get balanced sigraphs and also enumerated them. The following important lemma on balanced sigraphs is given by Zaslavsky:

Lemma 1. [37] *A sigraph in which every chordless cycle is positive, is balanced.*

A *marked sigraph* is an ordered pair $S_\mu = (S, \mu)$, where $S = (S^u, \sigma)$ is a sigraph and $\mu : V(S^u) \to \{+, -\}$ is a function from the vertex set $V(S^u)$ of S^u into the set $\{+, -\}$, called a *marking* of S. A cycle Z in S_μ is said to be *consistent* if it contains an even number of negative vertices. A given sigraph S is said to be *consistent* if every cycle in it is consistent [10, 11]. In particular, σ induces a unique marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where E_v is the set of edges incident at v in S, is called the *canonical marking* of S.

Now, if every vertex of a given sigraph S is canonically marked, then a cycle Z in S is said to be *canonically consistent* ($\mathcal{C}$-*consistent*) if it contains an even number of negative vertices and the given sigraph S is said be $\mathcal{C}$-*consistent* if every cycle in it is $\mathcal{C}$-consistent.

Let Γ be a group and B be a subset of Γ such that B does not contain identity of Γ. Assume $B^{-1} = \{b^{-1} : b \in B\} = B$. The *Cayley graph* $X' = Cay(\Gamma, B)$ is an undirected graph having vertex set $V(X') = \Gamma$ and edge set $E(X') = \{ab : ab^{-1} \in B\}$,

where $a, b \in \Gamma$. The Cayley graph X' is a regular graph of degree $|B|$. Its connected components are the right cosets of the subgroup generated by B. Therefore, if B generates Γ, then X' is a connected graph. The books on algebraic graph theory by Biggs [13] and by Godsil & Royle [19] provide many information regarding Cayley graphs.

For a positive integer $n > 1$, the *unitary Cayley graph* X_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a - b \in U_n$. The unitary Cayley graph X_n is also defined as, $X_n = Cay(Z_n, U_n)$. The structure and various properties of unitary Cayley graphs have been studied in literature (*see* [7], [8], [12], [14], [15], [16], [17], [18], [25], [26], [29]). The following theorem on bipartite unitary Cayley graphs is obtained by Dejter and Giudici:

Theorem 2. [15] *The unitary Cayley graph X_n, $n \geq 2$, is bipartite if and only if n is even.*

For a positive integer $n > 1$, the *unitary Cayley sigraph* $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$ is the sigraph, where $\mathcal{S}_n^u$ is the unitary Cayley graph and for an edge ab of $\mathcal{S}_n$,

$$\sigma(ab) = \begin{cases} + & \text{if } a \in U_n \text{ or } b \in U_n, \\ - & \text{otherwise.} \end{cases}$$

Two examples of unitary Cayley sigraphs are shown in **Figure 1**. Throughout the text, we consider $n \geq 2$.

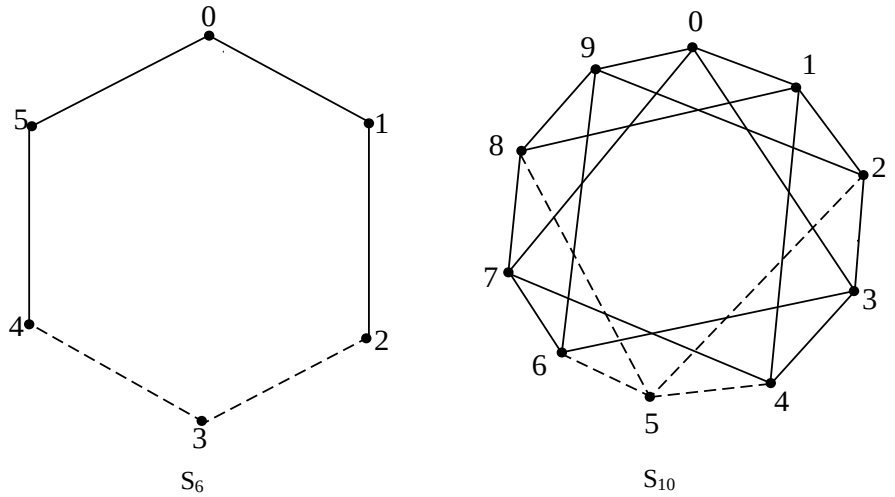

Figure 1: Unitary Cayley sigraphs for Z_6 and Z_{10}.

2 Balanced Unitary Cayley Sigraphs

In this section, we establish a characterization of balanced unitary Cayley sigraphs.

Lemma 3. *For the unitary Cayley sigraph $\mathcal{S}_n$, if $n = p^a$, where p is a prime number, then $\mathcal{S}_n$ is an all-positive sigraph.*

Proof. For the unitary Cayley sigraph $\mathcal{S}_n$, if $n = p^a$, then U_n consists of all the numbers less than n, which are not multiples of p. Suppose αp and βp are two numbers less than n and multiples of p. By the definition of unitary Cayley sigraph, we have a negative edge only when αp is adjacent to βp. But αp is not adjacent to βp since their difference $\alpha p - \beta p \notin U_n$. Thus, $\mathcal{S}_n$ is the all-positive sigraph.

□

Theorem 4. *The unitary Cayley sigraph $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$ is balanced if and only if either n is even or if n is odd, then it does not have more than one distinct prime factor.*

Proof. **Necessity**: Suppose the unitary Cayley sigraph $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$ is balanced. Assume that the conclusion is false. Suppose n is odd and it has at least two distinct prime factors. So, let $n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$, where all $p_1, p_2, \dots, p_m$ are distinct primes, $p_1 \neq 2$ and $p_1 < p_2 < \cdots < p_m$.

Case(i): There exist twin primes p_i and p_j for $1 \leq i < j \leq m$, that means $p_j - p_i = 2$. Since $(p_i + 1) - p_i = 1 \in U_n$, p_i and $p_i + 1$ are adjacent in $\mathcal{S}_n^u$. Next, $(p_i + 2) - (p_i + 1) = 1 \in U_n$, therefore $p_i + 1$ and $p_i + 2$ are adjacent in $\mathcal{S}_n^u$. Also, p_i and $p_i + 2$ are adjacent in $\mathcal{S}_n^u$ since $(p_i + 2) - p_i = 2 \in U_n$. Thus, consider the cycle

$$Z = (p_i, p_i + 1, p_i + 2 = p_j, p_i)$$

in $\mathcal{S}_n$. Clearly, p_i and p_j do not belong to U_n. Now, if $p_i + 1 \in U_n$, then Z has exactly one negative edge $p_i p_j$. Next, if $p_i + 1 \notin U_n$, then all the three edges in Z are negative. Thus, Z is a negative cycle in $\mathcal{S}_n$. This implies that $\mathcal{S}_n$ is not balanced, a contradiction to the hypothesis.

Case(ii): No two p_i's are twin primes. Now, $p_2 + (p_1 - 1)$ and p_2 are adjacent in $\mathcal{S}_n^u$ because $p_2 + (p_1 - 1) - p_2 = p_1 - 1 \in U_n$. Hence, consider the cycle

$$Z' = (p_2, p_2 + 1, p_2 + 2, \dots, p_2 + (p_1 - 1), p_2)$$

of length p_1 in $\mathcal{S}_n$. Since $p_1 < p_2$, there is a vertex in Z' which is multiple of p_1, say αp_1. Clearly, p_2 is adjacent to αp_1 because their difference $\alpha p_1 - p_2 < p_1$ and U_n contains all the numbers less than p_1. Now p_2 is adjacent to αp_1 with a negative edge since neither $p_2 \in U_n$ nor $\alpha p_1 \in U_n$. This implies, either the cycle

$$Z'' = (p_2, p_2 + 1, p_2 + 2, \dots, \alpha p_1, p_2)$$

or the cycle

$$Z''' = (\alpha p_1, \alpha p_1 + 1, \alpha p_1 + 2, \dots, p_2 + (p_1 - 1), p_2, \alpha p_1)$$

in $\mathcal{S}_n$ has exactly one negative edge. Thus, either Z'' or Z''' is a negative cycle in $\mathcal{S}_n$. This implies that $\mathcal{S}_n$ is not balanced, a contradiction to the hypothesis. So, by

contradiction, the conditions are satisfied.

Sufficiency: Suppose n is even. Then, U_n does not contain any multiple of 2. Then, by Theorem 2, $\mathcal{S}_n$ is bipartite, whence all its cycles are even. Hence, every cycle in $\mathcal{S}_n$ contains alternately either even-odd or odd-even labeled vertices. Without loss of generality, let

$$Z'''' = (e_1, o_1, e_2, o_2, \ldots, e_m, o_m, e_1)$$

be a cycle of even length in $\mathcal{S}_n$. Clearly, $e_i \notin U_n \ \forall \ i = 1, 2, \ldots, m$.

Case(i): Suppose $o_j \in U_n \ \forall \ j = 1, 2, \ldots, m$. Then, all the edges in Z'''' are positive.

Case(ii): Suppose $o_j \notin U_n$ for any $j = 1, 2, \ldots, m$. Then, Z'''' contains two negative edges $e_j o_j$ and $o_j e_{j+1}$ with respect to each $o_j \notin U_n$. Thus, Z'''' contains an even number of negative edges. Since Z'''' is an arbitrary cycle in $\mathcal{S}_n$, using Lemma 1, $\mathcal{S}_n$ is balanced.

Next, suppose n is odd and it does not have more than one distinct prime factor. That means, $n = p^a$. Now, using Lemma 3, $\mathcal{S}_n$ is an all-positive sigraph. Hence the theorem.

□

Corollary 5. *For the unitary Cayley sigraph $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$, its negation sigraph $\eta(\mathcal{S}_n)$ is balanced if and only if n is even.*

Proof. First, suppose $\eta(\mathcal{S}_n)$ is balanced. Assume that conclusion is false. Suppose n is odd. Then, $2 \in U_n$. Thus, we can consider a triangle $T : (0, 1, 2, 0)$ in $\mathcal{S}_n$. Since $1 \in U_n$ and $2 \in U_n$, all the edges of the triangle T are positive. That means, all the edges of the triangle T are negative in $\eta(\mathcal{S}_n)$. Thus, $\eta(\mathcal{S}_n)$ is unbalanced, which contradicts the hypothesis. Conversely, suppose n is even. Now due to Theorem 2, $\mathcal{S}_n^u$ is bipartite and due to Theorem 4, $\mathcal{S}_n$ is balanced. Thus, $\eta(\mathcal{S}_n)$ is balanced.

□

Theorem 6. [5] *For a sigraph S, its line sigraph $L(S)$ is balanced if and only if the following conditions hold:*

- *(i) for any cycle Z in S,*
  - *(a) if Z is all-negative, then Z has even length;*
  - *(b) if Z is heterogeneous, then Z has even number of negative sections with even length;*
- *(ii) for $v \in S$, if $d(v) > 2$, then there is at most one negative edge incident at v in S.*

Corollary 7. *For the unitary Cayley sigraph $\mathcal{S}_n$, its line sigraph $L(\mathcal{S}_n)$ is balanced if and only if $n = p^a$, where p is a prime number.*

Proof. Suppose $L(\mathcal{S}_n)$ is balanced for the unitary Cayley sigraph $\mathcal{S}_n$. Assume that the conclusion is false. Let n have at least two distinct prime factors. Suppose p_1 and p_2 are two smallest prime factors of n such that $p_1 < p_2$. Clearly, the vertex p_1 and the vertex $2p_1$ are adjacent to the vertex p_2 with a negative edge in $\mathcal{S}_n$. That means, $d^-(p_2) \geq 2$ and clearly, $d(p_2) > 2$ in $\mathcal{S}_n$. Thus, condition (ii) of Theorem 6 does not hold for $\mathcal{S}_n$, which implies that $L(\mathcal{S}_n)$ is unbalanced, a contradiction to the hypothesis. Hence $n = p^a$, where p is a prime number. Converse part can be proved easily using Lemma 3.

□

The $\times$*-line sigraph* $L_\times(S)$ of a sigraph $S = (S^u, \sigma)$ is a sigraph defined on the line graph $L(S^u)$ of the graph S^u by assigning to each edge ef of $L(S^u)$, the product of signs of the adjacent edges e and f of S. The *semi-total line graph* $T_1(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if (i) they are adjacent edges in G, or (ii) one is a vertex and the other is an edge in G incident to it. Let $S = (V,\ E,\ \sigma)$ be any sigraph. Its *semi-total line sigraph* $T_1(S)$ has $T_1(S^u)$ as its underlying graph and for any edge uv of $T_1(S^u)$,

$$\sigma_{T_1}(uv) = \begin{cases} \sigma(u)\sigma(v) & \text{if } u, v \in E, \\ \sigma(v) & \text{if } u \in V \text{ and } v \in E. \end{cases}$$

Theorem 8. [6] *The $\times$-line sigraph $L_\times(S)$ of a sigraph S is a balanced sigraph.*

Corollary 9. *For the unitary Cayley sigraph $\mathcal{S}_n$, its $\times$-line sigraph $L_\times(\mathcal{S}_n)$ is balanced.*

Theorem 10. [33] *The semi-total line sigraph $T_1(S)$ of a sigraph S is a balanced sigraph.*

Corollary 11. *For the unitary Cayley sigraph $\mathcal{S}_n$, its semi-total line sigraph $T_1(\mathcal{S}_n)$ is balanced.*

3 $\mathcal{C}$-Consistent Unitary Cayley Sigraphs

Beineke and Harary [10, 11] were the first to pose the problem of characterizing consistent marked graphs, which was subsequently settled by Acharya [1, 3], Rao [27] and Hoede [24]. Acharya and Sinha obtained consistency of sigraphs that satisfy certain sigraph equations in [4, 30]. Sinha and Garg discussed consistency of several sigraphs in [31, 32, 33]. In this section, we obtain a characterization of $\mathcal{C}$-consistent unitary Cayley sigraphs. Throughout the section, (a, b) denotes the gcd(a, b). Now, we require the following theorem by Hoede and Corollary 14, which play an important role in solving the problem.

Theorem 12. [24] *A marked graph G_μ is consistent if and only if for any spanning tree T of G all fundamental cycles with respect to T are consistent and all common paths of pairs of those fundamental cycles have end vertices carrying the same marks.*

Theorem 13. [28] *Let a, b and m be integers with m positive. The linear congruence*

$$ax \equiv b \pmod{m}$$

is soluble if and only if $(a,m)|b$. If x_0 is a solution, there are exactly (a,m) incongruent solutions given by $\{x_0 + tm/(a,m)\}$, where $t = 0, 1, \ldots, (a,m) - 1$.

Corollary 14. [28] *If $(a,m) = 1$ then the congruence*

$$ax \equiv b \pmod{m}$$

has exactly one incongruent solution.

Lemma 15. *In the unitary Cayley sigraph $\mathcal{S}_n$, if $n = 2p_1^{a_1}$, where p_1 is an odd prime, then the negative degree of the vertex 2 of $\mathcal{S}_n$ is odd.*

Proof. Suppose $n = 2p_1^{a_1}$ in $\mathcal{S}_n$, where p_1 is an odd prime. By the definition of $\mathcal{S}_n$, negative edges are incident at the vertex 2 of $\mathcal{S}_n$ only when 2 is adjacent to multiples of p_1. Since difference of 2 and any even multiple of p_1 is an even number and U_n does not contain an even number, the vertex 2 is not adjacent to any even multiple of p_1. Now, the number of odd multiples of p_1 are $p_1^{a_1-1}$. Since p_1 is an odd prime, $d^-(2)$ is odd.

□

Lemma 16. *In the unitary Cayley sigraph $\mathcal{S}_n$, if $n = 2p_1^{a_1}p_2^{a_2}$, where p_1 and p_2 are distinct odd primes, then the negative degree of the vertex 2 of $\mathcal{S}_n$ is odd.*

Proof. Given that $n = 2p_1^{a_1}p_2^{a_2}$ in $\mathcal{S}_n$, where p_1 and p_2 are distinct odd primes. By the definition of $\mathcal{S}_n$, negative edges are incident at the vertex 2 of $\mathcal{S}_n$ only when 2 is adjacent to odd multiples of p_1 or p_2. Suppose A_i is the set of odd multiples of p_i for $i = 1, 2$. Then,

$$|A_1| = p_1^{a_1-1}p_2^{a_2}, \tag{1}$$

$$|A_2| = p_1^{a_1}p_2^{a_2-1} \tag{2}$$

and

$$|A_1 \cap A_2| = p_1^{a_1-1}p_2^{a_2-1}. \tag{3}$$

Thus, using principle of inclusion and exclusion,

$$|A_1 \cup A_2| = p_1^{a_1-1}p_2^{a_2} + p_1^{a_1}p_2^{a_2-1} - p_1^{a_1-1}p_2^{a_2-1}. \tag{4}$$

Since there are some odd multiples of $p_1(p_2)$ whose difference with 2 is multiple of $p_2(p_1)$, such multiples of $p_1(p_2)$ are not adjacent with 2. These odd multiples of $p_1(p_2)$ are given by the two linear congruences,

$$p_1 x \equiv 2 \pmod{p_2} \tag{5}$$

and

$$p_2 y \equiv 2 \pmod{p_1}. \tag{6}$$

Solving first Eq. (5), by Corollary 14, there exists a unique incongruent solution, say x_0 of Eq. (5). But all the solutions of Eq. (5), for which $p_1 x - 2 < n$ are,

$$x_0 + 0(p_2), x_0 + 2(p_2), \ldots, x_0 + (2p_1^{a_1-1}p_2^{a_2-1} - 2)(p_2).$$

Thus, the total number of solutions of Eq. (5) are $p_1^{a_1-1}p_2^{a_2-1}$. Similarly, the total solutions of Eq. (6) are $p_1^{a_1-1}p_2^{a_2-1}$. Hence, the total number of negative edges incident at the vertex 2 are,

$$\begin{aligned} d^-(2) &= p_1^{a_1-1}p_2^{a_2} + p_1^{a_1}p_2^{a_2-1} - p_1^{a_1-1}p_2^{a_2-1} - p_1^{a_1-1}p_2^{a_2-1} - p_1^{a_1-1}p_2^{a_2-1} \\ &= p_1^{a_1-1}p_2^{a_2} + p_1^{a_1}p_2^{a_2-1} - 3p_1^{a_1-1}p_2^{a_2-1}. \end{aligned}$$

Since p_1 and p_2 are odd primes, it follows that $d^-(2)$ is odd.

□

Lemma 17. *In the unitary Cayley sigraph $\mathcal{S}_n$, if $n = p_1^{a_1}p_2^{a_2}$, where n is odd, then the negative degrees of the vertices of $\mathcal{S}_n$ that are multiples of p_1 or p_2 are even.*

Proof. Suppose $n = p_1^{a_1}p_2^{a_2}$ in $\mathcal{S}_n$, where n is odd. By the definition of $\mathcal{S}_n$, negative edges are incident at the vertex p_1 when p_1 is adjacent to multiples of p_2 which does not have p_1 as the factor. Thus,

$$\begin{aligned} d^-(p_1) &= p_1^{a_1}p_2^{a_2-1} - p_1^{a_1-1}p_2^{a_2-1} \\ &= p_1^{a_1-1}p_2^{a_2-1}(p_1 - 1). \end{aligned}$$

Since p_1 and p_2 are odd, $d^-(p_1)$ is even. This formula works for any multiple of p_1 except those which have p_2 as the factor. Similarly,

$$\begin{aligned} d^-(p_2) &= p_1^{a_1-1}p_2^{a_2} - p_1^{a_1-1}p_2^{a_2-1}. \\ &= p_1^{a_1-1}p_2^{a_2-1}(p_2 - 1). \end{aligned}$$

Since p_1 and p_2 are odd, $d^-(p_2)$ is even. This formula works for any multiple of p_2 except those which have p_1 as the factor. And the negative degrees of the vertices of $\mathcal{S}_n$ that are multiples of p_1p_2 is zero. Thus, the negative degrees of the vertices of $\mathcal{S}_n$ that are multiples of p_1 or p_2 is even.

□

Lemma 18. *In the unitary Cayley sigraph $\mathcal{S}_n$, if $n = 2^{a_0}p_1^{a_1}$, where $a_0 \geq 2$ and p_1 is an odd prime, then the negative degrees of the vertices of $\mathcal{S}_n$ that are multiples of 2 or p_1 are even.*

Proof. It can be proved easily, using the similar argument used in Lemma 17.

□

Lemma 19. *In the unitary Cayley sigraph* $\mathcal{S}_n$, *if* $n = 2^{a_0}p_1^{a_1}p_2^{a_2}$, *where* $a_0 \geq 2$ *and* p_1, p_2 *are distinct odd primes, then the negative degrees of the vertices of* $\mathcal{S}_n$ *that are multiples of* 2, p_1 *or* p_2 *are even.*

Proof. Using the similar argument used in Lemma 16,

$$d^-(2) = 2^{a_0-1}p_1^{a_1-1}p_2^{a_2-1}(p_1 + p_2 - 3).$$

Since $a_0 \geq 2$, $d^-(2)$ is even. Similarly,

$$d^-(p_1) = d^-(p_2) = 2^{a_0-1}p_1^{a_1-1}p_2^{a_2-1}(p_1p_2 - p_1 - p_2 + 1).$$

Since $a_0 \geq 2$, $d^-(p_1)$ and $d^-(p_2)$ are even.

□

Theorem 20. *The unitary Cayley sigraph* $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$, *where* n *has at most two distinct odd prime factors, is* $\mathcal{C}$*-consistent if and only if* n *is odd,* 2, 6 *or a multiple of* 4.

Proof. **Necessity**: Suppose the unitary Cayley sigraph $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$ is $\mathcal{C}$-consistent. Let on contrary, $n \equiv 2 \pmod 4$ with $n \neq 2$ and $n \neq 6$. Then, either $n = 2p_1^{a_1}$ or $n = 2p_1^{a_1}p_2^{a_2}$, where p_1 and p_2 are distinct odd primes.

Case(i): Suppose $n \equiv 0 \pmod 3$. Then, either $n = 2.3^{a_1}$ or $n = 2.3^{a_1}.p_2^{a_2}$. First, suppose $p_2 \neq 5$ and $p_2 \neq 7$. Then, due to Lemma 15 and Lemma 16,

$$\mu_\sigma(2) = -.$$

Since the vertex $7 \in U_n$, by the definition of $\mathcal{S}_n$, $d^-(7) = 0$. It follows,

$$\mu_\sigma(7) = +.$$

Now, the vertex 7 is adjacent to the vertex 2 since $7 - 2 = 5 \in U_n$. Consider two cycles, $Z' = (2, 3, 4, \ldots, 7, 2)$ and $Z'' = (7, 8, 9, \ldots, (n-1), 0, 1, 2, 7)$ in $\mathcal{S}_n$. Clearly, the cycles Z' and Z'' share the chord whose end vertices are 2 and 7. Now, if either Z' or Z'' is an $\mathcal{C}$-inconsistent cycle, then we have a contradiction to the hypothesis. Therefore, Z' and Z'' are both $\mathcal{C}$-consistent cycles. However, the end vertices 2 and 7 of their common chord are marked oppositely under the canonical marking and this contradicts Theorem 12.

Now, if $n = 2.3^{a_1}.p_2^{a_2}$, where either $p_2 = 5$ or $p_2 = 7$, then since the vertex $13 \in U_n$, by the definition of $\mathcal{S}_n$, $d^-(13) = 0$. It follows,

$$\mu_\sigma(13) = +.$$

Now, the vertex 13 is adjacent to the vertex 2 since $13 - 2 = 11 \in U_n$. Then, consider the two cycles, $Z''' = (2, 3, 4, \ldots, 13, 2)$ and $Z'''' = (13, 14, 15, \ldots, (n-1), 0, 1, 2, 13)$ in

$\mathcal{S}_n$. Clearly, the cycles Z''' and Z'''' share the chord whose end vertices are 2 and 13. As argued above, Z''' and Z'''' are both $\mathcal{C}$-consistent cycles. However, the end vertices 2 and 13 of their common chord are marked oppositely under the canonical marking, a contradiction to Theorem 12.

Case(ii): Suppose either $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. That means, 3 does not divide n, which implies that the vertex $3 \in U_n$. Now, consider a cycle $Z = (0, 1, 2, 3, 0)$ in $\mathcal{S}_n$. Since $1 \in U_n$ and $3 \in U_n$, by the definition of $\mathcal{S}_n$, $d^-(1) = d^-(3) = 0$. It follows that in the cycle Z,

$$\mu_\sigma(1) = \mu_\sigma(3) = +.$$

Since the vertex 0 is adjacent to those vertices which belong to U_n, $d^-(0) = 0$. That means,

$$\mu_\sigma(0) = +.$$

Now due to Lemma 15 and Lemma 16,

$$\mu_\sigma(2) = -.$$

Thus, the cycle Z is $\mathcal{C}$-inconsistent. Hence $\mathcal{S}_n$ is not $\mathcal{C}$-consistent, a contradiction to the hypothesis. Thus, the result follows.

Sufficiency: Suppose n is odd, 2, 6 or a multiple of 4.

Case(i): Let n be odd, and $n = p_1^{a_1} p_2^{a_2}$, where p_1 and p_2 are distinct odd primes. By the definition of $\mathcal{S}_n$, there is a negative edge in $\mathcal{S}_n$ only when both the end vertices of the edge are multiples of either p_1 or p_2. Thus using Lemma 17, all the vertices of $\mathcal{S}_n$ are marked positively under the canonical marking. Hence, $\mathcal{S}_n$ is $\mathcal{C}$-consistent.

Case(ii): Suppose $n = 2, 6$ in $\mathcal{S}_n$. Then, we can easily verify that $\mathcal{S}_2$ and $\mathcal{S}_6$ are $\mathcal{C}$-consistent.

Case(iii): Suppose n is a multiple of 4. Then, let $n = 2^{a_0} p_1^{a_1}$ or $n = 2^{a_0} p_1^{a_1} p_2^{a_2}$, where $a_0 \geq 2$ and p_1, p_2 are distinct odd primes. By the definition of $\mathcal{S}_n$, there is a negative edge in $\mathcal{S}_n$ only when both the end vertices of the edge are either multiples of 2, p_1 or p_2. Thus, using Lemma 18 and Lemma 19, all the vertices of $\mathcal{S}_n$ are marked positively under the canonical marking. Hence, $\mathcal{S}_n$ is $\mathcal{C}$-consistent.

□

Corollary 21. *For the unitary Cayley sigraph $\mathcal{S}_n = (\mathcal{S}_n^u, \sigma)$, its negation sigraph $\eta(\mathcal{S}_n)$ is $\mathcal{C}$-consistent if and only if $\mathcal{S}_n$ is $\mathcal{C}$-consistent.*

Proof. Suppose $\eta(\mathcal{S}_n)$ is $\mathcal{C}$-consistent. That means, each cycle of $\eta(\mathcal{S}_n)$ consists of an even number of vertices whose negative degree is odd. Since degree of a vertex in $\mathcal{S}_n$ and hence in $\eta(\mathcal{S}_n)$ is even, an even number of vertices are left in each cycle of $\eta(\mathcal{S}_n)$, whose positive degree is odd. Thus, there are an even number of vertices in $\mathcal{S}_n$ whose

negative degree is odd. Hence, $\mathcal{S}_n$ is $\mathcal{C}$-consistent. Converse part can be proved in a similar manner.

□

4 Conclusion

In this paper, we have obtained a characterization of balanced unitary Cayley sigraphs and a characterization of canonically consistent unitary Cayley sigraphs. But the problem of canonically consistent unitary Cayley sigraphs is solved for n with at most two distinct odd prime factors. One can think the problem for general n. In our opinion, our result would also work for general n.

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