Disproof of the List Hadwiger Conjecture

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Abstract

The List Hadwiger Conjecture asserts that every K_t-minor-free graph is t-choosable. We disprove this conjecture by constructing a K_{3t+2}-minor-free graph that is not $4t$-choosable for every integer $t \geq 1$.

1 Introduction

In 1943, Hadwiger [6] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory; see [28] for a survey[1].

Hadwiger Conjecture. *Every K_t-minor-free graph is $(t-1)$-colourable.*

The Hadwiger Conjecture holds for $t \leq 6$ (see [3, 6, 19, 20, 30]) and is open for $t \geq 7$. In fact, the following more general conjecture is open.

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[1]See [2] for undefined graph-theoretic terminology. Let $[a, b] := \{a, a+1, \ldots, b\}$.

Weak Hadwiger Conjecture. *Every K_t-minor-free graph is ct-colourable for some constant $c \geq 1$.*

It is natural to consider analogous conjectures for list colourings[2]. First, consider the choosability of planar graphs. Erdős et al. [5] conjectured that some planar graph is not 4-choosable, and that every planar graph is 5-choosable. The first conjecture was verified by Voigt [29] and the second by Thomassen [27]. Incidentally, Borowiecki [1] asked whether every K_t-minor-free graph is $(t-1)$-choosable, which is true for $t \leq 4$ but false for $t = 5$ by Voigt's example. The following natural conjecture arises (see [10, 32], and see [34] for related conjectures).

List Hadwiger Conjecture. *Every K_t-minor-free graph is t-choosable.*

The List Hadwiger Conjecture holds for $t \leq 5$ (see [7, 22, 33]). Again the following more general conjecture, first stated by Kawarabayashi and Mohar [10], is open.

Weak List Hadwiger Conjecture. *Every K_t-minor-free graph is ct-choosable for some constant $c \geq 1$.*

In this paper we disprove the List Hadwiger Conjecture for $t \geq 8$, and prove that $c \geq \frac{4}{3}$ in the Weak List Hadwiger Conjecture.

Theorem 1. *For every integer $t \geq 1$,*
(a) *there is a K_{3t+2}-minor-free graph that is not $4t$-choosable.*
(b) *there is a K_{3t+1}-minor-free graph that is not $(4t-2)$-choosable,*
(c) *there is a K_{3t}-minor-free graph that is not $(4t-3)$-choosable.*

Before proving Theorem 1, note that adding a dominant vertex to a graph does not necessarily increase the choice number (as it does for the chromatic number). For example, $K_{3,3}$ is 3-choosable but not 2-choosable. Adding one dominant vertex to $K_{3,3}$ gives $K_{1,3,3}$, which again is 3-choosable [18]. In fact, this property holds for infinitely many complete bipartite graphs [18]; also see [21]. On the other hand, a referee made the following observation which readily implies that parts (b) and (c) of Theorem 1 are consequences of part (a). That said, parts (b) and (c) can also be proved using the same method as that used to prove part (a), and that is what we do.

Proposition 2. *If G is a K_t-minor-free graph that is not k-choosable, then there is a K_{t+1}-minor-free graph that is not $(k+1)$-choosable.*

[2]A *list-assignment* of a graph G is a function L that assigns to each vertex v of G a set $L(v)$ of colours. G is *L-colourable* if there is a colouring of G such that the colour assigned to each vertex v is in $L(v)$. G is *k-choosable* if G is L-colourable for every list-assignment L with $|L(v)| \geq k$ for each vertex v of G. The *choice number* of G is the minimum integer k such that G is k-choosable. If G is k-choosable then G is also k-colourable—just use the same set of k colours for each vertex. Thus the choice number of G is at least the chromatic number of G. See [34] for a survey on list colouring.

Proof. Let L be a list-assignment that proves that G is not k-choosable. Assume that no integer in $[1, k+1]$ appears in the lists of L. For $i \in [1, k+1]$, let G'_i be a copy of G with list-assignment $L'(v) := L(v) \cup \{i\}$ for each vertex v of G'_i. In every L'-colouring of G'_i some vertex is coloured i. Let G' be the graph obtained from the disjoint union of $G'_1, \dots, G'_{k+1}$ by adding a dominant vertex u. Let $L'(u) := [1, k+1]$. Clearly G' is K_{t+1}-minor-free. Say G' is L'-colourable, and u is coloured $i \in L'(u) = [1, k+1]$. Then no vertex in G'_i is coloured i, which is a contradiction. Hence G' is not L'-colourable, and G' is not $(k+1)$-choosable, as desired. □

2 Proof of Theorem 1

Let G_1 and G_2 be graphs, and let S_i be a k-clique in each G_i. Let G be a graph obtained from the disjoint union of G_1 and G_2 by pairing the vertices in S_1 and S_2 and identifying each pair. Then G is said to be obtained by *pasting* G_1 and G_2 on S_1 and S_2. The following lemma is well known.

Lemma 3. *Let G_1 and G_2 be K_t-minor-free graphs. Let S_i be a k-clique in each G_i. Let G be a pasting of G_1 and G_2 on S_1 and S_2. Then G is K_t-minor-free.*

Proof. Suppose on the contrary that K_{t+1} is a minor of G. Let $X_1, \dots, X_{t+1}$ be the corresponding branch sets. If some X_i does not intersect G_1 and some X_j does not intersect G_2, then no edge joins X_i and X_j, which is a contradiction. Thus, without loss of generality, each X_i intersects G_1. Let $X'_i := G_1[X_i]$. Since S_1 is a clique, X'_i is connected. Thus $X'_1, \dots, X'_{t+1}$ are the branch sets of a K_{t+1}-minor in G_1. This contradiction proves that G is K_t-minor-free. □

Let $K_{r\times 2}$ be the complete r-partite graph with r colour classes of size 2. Let $K_{1,r\times 2}$ be the complete $(r+1)$-partite graph with r colour classes of size 2 and one colour class of size 1. That is, $K_{r\times 2}$ and $K_{1,r\times 2}$ are respectively obtained from K_{2r} and K_{2r+1} by deleting a matching of r edges. The following lemma will be useful.

Lemma 4 ([8, 31])**.** *$K_{r\times 2}$ is $K_{\lfloor 3r/2\rfloor+1}$-minor-free, and $K_{1,r\times 2}$ is $K_{\lfloor 3r/2\rfloor+2}$-minor-free.*

Proof of Theorem 1. Our goal is to construct a K_p-minor-free graph and a non-achievable list assignment with q colours per vertex, where the integers p, q and r and a graph H are defined in the following table. Let $\{v_1w_1, \dots, v_rw_r\}$ be the deleted matching in H. By Lemma 4, the calculation in the table shows that H is K_p-minor-free.

case	p	q	r	H	
(a)	$3t+2$	$4t$	$2t+1$	$K_{r\times 2}$	$\lfloor \frac{3}{2}r \rfloor + 1 = 3t+2 = p$
(b)	$3t+1$	$4t-2$	$2t$	$K_{r\times 2}$	$\lfloor \frac{3}{2}r \rfloor + 1 = 3t+1 = p$
(c)	$3t$	$4t-3$	$2t-1$	$K_{1,r\times 2}$	$\lfloor \frac{3}{2}r \rfloor + 2 = 3t = p$

For each vector $(c_1, \dots, c_r) \in [1, q]^r$, let $H(c_1, \dots, c_r)$ be a copy of H with the following list assignment. For each $i \in [1, r]$, let $L(w_i) := [1, q+1] \setminus \{c_i\}$. Let $L(u) := [1, q]$ for

each remaining vertex u. There are $q+1$ colours in total, and $|V(H)| = q+2$. Thus in every L-colouring of H, two non-adjacent vertices receive the same colour. That is, $\mathrm{col}(v_i) = \mathrm{col}(w_i)$ for some $i \in [1,r]$. Since each $c_i \notin L(w_i)$, it is not the case that each vertex v_i is coloured c_i.

Let G be the graph obtained by pasting all the graphs $H(c_1,\dots,c_r)$, where $(c_1,\dots,c_r) \in [1,q]^r$, on the clique $\{v_1,\dots,v_r\}$. The list assignment L is well defined for G since $L(v_i) = [1,q]$. By Lemma 3, G is K_p-minor-free. Suppose that G is L-colourable. Let c_i be the colour assigned to each vertex v_i. Thus $c_i \in L(v_i) = [1,q]$. Hence, as proved above, the copy $H(c_1,\dots,c_r)$ is not L-colourable. This contradiction proves that G is not L-colourable. Each vertex of G has a list of q colours in L. Therefore G is not q-choosable. (It is easily seen that G is q-degenerate[3], implying G is $(q+1)$-choosable.) □

Note that this proof was inspired by the construction of a non-4-choosable planar graph by Mirzakhani [17].

3 Conclusion

Theorem 1 disproves the List Hadwiger Conjecture. However, list colourings remain a viable approach for attacking Hadwiger's Conjecture. Indeed, list colourings provide potential routes around some of the known obstacles, such as large minimum degree, and lack of exact structure theorems; see [10, 13, 32, 33].

The following table gives the best known lower and upper bounds on the maximum choice number of K_t-minor-free graphs. Each lower bound is a special case of Theorem 1. Each upper bound (except $t=5$) follows from the following degeneracy results. Every K_3-minor-free graph (that is, every forest) is 1-degenerate. Dirac [4] proved that every K_4-minor-free graph is 2-degenerate. Mader [16] proved that for $t \le 7$, every K_t-minor-free graph is $(2t-5)$-degenerate. Jørgensen [9] and Song and Thomas [23] proved the same result for $t=8$ and $t=9$ respectively. Song [24] proved that every K_{10}-minor-free graph is 21-degenerate, and that every K_{11}-minor-free graph is 25-degenerate. In general, Kostochka [14, 15] and Thomason [25, 26] independently proved that every K_t-minor-free graph is $\mathcal{O}(t\sqrt{\log t})$-degenerate.

t	3	4	5	6	7	8	9	10	11	$\cdots$	t
lower bound	2	3	5	6	7	9	10	11	13	$\cdots$	$\frac{4}{3}t - c$
upper bound	2	3	5	8	10	12	14	22	26	$\cdots$	$\mathcal{O}(t\sqrt{\log t})$

The following natural open problem arises: Is every K_6-minor-free graph 6-choosable? It is even open whether every K_6-minor-free graph is 7-choosable. This would be implied if every K_6-minor-free graph is 6-degenerate (and we conjecture that this is true). Equivalently, we conjecture that every graph with minimum degree at least 7 contain a K_6-minor.

[3] A graph is d-degenerate if every subgraph has minimum degree at most d. Clearly every d-degenerate graph is $(d+1)$-choosable.

It is even open whether every 7-connected graph contains a K_6-minor. This would be implied by Jørgensen's conjecture, which asserts that every 6-connected K_6-minor-free graph is apex[4]. Jørgensen's conjecture was recently proved for sufficiently large graphs [11, 12].

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